Number Theory, edited by R. Mollin, p. 427-439, de Gruyter 1990.
Proceedings of the First CNTA Conference, Banff, Alberta, April 1988.

On Simple Zeros of Certain L-Series

M. Ram Murty [1]

1. Introduction

Let E be an elliptic curve defined over $\mathbb{Q}$ and denote by Ш the Tate–Shafarevic group of $E/\mathbb{Q}$. It is an outstanding unresolved conjecture that this group is finite. If the rank of the Mordell–Weil group of E over $\mathbb{Q}$ is zero, and E has complex multiplication, then K. Rubin [12] showed that Ш is finite. Recently, V. Kolyvagin [7] (see also Washington [15] for an exposition of Kolyvagin's work) showed that if E is a modular elliptic curve over $\mathbb{Q}$ with Mordell–Weil rank equal to zero, then Ш is finite, *provided* there is a "quadratic twist" of the L-series of the given elliptic curve having a simple zero at $s = 1$. In any given case, it is not difficult to produce such a twist. To demonstrate that, in general, such a twist exists seems to be difficult.

The purpose of this paper is to show that on the generalized Riemann hypothesis (GRH), such a twist always exists. Our method has other applications. For instance, if we do not confine ourselves to quadratic twists and consider twisting by any character $\chi \pmod q$, then the density of such twists with a zero at $s = 1$ is at most 1/2, under the same hypothesis. The method applied to a classical context shows that if $L(s,\chi)$ is the classical Dirichlet L-series, then $L(1/2,\chi) \neq 0$ for at least $\phi(q)/2$ characters $\chi \pmod q$. (It is generally conjectured that $L(1/2,\chi) \neq 0$ for all characters $\chi \pmod q$, but this is, as yet, unproved. I am not sure of the source of this conjecture but I think it originates from S. Chowla.) To state the results more precisely, we need the following background.

1 Research partially supported by an NSERC grant.

Let f be a cusp form of weight 2 for $\Gamma_0(N)$. Let

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

be its Fourier expansion at the cusp $i\infty$. Let

$$L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and

$$L(f, D, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}\left(\frac{D}{n}\right)$$

where $\left(\frac{D}{n}\right)$ denotes the Legendre symbol. It is well-known that [13] by the theory of modular forms, both $L(f, s)$ and $L(f, D, s)$ have an analytic continuation to the entire complex plane. Moreover, if D is a fundamental discriminant (that is, D is the discriminant of a quadratic field) and f is an eigenfunction of the Hecke operators, then $L(f, s)$ and $L(f, D, s)$ can be written as Euler products and have functional equations. More precisely, if we let

$$\Lambda(s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f, s)$$

then $\Lambda(s)$ is entire and satisfies

$$\Lambda(s) = w\,\Lambda(2-s)$$

where $w = \pm 1$. If $w = -1$, then $L(f, s)$ has a zero of odd order at $s = 1$ and if $w = 1$, it has a zero of even order at $s = 1$. As $\Gamma(s)$ has simple poles at $s = 0$, $-1, \ldots$, $L(f, s)$ has (trivial) zeros at these points. If we let

$$\Lambda(s, D) = D^s N^{s/2}(2\pi)^{-s}\Gamma(s)L(f, D, s)$$

then $\Lambda(s, D)$ is entire and satisfies

$$\Lambda(s, D) = w\left(\frac{D}{-N}\right)\Lambda(2-s, D).$$

Again, $L(f, s)$ has trivial zeros at $s = 0, -1, -2, \ldots$. Moreover, $L(f, s)$ has an Euler product of the form:

$$L(f, s) = \prod_{p \mid N}\left(1 - \frac{a_p}{p^s}\right)^{-1} \prod_{p \nmid N}\left(1 - \frac{\alpha_p}{p^s}\right)^{-1}\left(1 - \frac{\bar{\alpha}_p}{p^s}\right)^{-1}.$$

$L(f, D, s)$ has a similar Euler product.

