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On Artin L-Functions

Maruti Ram Murty[1]

§1. Introduction.

The celebrated conjecture of Emil Artin about the holomorphy of his non-abelian L-series has inspired a vast amount of development in number theory, algebraic geometry and representation theory. There exist at least four different programs to approach this conjecture. Most notable amongst these is the famous Langlands program. One of the objectives of the Langlands program has been to create the theoretical framework with which to attack Artin's conjecture. The notions of base change, automorphic induction and converse theory provide the conceptual tools to be developed and applied towards this goal. There are already excellent descriptions of this approach in the literature, such as Gelbart [Ge], Murty [Mu1], Prasad and Yogananda [PY] and the recent paper by Rogawski [Rog]. Therefore, we shall not deal with this approach in this survey.

A second method is the program initiated by Serre [Se2]. Indeed, Khare [Kh] has recently shown that Serre's conjectures imply Artin's conjecture for two-dimensional, complex, odd representations over $\mathbb{Q}$. After the spectacular success of Wiles, Buzzard and Taylor [BT] proved a theorem that makes a significant advance towards the A_5 case of Artin's conjecture. We will refer the reader to these papers as well as [ST] for an insight into these new p-adic methods. In this paper, we shall focus more on the analytic aspects of Artin L-series and describe the approach implied by the recently formulated conjectures of Selberg concerning general L-functions with Euler products and functional equations. At the end of the paper, we discuss certain group-theoretic considerations

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that give some information on the location of poles (if any) of Artin L-functions. These are the third and fourth methods implied above.

The Selberg conjectures predict a certain 'orthogonality' principle and a remarkable unique factorization theorem for decomposing general L-functions into 'primitive' functions from which both the Artin conjecture and the Langlands reciprocity conjecture (at least in the solvable case) follow. Perhaps a final resolution of Artin's conjecture will involve a marriage of the two approaches.

The group-theoretic approach initiated by Heilbronn [H] and Stark [St] fuses group theory and analytic number theory and has been successfully developed by Murty [VKM, Mu] and Foote-Murty [FM]. The new results of this paper are contained in section 3 where we push this theme further and refine the results of Foote-Murty [FM].

The notion of an Artin L-function can be defined for any global field. In the case of a function field over a finite field, the holomorphy of Artin L-series is known and is due to Weil [W]. However, the reciprocity conjecture of Langlands has been settled only for dimensions 1 (class field theory) and 2 (Drinfeld theory). It would be interesting to formulate the Selberg approach in the function field context. In this paper, we shall deal exclusively with the number field setting.

Let K/k be a Galois extension of algebraic number fields with $\mathrm{Gal}(K/k) = G$. Let V be a finite dimensional vector space over $\mathbb{C}$ and $\phi : G \to GL(V)$ a representation. An Artin L-function is a meromorphic function

$$L(s, \phi; K/k)$$

attached to this data. For each prime ideal $\wp$ of K, let

$$I_\wp = \{\sigma \in G : \sigma(x) \equiv x \pmod{\wp}\}$$

be its inertia group and

$$D_\wp = \{\sigma \in G : \wp^\sigma = \wp\}$$

its decomposition group. The inertia group is a normal subgroup of the decomposition group and the quotient $D_\wp/I_\wp$ is cyclic generated by the Frobenius automorphism $\sigma_\wp$. This automorphism has the property that

$$\sigma_\wp(x) \equiv x^{N(\mathfrak{p})} \pmod{\wp}$$

where $\mathfrak{p} = \wp \cap k$ and N is the absolute norm from k to $\mathbb{Q}$. For any $\wp | \mathfrak{p}$, the Frobenius elements $\sigma_\wp$ are well-defined modulo $I_\wp$ and are all conjugate.

When the inertia is trivial, which is the case for all $\mathfrak{p}$ unramified in K, the conjugacy class of $\sigma_\wp$ is called the **Artin symbol** $\sigma_\mathfrak{p}$. In all cases, it is a well-defined conjugacy class modulo inertia. The Artin L-function is defined as

$$L(s, \phi; K/k) = \prod_{\mathfrak{p}} \det\left(1 - \phi(\sigma_\mathfrak{p}) N(\mathfrak{p})^{-s} | V^{I_\wp}\right)^{-1}.$$

We may sometimes abbreviate this as $L(s, \phi)$. Since $\chi = Tr\,\phi$ determines ϕ upto equivalence, we also write $L(s, \chi; K/k)$ for $L(s, \phi; K/k)$.

