**MATH 381: Assignment 4 (due: March 13, 2020)**

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1. Show that if $2^n+1$ is a prime number, then $n$ is a power of 2. [Hint: write $n = 2^k m$ with $m$ odd. If $m > 1$, show that $2^{2^k} + 1$ divides $2^n + 1$.]

2. Given an ellipse with semi-major axis length $a$ and semi-minor axis length $b$, show that the eccentricity $e$ is given by

$$e = \sqrt{1 - \frac{b^2}{a^2}}.$$

[Recall that if $e$ is the eccentricity, then $(\pm ae, 0)$ are the two focal points of the ellipse in Cartesian co-ordinates. This fact was used in Newton's derivation of Kepler's third law. ]

3. Prove that

$$\left| \frac{\pi}{4} - \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \right| \leq \frac{1}{2n+3}.$$

Deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots .$$

4. Show that

$$\prod_{k=1}^{n} \frac{(2k)(2k)}{(2k-1)(2k+1)} = \frac{2^{4n}}{2n+1} \binom{2n}{n}^{-2}.$$

[This fact was needed in our proof of Wallis's formula for $\pi$.]

5. Write a short essay (minimum 1 page; maximum 2 pages, typed in 12 point font, double spaced) tracing the evolution of calculus. Discuss the role physics played in this evolution?