If E is a modular elliptic curve of conductor N, then the L series of the elliptic curve is given by $L(f, s)$ for some cusp eigenform f on $\Gamma_0(N)$. In his proof of the finiteness of Ш, Kolyvagin needs to show that there is a fundamental discriminant D of an imaginary quadratic field K such that all of the prime divisors of N split in K and $L(f, D, s)$ has a simple zero at $s = 1$. Accordingly, we prove:

Theorem 1. *Let N be a fixed natural number. Suppose that for each fundamental discriminant D, $L(f, D, s)$ and the classical Dirichlet L-series*

$$L(D, s) = \sum_{n=1}^{\infty}\left(\frac{D}{n}\right)\frac{1}{n^s}$$

satisfy GRH. That is, all of the non-trivial zeros of $L(f, D, s)$ and $L(D, s)$ satisfy $\Re s = 1$ and $\Re s = 1/2$ respectively. Then, there is a fundamental discriminant D of an imaginary quadratic field K such that $L(f, D, s)$ has a simple zero at $s = 1$ and all the prime divisors of N split in K.

Let χ be a Dirichlet character mod q and set

$$L(f, \chi, s) = \sum_{n=1}^{\infty}\frac{a_n\,\chi(n)}{n^s}.$$

Then

Theorem 2. *Suppose each $L(f, \chi, s)$ satisfies GRH. Then $L(f, \chi, 1) = 0$ for at most $\phi(q)/2$ characters $\chi \pmod q$.*

Corollary. *Under GRH, $L(f, \chi, 1) \neq 0$ for at least $\phi(q)/2$ characters $\chi \pmod q$.*

An analogous method in the classical situation leads to the interesting:

Theorem 3. *The number of* $\chi \pmod q$ *such that*

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{\sqrt{n}} \neq 0$$

is at least $\phi(q)/2$ *on the GRH.*

2. Lemmas

Our proof is based on Weil's explicit formula. Goldfeld [5] had utilized a similar method to show that under GRH, the order of the zero of $L(f, D, s)$ at $s = 1$ is bounded by 3 for almost all D. Our proof is a modification of his method.

Throughout the rest of the paper, f is a normalized cusp eigenform of weight 2 on $\Gamma_0(N)$. $L(f, s)$ and $L(f, D, s)$ are then as defined in section 1. Let us define

$$c_n = \begin{cases} \alpha_p^m + \bar{\alpha}_p^m & \text{if } n = p^m \text{ and } p \nmid N \\ a_p^m & n = p^m \text{ and } p \mid N \\ 0 & \text{otherwise.} \end{cases}$$

Then we can write

$$\frac{L'}{L}(f, s) = \sum_{n=1}^{\infty} \frac{c_n \Lambda(n)}{n^s}$$

Also,

$$\frac{L'}{L}(f, D, s) = \sum_{n=1}^{\infty} \frac{c_n \Lambda(n)}{n^s} \left(\frac{D}{n}\right).$$

Lemma 1. *(Weil's explicit formula.) Let* $F: \mathbb{R} \to \mathbb{R}$ *satisfy the following conditions:*

- (A) *there is an* $\epsilon > 0$ *such that* $F(x)\exp((1+\epsilon)x)$ *is integrable and of bounded variation,*
- (B) *the function*

$$\frac{F(x) - F(0)}{x}$$

is of bounded variation.

Define

$$\phi(\gamma) = \int_{-\infty}^{\infty} F(x) e^{i\gamma x}\, dx .$$

Then

$$\sum_{\gamma} \phi(\gamma) = 2F(0)\log\frac{\sqrt{N}}{2\pi} - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(1+it)\phi(t)\,dt$$

$$- 2\sum_{n=1}^{\infty}\frac{c_n}{n}\Lambda(n)F(\log n) \qquad (1)$$

where the sum on the left hand side is over γ *such that* $L(f, 1+i\gamma)=0$ *and* $1 \leq \Re(1+i\gamma) \leq 3/2$.