Artin [A] showed that these L-functions satisfy the following functorial properties:

(1) $L(s, \phi_1 \oplus \phi_2) = L(s, \phi_1) L(s, \phi_2)$.

If H is a subgroup of G and τ is a representation of H, then

(2) $L(s, \tau; K/K^H) = L(s, Ind_H^G \tau; K/k)$.

We have the famous **Artin's Conjecture:** for any irreducible $\phi \neq 1$, $L(s, \phi; K/k)$ extends to an entire function of s. In the case ϕ is one-dimensional, Artin proved his conjecture by establishing what is now called the **Artin reciprocity law**. This states that if ϕ is one-dimensional, there is a Hecke character π_ϕ of k such that

$$L(s, \pi_\phi) = L(s, \phi; K/k).$$

This theorem is considered to be one of the masterpieces of class field theory. It embodies all the classical reciprocity laws such as quadratic, cubic and higher power reciprocity laws.

Brauer[B] showed that $L(s, \phi; K/k)$ extends to a meromorphic function for all $s \in \mathbb{C}$ using his famous induction theorem and Artin's reciprocity law. **Brauer's induction theorem** states that any character χ of a finite group can be written as

$$\chi = \sum_i n_i Ind_{H_i}^G \psi_i$$

where H_i's are nilpotent and ψ_i are one-dimensional and the n_i are integers. This immediately implies Brauer's theorem since

$$\begin{aligned} L(s,\chi,K/k) &= \prod_i L(s, Ind_{H_i}^G \psi_i; K/k)^{n_i} \\ &= \prod_i L(s,\psi_i; K/K^{H_i})^{n_i} \quad \text{by (2)} \\ &= \prod_i L(s,\pi_{\psi_i})^{n_i} \quad \text{by Artin reciprocity.} \end{aligned}$$

Brauer's theorem also implies Artin's conjecture if G is nilpotent or supersolvable since in these two cases every character of G is monomial (see for example Serre [Ser]).

In the case of two-dimensional representations we have the important **Langlands-Tunnell theorem**: if ϕ is 2-dimensional with **solvable** image, then Artin's conjecture is true and in this case $L(s,\phi,K/k)$ is equal to the (Jacquet-Langlands) L-function of an automorphic representation of $GL_2(\mathbb{A}_k)$, where $\mathbb{A}_k$ is the adele ring of k.

The Langlands reciprocity conjecture is also referred to as the **Strong Artin Conjecture** in the literature. This conjecture forecasts that for each irreducible representation ϕ of G of degree n, there exists a cuspidal automorphic representation $\pi = \pi_\phi$ of $GL_n(\mathbb{A}_k)$ such that

$$L(s,\phi;K/k) = L(s,\pi).$$

Some progress has been made towards this conjecture. For instance, there is the theorem of Arthur and Clozel [AC]: the strong Artin conjecture is true for nilpotent extensions. There is a recent theorem of Yuanli Zhang [Z]: the strong Artin conjecture is true if G is a Frobenius group. Some examples of the "icosahedral" case of Artin's conjecture are due to J. Buhler [B].

The strong Artin conjecture associates an automorphic form for each complex linear representation of $G_{\mathbb{Q}}$. In the other direction, there is the **Deligne-Serre theorem**: given any newform f of weight 1 and level N, there is a two dimensional $G_{\mathbb{Q}}$ representation ϕ such that

$$L(s,f) = L(s,\phi).$$

The converse theorem in the simplest case as described by Weil [W2] shows that Artin's conjecture for two dimensional Galois representations implies the strong Artin conjecture. A suitable converse theory would give in general that Artin's conjecture implies strong Artin conjecture.

See for example the paper by Ramakrishnan [R] for a discussion of this theme.

§2. Analytic theory of Artin L-series.

We will now study the behaviour of Artin L-series at a given point in complex plane. These ideas originate with Heilbronn [H], Stark [St] and Kumar Murty [Mu] and involve a beautiful interplay of character theory of finite groups and the Artin L-function formalism from which many important deductions such as the Aramata-Brauer theorem can be made. The most important of these deductions seems to be the relationship between the zeros of the Dedekind zeta function and Artin L-series.

Fix $s_0 \in \mathbb{C}$. We want to study the behaviour of Artin L-functions at s_0. Let $G = \mathrm{Gal}(K/k)$ and H a subgroup of G. We define the function

$$\theta_H(g) = \sum_{\psi \in \hat{H}} n_\psi \psi(g)$$

where

$$n_\psi = ord_{s=s_0} L(s,\psi;K/K^H).$$

Since $L(s,\psi;K/K^H) = L(s, Ind_H^G\psi; K/k)$ we get

$$n_\psi = \sum_{\chi\in\hat{G}} (Ind_H^G\psi,\chi) n_\chi.$$

Proposition 1 (Heilbronn-Stark).

$$\theta_G|_H = \theta_H.$$

Proof. By Frobenius reciprocity,

$$\begin{aligned} \theta_G|_H &= \sum_{\chi\in\hat{G}} n_\chi \chi|_H \\ &= \sum_{\chi\in\hat{G}} n_\chi \sum_{\psi\in\hat{H}} (\psi,\chi|_H)\psi \\ &= \sum_{\psi\in\hat{H}} \Big(\sum_{\chi\in\hat{G}} (Ind_H^G\psi,\chi) n_\chi\Big)\psi \\ &= \sum_{\psi\in\hat{H}} n_\psi\psi = \theta_H. \end{aligned}$$

This completes the proof.

Now recall the Artin-Takagi factorization:

$$\zeta_K(s) = \prod_{\psi\in\hat{H}} L(s,\psi;K/K^H)^{\psi(1)}.$$

We can use this to deduce:

Corollary 2. *If H is abelian,*

$$|\theta_H(g)| \le ord_{s=s_0}\zeta_K(s).$$

Proof.

$$|\theta_H(g)| \le \sum_{\psi\in\hat{H}} n_\psi = ord_{s=s_0}\zeta_K(s).$$

Theorem 3 (Kumar Murty).

$$\sum_{\chi\in\hat{G}} n_\chi^2 \le (ord_{s=s_0}\zeta_K(s))^2.$$

Proof. We have,

$$\begin{aligned}\sum_{\chi\in\hat{G}} n_\chi^2 &= (\theta_G,\theta_G)\\ &= \frac{1}{|G|}\sum_{g\in G}|\theta_G(g)|^2\\ &= \frac{1}{|G|}\sum_{g\in G}|\theta_{\langle g\rangle}(g)|^2\\ &\le \frac{1}{|G|}\sum_{g\in G}(ord_{s=s_0}\zeta_K(s))^2\\ &\le (ord_{s=s_0}\zeta_K(s))^2.\end{aligned}$$

This completes the proof.

Corollary 4. *$\zeta_K(s)L(s,\phi;K/k)$ is regular for $s\ne 1$.*

Proof. Let us write χ for the character of the representation ϕ. By the inequality of Theorem 3, we have $|n_\chi| \le ord_{s=s_0}\zeta_K(s)$ from which the result is immediate.

Corollary 5. *$L(s,\phi;K/k)$ is analytic and non-zero for* $\mathrm{Re}(s)=1$.

Proof. Since the Dedekind zeta function doesn't vanish on the line $\mathrm{Re}(s)=1$, we get immediately that every Artin L-function is regular on the line $\mathrm{Re}(s)=1$.

Corollary 6 (Aramata-Brauer). *$\zeta_K(s)/\zeta_k(s)$ is entire.*

Proof. Since the trivial character corresponds to the zeta function of the base field, the result is immediate from Theorem 3.

Corollary 7. *All the poles (if any) of an Artin L-function $L(s,\phi;K/k)$ are contained in the zeros of $\zeta_K(s)$.*

Proof. This is clear from Theorem 3.

Corollary 8 (Stark). *If $s=s_0$ is a zero of $\zeta_K(s)$ of order ≤ 1, then any Artin L-function is analytic at $s=s_0$.*

Proof. Any simple zero of $\zeta_K(s)$ must arise from at most one character. If this comes from a pole of an Artin L-series attached to a non-abelian character, then the Artin-Takagi facatorization gives a contradiction, for then we would have a pole of $\zeta_K(s)$ at a point s unequal to 1 and this is a contradiction. Thus the zero must come from an abelian L-series and by the reciprocity law, the L-series is a Hecke L-series which is entire. This completes the proof.