Remark. An analogous formula holds for $L(f, D, s)$. Namely,

$$\sum_{\gamma} \phi(\gamma) = 2F(0)\log\frac{\sqrt{N}\,D}{2\pi} - \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(1+it)\phi(t)\,dt$$

$$- 2\sum_{n=1}^{\infty}\frac{c_n}{n}\left(\frac{D}{n}\right)\Lambda(n)F(\log n) \qquad (2)$$

where the sum on the left hand side is over γ satisfying $L(f, D, 1+i\gamma) = 0$ and $1 \leq \Re(1+i\gamma) \leq 3/2$.

Proof. See Mestre [8], p. 215.

Lemma 2. *Let* $T > 0$, *and define*

$$F(x) = \begin{cases} 2T - |x| & \text{if } |x| \leq 2T \\ 0 & \text{otherwise.} \end{cases}$$

Then, F satisfies the conditions of Lemma 1 and

$$\phi(\gamma) = \left(\frac{2\sin(\gamma T)}{\gamma}\right)^2 .$$

Proof. It is clear that F satisfies (A) and (B) of Lemma 1. The computation of ϕ follows from the straightforward integration.

Lemma 3. *Let* $T > 1$. *Then,*

$$\int_0^{\infty} \frac{\Gamma'}{\Gamma}(1+it)\left(\frac{\sin(Tt)}{t}\right)^2 dt \ll T.$$

Proof. We use the well-known estimate (see Davenport [4], p. 73)

$$\frac{\Gamma'}{\Gamma}(1+it) = O(\log(|t|+2)),$$

and decompose the integral

$$\int_0^{\infty} = \int_0^{1/T} + \int_{1/T}^{\infty}.$$

Since, $\sin x \le x$ if $x > 0$, the first integral is $O(T)$ as the gamma function is bounded in this range. The second integral is

$$\ll \int_{1/T}^{\infty} \frac{\log(t+2)}{t^2} dt \le T \log(2+\frac{1}{T}) + \int_{1/T}^{\infty} \frac{dt}{t(t+2)} \ll T,$$

on using the cited property of the gamma function.

Lemma 4. *Let* p *be a prime and* N *a natural number. Assuming GRH,*

$$\sum_{\substack{q \le z \\ q \equiv 3 (\mathrm{mod}\ 4)}} \left(\frac{-q}{p}\right) = O(z^{\frac{1}{2}} \log(pz)),$$

where the summations over primes $q \equiv 3(\mathrm{mod}\ 4)$ *satisfying* $(-q/r) = 1$ *for all prime divisors* r *of* N.

Proof. Let υ denote the number of prime factors of N. We can write the sum on the left hand side as

$$\frac{1}{2^{\upsilon+1}} \sum_{q \le z} \left(\frac{-q}{p}\right)\left(1 - \left(\frac{-1}{q}\right)\right) \prod_{r | N} (1 + (-q/r))$$

and by the classical estimates, (see Davenport [4], p. 125) the result follows.

Lemma 5.

$$\sum_{p^{2m}\le x} \frac{\alpha_p^{2m} + \bar{\alpha}_p^{2m}}{p^{2m}} (\log p) \log \frac{x}{p^{2m}} \sim -\frac{1}{4}\log^2 x$$

as $x \to \infty$.

Proof. Set

$$L_2(f, s) = \prod_{p|N} (1 - \frac{a_p^2}{p^s})^{-1} \prod_{p \nmid N} (1 - \frac{\alpha_p^2}{p^s})^{-1} (1 - \frac{\bar{\alpha}_p^2}{p^s})^{-1}.$$