Corollary 9 (Foote-K. Murty). *If K/k is solvable with group G and $s=s_0$ is a zero of $\zeta_K(s)$ of order $\le p-2$ where p is the second smallest prime divisor of $|G|$, then any Artin L-function $L(s,\phi;K/k)$ is analytic at $s=s_0$.*

Proof. We refer the reader to [KM].

Corollary 10. *If RH holds for $\zeta_K(s)$, then Artin L-functions are analytic for* $\mathrm{Re}(s)>1/2$.

Proof. This is clear from Theorem 3.

In section 3 below, we will discuss variations on this theme.

Brauer's theorem also implies non-vanishing of Artin L-functions on the line $\mathrm{Re}(s)=1$ since Hecke's L-functions don't vanish there. However, the non-vanishing of Artin L-functions can also be deduced from Theorem 3 above. The classical trigonometric inequality $3+4\cos\theta+\cos 2\theta\ge 0$ can be used to show the non-vanishing of the Dedekind zeta function on the line $\mathrm{Re}(s) = 1$. From Theorem 3, the holomorphy and non-vanishing of Artin L-functions on the line $\mathrm{Re}(s) = 1$ is now immediate. (The reader may consult the monograph [MM2] for details.) By the Wiener-Ikehara Tauberian theorem, this implies

$$\sum_{N(\mathfrak{p})\le x} \chi(\sigma_\mathfrak{p}) = o(x/\log x)$$

as $x \to \infty$ and χ irreducible $\neq 1$.

One can also derive the Chebotarev density theorem from this. For each conjugacy class C of G, set

$$\pi_C(x) = \#\{N(\mathfrak{p}) \le x : \mathfrak{p} \text{ unramified, } \sigma_\mathfrak{p} \in C\}$$

and

$$\pi(x,\chi) = \sum_{N(\mathfrak{p})\le x} \chi(\sigma_\mathfrak{p})$$

so that the orthogonality relations give

$$\pi_C(x) = \frac{|C|}{|G|}\sum_\chi \bar{\chi}(g_C)\pi(x,\chi)$$

where $g_C \in C$. We then have the **Chebotarev density theorem**. As $x \to \infty$,

$$\pi_C(x) \sim \frac{|C|}{|G|}\pi(x)$$

where $\pi(x)$ is the number of primes $\le x$. This theorem generalizes Dirichlet's theorem about primes in arithmetic progressions. One important consequence of the Chebotarev density theorem is that if ϕ_1 and ϕ_2 are two representations of G and

$$\mathrm{Tr}\,\phi_1(\sigma_\mathfrak{p}) = \mathrm{Tr}\,\phi_2(\sigma_\mathfrak{p})$$

for all but finitely many ideals $\mathfrak{p}$ of k, then ϕ_1 and ϕ_2 are isomorphic.

In many questions of analytic number theory, one needs a stronger version of this theorem with effective error terms. The relationship between the effective versions of Chebotarev density theorem and Artin's conjecture is exemplified by the following theorem.

Theorem 11 (Effective Chebotarev) : (K. & R. Murty, Saradha [MMS]). *Assume Artin's conjecture and that $\zeta_K(s)$ has all its*

zeros on $\mathrm{Re}(s) = 1/2$, *then*

$$|\pi_C(x) - \frac{|C|}{|G|}\pi(x)| \ll |C|^{1/2}x^{1/2}[K:\mathbb{Q}](\log M(K/k)x)$$

where

$$M(K/k) = d_k^{1/[k:\mathbb{Q}]} \prod_{p\in P(K/k)} p$$

and $P(K/k)$ is the set of rational primes p for which there is a prime ideal $\mathfrak{p}|p$ in k which ramifies in K.

The above discussion leads to the following question. Does the Riemann hypothesis for Dedekind zeta functions imply Artin's conjecture? K. Murty's theorem shows that if there are any poles, they must be on the line $\mathrm{Re}(s) = 1/2$. It seems difficult to answer this question. However, recently in joint work with A. Perelli, it was proved that a 'pair-correlation' conjecture for zeros of functions in the Selberg class leads to a proof of the Selberg conjectures. This in turn implies Artin's conjecture and the Langlands reciprocity conjecture (in the solvable case) as described by Murty [Mu2].

We begin by describing the **Pair Correlation Conjecture**. Assume RH for the Riemann zeta function. Recall the pair correlation conjecture formulated by Montgomery [Mo]. Write $1/2 + i\gamma$ for a typical zero of $\zeta(s)$. If $0 \notin [\alpha,\beta]$ and $T \to \infty$, then

$$\#\{0 < \gamma,\gamma' < T : \alpha \le \frac{(\gamma-\gamma')\log T}{2\pi} \le \beta\}$$
$$\sim (\frac{T}{2\pi}\log T)\int_\alpha^\beta (1 - \frac{\sin^2 \pi u}{\pi^2 u^2})du.$$

This is consistent with the pair-correlation function of eigenvalues of a random Hermitian operator.

Selberg's Class consists of Dirichlet series

$$F(s) = \sum_{n=1}^\infty \frac{a_n(F)}{n^s}$$

which are absolutely convergent in $\mathrm{Re}(s) > 1$ such that $(s-1)^m F(s)$ admits an analytic continuation to an entire function for some non-negative integer m, and satisfies a functional equation and an Euler product condition. More precisely, the functional equation has the form:

$$\Phi(s) := Q^s \prod_{i=1}^{d} \Gamma(\lambda_i s + \mu_i) F(s)$$
$$= w\overline{\Phi}(1 - \bar{s})$$

where $|w| = 1$, $Q > 0$, and $\lambda_i > 0$, $\mathrm{Re}(\mu_i) \geq 0$ and the Euler product is of the form

$$F(s) = \prod_p F_p(s)$$

with

$$F_p(s) = \exp\Big(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\Big)$$

and $b(p^k) = O(p^{k\theta})$ for some $0 \leq \theta < 1/2$. In addition, $a_1(F) = 1$ and $a_n(F) \ll n^{\epsilon}$.

An element F in the Selberg class $\mathcal{S}$ is called **primitive** if it cannot be factored as a product of two elements non-trivially. Selberg [Sel] proved that every element of $\mathcal{S}$ can be written as a product of primitive elements. Is this factorization unique? Selberg conjectures yes.

For two primitive functions F_1 and F_2, Selberg conjectures

$$\sum_{p \leq x} \frac{a_p(F_1)\overline{a_p(F_2)}}{p} = \log\log x + O(1) \quad \text{if } F_1 = F_2$$
$$= O(1) \quad \text{if } F_1 \neq F_2.$$

This conjecture implies the unique factorization conjecture (see [CG] and [Mu2]).

Theorem 12 (R. Murty). *Selberg's conjecture implies Artin's conjecture and for $K/\mathbb{Q}$ solvable, it implies the strong Artin conjecture.*

Proof. We refer the reader to [Mu2] for details. We outline the highlights below. By the classical factorization of Artin and Takagi, we have, on the one hand,

$$\zeta_K(s) = \prod_{\phi \in \hat{G}} L(s, \phi; K/\mathbb{Q})^{\phi(1)}$$

and on the other, by the result of Arthur and Clozel [AC]

$$\zeta_K(s) = \prod_i L(s, \pi_i)^{e_i}$$

Selberg conjectures imply each $L(s, \phi; K/\mathbb{Q})$ is entire and primitive. (This is an application of Chebotarev density theorem.)

Selberg conjectures also imply $L(s, \pi_i)$ are primitive and so by unique factorisation, we get that each $L(s, \phi; K/\mathbb{Q})$ is an $L(s, \pi_i)$ which is the strong Artin conjecture. This proves Theorem 12.

For a primitive function F, it seems reasonable to conjecture that if $0 \notin [\alpha, \beta]$, then

$$\#\{0 \leq |\gamma_1|, |\gamma_2| \leq T : \alpha \leq \frac{(\gamma_1 - \gamma_2) d_F \log T}{2\pi} \leq \beta\}$$
$$\sim \frac{T \log T}{2\pi} \int_\alpha^\beta \Big(1 - \Big(\frac{\sin \pi u}{\pi u}\Big)^2\Big) du.$$

There should be very little correlation between zeros of two distinct primitive functions.