It is well-known [14] that $\zeta(s-1)L_2(f, s)$ is entire. Therefore, $L_2(f, s+1)$ has a simple zero at $s = 1$. Moreover, $L_2(f, s)$ does not vanish on the line $\Re s = 1$ (see Rankin [11]). Therefore, if we write $\alpha_p = \sqrt{p}\exp(i\,\theta_p)$, then by the classical Tauberian theorem,

$$\sum_{p^m \le x} 2(\cos 2m\theta_p) \log p \sim -x .$$

By partial summation, it follows that

$$\sum_{p^{2m}\le x} \frac{2\cos 2m\theta_p}{p^m} \log p \sim -\int_1^{\sqrt{x}} \frac{dt}{t} \sim -\frac{1}{2}\log x ,$$

as $x \to \infty$. Similarly,

$$\sum_{p^{2m}\le x} \frac{2\cos 2m\theta_p}{p^m} (\log p)(\log p^{2m}) \sim \frac{-\log^2 x}{4}.$$

Thus,

$$\sum_{p^{2m}\le x} \frac{2\cos 2m\theta_p}{p^m} \log p \ \log \frac{x}{p^{2m}} \sim \frac{-\log^2 x}{4}.$$

3. Proof of Theorem 1

We shall choose

$$F(u) = \begin{cases} 2T - |u| & \text{if } |u| \le 2T \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by Lemma 2,

$$\phi(\gamma) = \left(\frac{2\sin(\gamma T)}{\gamma}\right)^2$$

so that if $\gamma \in \mathbb{R}$, then $\phi(\gamma) \geq 0$. We will choose $T = \log\sqrt{x}$. Let q be a prime $\equiv 3 \pmod 4$. Then $-q$ is a fundamental discriminant. Let r_q be the order of zero at $s = 1$ of $L(f, -q, s)$. With the above choice of F and ϕ, we apply the explicit formula (Lemma 1) to $L(f, -q, s)$. Assuming GRH, all of the γ are real and so, we obtain the inequality

$$r_q(\log x)^2 \leq 2\left(\log\frac{\sqrt{N}\, q}{2\pi}\right)(\log x) + \frac{1}{\pi}\int_{-\infty}^{\infty}\left|\frac{\Gamma'}{\Gamma}(1+it)\right|\left(\frac{\sin^2(t\log\sqrt{x})}{t^2}\right)dt$$
$$- 2\sum_{n \leq x}\frac{c_n}{n}\left(\frac{-q}{n}\right)\Lambda(n)\log\frac{x}{n}.$$

The integral is, by Lemma 3,

$$\ll \log x .$$

Let $P(z)$ denote the number of primes $q \leq z$ satisfying $q \equiv 3 \pmod 4$, $(-q/r) = 1$ for all prime divisors r of N. We obtain

$$\sum_{q \leq x}{}' r_q(\log x)^2 \leq 2(\log z)(\log x)P(z) + O(P(z)\log x)$$
$$- 2\sum_{n \leq x}\frac{c_n}{n}\Lambda(n)\log\frac{x}{n}\sum_{q \leq z}{}'\left(\frac{-q}{n}\right)$$

where the dash on the summation indicates we sum over primes $q \leq z$, $q \equiv 3 \pmod 4$, satisfying $(-q/r) = 1$ for all prime divisors r of N. The last sum is by Lemma 4,

$$\sum_{q \leq x}{}'\left(\frac{-q}{n}\right) \ll z^{\frac{1}{2}}\log(nz)$$

if n is prime. If $n = p^m$ and m is odd, a similar estimate holds. If m is even, the inner sum is equal to $P(z)$. Thus, if we split the sum over $n = p^m$ into two parts S_1 and S_2, where S_1 corresponds to the odd powers and S_2 corresponds to the even powers, then it is now clear that

$$S_1 \ll x^{\frac{1}{2}+\epsilon} z^{\frac{1}{2}}.$$

The remaining sum S_2 is equal to

$$P(z) \sum_{p^{2m} \le x} \frac{\alpha_p^{2m} + \bar{\alpha}_p^{2m}}{p^{2m}} (\log p) \log \frac{x}{p^{2m}}$$

which by Lemma 5 is

$$\approx -\frac{1}{4} (\log^2 x) P(z).$$

We therefore obtain the inequality

$$\sum_{q \le z}{}' r_q (\log x)^2 \le 2P(z) \log z \log x + \frac{P(z)(\log^2 x)}{2}$$

$$+ O(P(z) \log x) + O(z^{\frac{1}{2}} x^{\frac{1}{2}+\epsilon}).$$

Hence,

$$\sum_{q \le z}{}' r_q \le \frac{2P(z) \log z}{\log x} + \frac{1}{2} P(z) + O\left(\frac{P(z)}{\log x}\right) + O(z^{\frac{1}{2}} x^{\frac{1}{2}+\epsilon}).$$