Turning to the analogue of the pair correlation function of Montgomery, for functions in $\mathcal{S}$, we define for $F, G \in \mathcal{S}$, the correlation function

$$\mathcal{F}_{F,G}(\alpha, T) = \frac{\pi}{T \log X} \sum_{-T \leq \gamma_F, \gamma_G \leq T} X^{i\alpha(\gamma_F - \gamma_G)} w(\gamma_F - \gamma_G)$$

where $w(u) = 4/(4+u^2)$, $d_F = 2\sum_{i=1}^{d} \lambda_i$, $X = T^{d_F}$, $1/2 + i\gamma_F$ and $1/2 + i\gamma_G$ run through the zeroes of F and G respectively. Now assume the GRH for each $F \in \mathcal{S}$. The pair correlation conjecture is that for F, G primitive functions of $\mathcal{S}$, we have

$$\mathcal{F}_{F,G}(\alpha) = \begin{cases} \delta_{F,G}|\alpha| + d_G T^{-2|\alpha| d_F} \log T (1 + o(1)) + o(1) & \text{if } |\alpha| \leq 1 \\ \delta_{F,G} + o(1) & \text{if } |\alpha| \geq 1 \end{cases}$$

uniformly for α in any bounded interval. (Here $\delta_{F,G} = 1$ if $F = G$ and 0 otherwise.) This conjecture implies that almost all zeros of a primitive function are simple [MP].

Theorem 13 (R. Murty and A. Perelli). *Assume that each element of the Selberg class satisfies the Riemann hypothesis. In addition,* *suppose a pair correlation conjecture for primitive elements. Then Selberg's conjectures are true and thus the unique factorization conjecture follows.*

Therefore, RH and a form of the pair correlation conjecture for elements of the Selberg class imply Artin's conjecture and the strong Artin conjecture for solvable Galois extensions of $\mathbb{Q}$.

§3. Variations.

The above discussion can be generalised in various ways. We begin by describing a general formalism first outlined in [MM1]. Let G be a finite group. For every subgroup H of G and complex character ψ of H, we attach a complex number $n(H,\psi)$ satisfying the following properties:

(3) $n(H,\psi+\psi') = n(H,\psi) + n(H,\psi')$,

(4) $n(H, Ind_H^G\psi) = n(H,\psi)$.

Defining

$$\theta_H = \sum_{\psi} n(H,\psi)\psi$$

where the sum is over all irreducible characters ψ of H leads to the generalisation of the Heilbronn-Stark lemma above. Namely, $\theta_G|_H = \theta_H$. And this leads, in a purely formal way to the fundamental inequality

$$\sum_{\chi} |n(G,\chi)|^2 \leq n(G,reg)^2,$$

whenever $n(H,\psi) \geq 0$ for every cyclic subgroup H of G and reg denotes the regular representation. The formalism can be applied to the case G is the Galois group of a normal extension K/k and $n(H,\psi)$ is the order of the zero at $s = s_0$ of the Artin L-series attached to the Galois extension K/K^H. It can also be applied to the situation when E is an elliptic curve over k and $n(H,\psi)$ corresponds to the order of the zero at $s = s_0$ of a twist by ψ of the L-function of the elliptic curve over K^H. This was addressed in [MM1] and we refer the reader to that paper for further details. We will sharpen the above discussion in the following way.

It is clear that

$$(\theta_G - n(G,1)1, \theta_G - n(G,1)1) = \sum_{\chi\neq 1} n(G,\chi)^2.$$

Since $\theta_G|_H = \theta_H$, we observe that

$$\begin{aligned}\theta_G(g) - n(G,1) &= \theta_{\langle g\rangle}(g) - n(G,1)\\ &= \sum_{\psi} n(\langle g\rangle,\psi)\psi(g) - n(G,1)\\ &= n(\langle g\rangle,1) - n(G,1) + \sum_{\psi\neq 1} n(\langle g\rangle,\psi)\psi(g).\end{aligned}$$

If we suppose that for every cyclic H, we have $n(H,1) \geq n(G,1)$, then

$$|\theta_G(g) - n(G,1)| \leq n(G,reg) - n(G,1).$$

This proves:

Theorem 14. *Suppose $n(H,1) \geq n(G,1)$ for every cyclic H. Then*

$$\sum_{\chi\neq 1} |n(G,\chi)|^2 \leq (n(G,reg) - n(G,1))^2.$$

Applying this to the context of zeros discussed in the previous section gives the following sharper form of Theorem 2.

Corollary 15. *Assume $\zeta_{K^H}(s)/\zeta_k(s)$ is regular at $s = s_0$ for every cyclic subgroup H of G. Then,*

$$\sum_{\chi\neq 1} n_\chi^2 \leq (ord_{s=s_0}\zeta_K(s)/\zeta_k(s))^2.$$

The hypothesis of Corollary 15 is satisfied when K/k is solvable (see [Mu], Uchida [U] or van der Waall [VW]). This gives

Theorem 16. *Let K/k be solvable. Then,*

$$\sum_{\chi\neq 1} n_\chi^2 \leq (ord_{s=s_0}\zeta_K(s)/\zeta_k(s))^2.$$

Another variation is obtained by combining two functions in the above discussion. Suppose we have numbers $n_i(H,\psi)$ satisfying (3) and (4). Let

$$\theta_G^{(i)}(g) = \sum_{\chi} n_i(G,\chi)\chi(g).$$

Then, we easily see directly, or by applying the Cauchy-Schwarz inequality:

Proposition 17. *Suppose $n_i(H,\psi) \geq 0$ for every cyclic subgroup H of G and every irreducible ψ of H. Then,*

$$|\sum_{\chi} n_1(G,\chi)\overline{n_2(G,\chi)}| \leq n_1(G,reg)n_2(G,reg).$$

Applied to zeros, this leads to

Corollary 18. *For $s_0, s_0' \in \mathbb{C}$, we have*

$$\sum_{\chi} n_\chi(s_0)n_\chi(s_0') \leq (ord_{s=s_0}\zeta_K(s))(ord_{s=s_0'}\zeta_K(s)).$$

The corresponding version of Theorem 14 is:

Theorem 19. *Suppose $n_i(H,1) \geq n_i(G,1)$ and $n_i(H,\psi) \geq 0$ for every cyclic subgroup H of G. Then*

$$\sum_{\chi\neq 1} n_1(G,\chi)\overline{n_2(G,\chi)} \leq (n_1(G,reg) - n_1(G,1))(n_2(G,reg) - n_2(G,1)).$$

Corollary 20. *Suppose $\zeta_{K^H}(s)/\zeta_k(s)$ is regular at $s = s_0, s_0'$ for every cyclic subgroup H of G. Then*

$$\sum_{\chi\neq 1} n_\chi(s_0)n_\chi(s_0') \leq (ord_{s=s_0}\zeta_K(s)/\zeta_k(s))(ord_{s=s_0'}\zeta_K(s)/\zeta_k(s)).$$

Corollary 21. *If K/k is solvable, then*

$$\sum_{\chi\neq 1} n_\chi(s_0)n_\chi(s_0') \leq (ord_{s=s_0}\zeta_K(s)/\zeta_k(s))(ord_{s=s_0'}\zeta_K(s)/\zeta_k(s)).$$

An interesting special case is if $s_0' = 1 - s_0$. Then, by the functional equation, $n_\chi(1-s_0) = n_{\overline{\chi}}(s_0)$ so we deduce:

Corollary 22. *If K/k is solvable, then*

$$\sum_{\chi\neq 1} n_\chi(s_0)n_{\overline{\chi}}(s_0) \leq (ord_{s=s_0}\zeta_K(s)/\zeta_k(s))^2.$$

To conclude, let us observe that Theorem 16 implies that all the poles (if any) of a non-abelian Artin L-function corresponding to a solvable extension K/k are contained in the zeros of $\zeta_K(s)/\zeta_k(s)$. One can

also deduce the same result provided that the base change of a zeta function to any arbitrary finite extension of $\mathbb{Q}$ does indeed correspond to an automorphic representation as predicted by Langlands. At present, this is not known.

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Department of Mathematics
Queen's University
Kingston, K7L 3N6
Ontario, Canada
E-mail address: `murty@mast.queensu.ca`