If there is no q satisfying the hypothesis, then each $r_q \ge 3$ in the above sum. We will choose $z = x^{\alpha}$, where α satisfies

$$1 < \alpha < \frac{5}{4}.$$

Dividing the entire expression by $P(z)$ we get that

$$3 \le 2\alpha + .5 + O\left(\frac{1}{\log x}\right) + O(x^{-\frac{\alpha}{2}+\frac{1}{2}+\epsilon})$$

which is a contradiction since $\alpha < 5/4$. This proves the result.

4. Proof of Theorem 2

For the sake of notational simplicity, we shall consider the case when q is prime. The general case is only slightly complicated in that we must deal carefully with imprimitive characters. The case when q is prime, we have only one imprimitive character, namely the principal character. Let r_χ denote the order of the zero at $s = 1$

of $L(f, \chi, s)$. From Lemma 1, and the choice of ϕ as in Lemma 2, with $T = (\log x)/2$, it follows that

$$\sum_{\chi \bmod q} r_\chi (\log x)^2 \leq \phi(q)(\log x) \log \frac{\sqrt{N}\, q}{2\pi}$$
$$+ 2\phi(q) \sum_{\substack{n \leq x \\ n \equiv 1 (\bmod q)}} \frac{\Lambda(n)}{\sqrt{n}} \log \frac{x}{n} - \frac{(\log x)(\log q)}{\sqrt{q}},$$

where the last term is the correction term in the explicit formula for $L(f, \chi, s)$ when χ is the principal character. But,

$$\sum_{\substack{n \leq x \\ n \equiv 1 (\bmod q)}} \frac{\Lambda(n)}{\sqrt{n}} \log \frac{x}{n} \ll \log^2 x \sum_{\substack{n \leq x \\ n \equiv 1 (\bmod q)}} \frac{1}{\sqrt{n}}.$$

Since the number of primes $p \leq x$, $p \equiv 1 \pmod q$ is

$$\ll \frac{x}{\phi(q) \log(x/q)}, \quad q < x,$$

by the Brun–Titchmarsh theorem, the above sum is seen to be

$$\ll \frac{\sqrt{x} \log^2 x}{\phi(q) \log(x/q)}.$$

Thus, choosing $x = \phi(q)^2$, we obtain

$$\sum_{\chi \bmod q} r_\chi \leq \frac{\phi(q)}{2} + O\left(\frac{\phi(q)}{\log q}\right).$$

This proves the result.

The corollary is now immediate. To establish Theorem 3, there is only a slight variation on the above proof. The explicit formula is applied to the Dedekind zeta function of the cyclotomic field $\mathbb{Q}(\zeta_q)$. As the zeta function has a simple pole at $s = 1$, there is an additional contribution of $O(x^{1/2})$ arising from $\phi(0)$ and $\phi(1)$. This causes no problem upon division by $\log^2 x$. Moreover, the analogue of the sum dealt with above can be discarded as it appears with a negative sign. Since the proof is now completely analogous to the above, we leave the details to the reader.

5. Concluding Remarks

To eliminate the GRH from the above proofs seems to be difficult. In Theorem 1, it is used in two places, though, there is some possibility of removing it. More precisely, the use of GRH in Lemma 4 can be eliminated if instead of summing over prime discriminants, we sum over all fundamental discriminants satisfying the quadratic conditions of Kolyvagin. In such a case, we would require

$$\sum_{D \le z}{}' \left(\frac{D}{n}\right) \ll z^{\frac{1}{2}} n^{\frac{1}{8}+\epsilon}$$

when n is prime. This is only slightly stronger than the Burgess estimate [3].

The second usage of the GRH is in establishing the positivity of $\phi(\gamma)$ as $1 + i\gamma$ runs over the zeros of $L(f, D, s)$. This can be circumvented by using a remark of Odlyzko (see Poitou [10], p. 148). We only need a function ϕ which satisfies $\Re\phi(\gamma) \ge 0$. Thus if $F(x)$ is a non-negative function satisfying the conditions of Lemma 1, then

$$\frac{F(x)}{\cosh x}$$

has the property that its Fourier transform has non-negative real part. This follows from the fact that the real part of the analytic function

$$\int_{-\infty}^{\infty} \frac{F(x)}{\cosh x} e^{(s-1)x}\, dx$$

is a harmonic function which is non-negative on $\Re s = 0$ and $\Re s = 2$ and hence non-negative throughout the critical strip. But this modification in the above proof eliminates one of the log factors and the resulting inequality is insufficient to deduce anything significant about the orders of zeros.

But in the case of Theorems 2 and 3, there is some possibility though this requires a great deal of technical prowess. Indeed Theorem 3 was recently established without

GRH by R. Balasubramanian and V. Kumar Murty [2] with a constant smaller than 1/2. It would be interesting to determine if by these methods one could prove that almost all of $L(s, \chi)$ with $\chi \pmod q$ do not vanish at $s = 1/2$.

References

[1] *R. Ayoub*, An Introduction to the Analytic Theory of Numbers. Amer. Math. Soc. Math. Surveys **10** (1963), Providence.

[2] *R. Balasubramanian* and *V. Kumar Murty*, Zeros of the Dirichlet L-functions (in preparation).

[3] *D.A. Burgess*, On character sums and L-series II. Proc. London Math. (3), **13** (1963), 524—536.

[4] *H. Davenport*, Multiplicative Number Theory. Springer–Verlag.

[5] *D. Goldfeld*, Conjectures on elliptic curves over quadratic fields. Proc. Southern Illinois Number Theory Conf., Springer Lecture Notes **751** (1979), Springer–Verlag.

[6] *D. Goldfeld* and *C. Viola*, Mean values of L-functions associated to elliptic, Fermat, and other curves at the centre of the critical strip. J. Number Theory **11** (1979), 305—320.

[7] *V.A. Kolyvagin*, On the Mordell–Weil and Shafarevic–Tate groups of elliptic Weil curves. Preprint (in Russian).

[8] *J.-F. Mestre*, Formules explicites et minorations de conducteurs de variétés algébriques. Comp. Math. **58** (1986), 209—232.

[9] *M.R. Murty, V.K. Murty* and *N. Saradha*, Modular forms and the Chebotarev density theorem. Amer. J. Math. **110** (1988), 253—281.

[10] *G. Poitou*, Minorations de discriminants. Séminaire Boubaki, Exposé 479, Springer Lecture Notes **567** (1976), Springer–Verlag.

[11] *R.A. Rankin*, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetic functions. Proc. Camb. Phil. Soc. **35** (1939), 357–375.

[12] *K. Rubin*, Tate–Shafarevic groups and the L-functions of elliptic curves with complex multiplication. Inven. Math. **89** (1987), 527—560.

[13] *G. Shimura*, An introduction to the arithmetic theory of automorphic functions. Publications of Math. Soc. of Japan **11** (1971), Princeton University Press.

[14] *G. Shimura*, On the holomorphy of certain Dirichlet series. Proc. London Math. Soc. **31** (1975), 79—98.

[15] *L. Washington*, Number fields and elliptic curves. Proc. NATO ASI on Number Theory and Applications, Banff, Canada, Kluwer Academic Publishers; Editor: R.A. Mollin, 245—278.

Department of Mathematics, McGill University, Montréal, H3A 2K6 CANADA