http://dx.doi.org/10.1090/conm/143/01014

Contemporary Mathematics
Volume 143, 1993

BASE CHANGE AND THE BIRCH-SWINNERTON-DYER CONJECTURE

M. RAM MURTY AND V. KUMAR MURTY*

Dedicated to the memory of Emil Grosswald

1. Introduction.

Let k be a number field and E an elliptic curve defined over k. It is well known that the set of $k-$ rational points $E(k)$ is a finitely generated abelian group. Let $L_k(s)$ denote the L-series of E over k. This is defined for $Re(s) > 3/2$ as an Euler product:

$$L_k(s) = \prod_v L_v(s)$$

where v runs through all the finite places of k and for v coprime to the conductor of E,

$$L_v(s) = (1 - a_v Nv^{-s} + Nv^{1-2s})^{-1}$$

where

$$Nv + 1 - a_v$$

represents the number of points of E mod v and Nv is the absolute norm of v. Birch and Swinnerton-Dyer conjecture that $L_k(s)$ extends to an entire function, satisfies a suitable functional equation and at $s = 1$ has a zero of order equal to the rank of the group $E(k)$.

If K is a finite extension of k, it is evident that

$$\text{rank } E(K) \geq \text{rank } E(k).$$

1991 *Mathematics Subject Classification.* Primary 11S40; Secondary 11R39.
This paper is in final form and no version of it will be submitted for publication elsewhere.
*Research of both authors partially supported by grants from NSERC

This observation motivates several natural questions concerning $L_K(s)$. Given that $L_k(s)$ extends to an entire function, does it follow that $L_K(s)$ extends to an entire function ? This is the first question. As we shall see below, this is related to the problem of base change in the Langlands program. An affirmative answer to this question leads us to inquire whether

$$ord_{s=1}\ L_K(s) \geq ord_{s=1}\ L_k(s)$$

whenever $K \supseteq k$.

In this paper, we prove the following theorems.

Theorem 1. *Let E be an elliptic curve defined over k. Suppose that E has complex multiplication (CM) and K is a finite extension of k. Then, $L_K(s)$ extends to an entire function of s. If K is Galois over k or if K is contained in a solvable extension of k, then $L_K(s)/L_k(s)$ is entire. In particular,*

$$ord_{s=1}\ L_K(s) \geq ord_{s=1}\ L_k(s).$$

A classical conjecture of Taniyama predicts that every elliptic curve defined over $\mathbb{Q}$ occurs as a factor of the Jacobian of the modular curve. For CM elliptic curves, Shimura [11, 12] proved Taniyama's conjecture. If such is the case in general, one can prove that for E defined over $\mathbb{Q}$ and K a finite solvable extension of $\mathbb{Q}$, we have an analytic continuation of $L_K(s)$. More generally, denote by $\mathbb{A}_k$ the adele ring of k and let $\mathfrak{A}(GL_2(\mathbb{A}_k))$ be the space of automorphic representations of $GL_2(\mathbb{A}_k)$. Langlands [8] has shown how to attach an L-function $L(s,\pi)$ for each $\pi \in \mathfrak{A}(GL_2(\mathbb{A}_k))$. He established the analytic continuation and the functional equation for each $L(s,\pi)$. It is suspected that $L_k(s)$ (after a suitable translation) is equal to $L(s,\pi)$ for some cuspidal $\pi \in \mathfrak{A}(GL_2(\mathbb{A}_k))$. This we shall call the Taniyama conjecture over k (or the generalised Taniyama conjecture).

Theorem 2. *Suppose that E satisfies the Taniyama conjecture over k. If K is a solvable extension of k, then $L_K(s)$ extends to an entire function and $L_K(s)/L_k(s)$ is entire. In particular,*

$$ord_{s=1}\ L_K(s) \geq\ ord_{s=1}\ L_k(s).$$

The theorems are highly reminiscent of the classical Aramata-Brauer theorem that $\zeta_K(s)/\zeta_k(s)$ is entire if K/k is a Galois extension. This was also proved in the case when K is contained in a solvable extension of k by Uchida [14] and van der Waall [15]. It is a conjecture of Dedekind that $\zeta_K(s)/\zeta_k(s)$ is always entire if K is a finite algebraic extension of k. A similar conjecture can be made here for $L_K(s)/L_k(s)$ the truth of which would follow from the Langlands program.

There are two key ingredients in the proof of these theorems. The first is the theory of base change as initiated by Saito, Shintani, Langlands [9] and further developed by Arthur-Clozel [1]. As we shall see, the constraints of Theorem 2 are forced upon us by the present state of development of this theory. The second ingredient is a zero-counting formalism implicit in the work of Heilbronn [5] and Stark [13]. This formalism was made more explicit and applied to Artin's holomorphy conjecture in Kumar Murty [10] and R. Foote and K. Murty [4]. We will begin with the formalism first and then discuss base change. In the final section, we apply these ideas to complete the proofs of the theorems.

2. Formalism.

Our approach applies in a wider context of an L-function formalism which is satisfied by a variety of objects in number theory and algebraic geometry. Let G be a finite group. For every subgroup H of G and complex character ψ of H, we attach a complex number $n(H,\psi)$ satisfying the following properties:

(1) Additivity: $n(H,\psi+\psi') = n(H,\psi) + n(H,\psi')$,
(2) Invariance under induction: $n(G, Ind_H^G\ \psi) = n(H,\psi)$.

The formalism can be applied to the case when G is the Galois group of a normal extension K/k and $n(H,\psi)$ is the order of the zero at $s = s_0$ of the Artin L-series attached to ψ corresponding to the Galois extension K/K^H. It can also be applied to the situation when E is an elliptic curve over k and $n(H,\psi)$ corresponds to the order of the zero at $s = s_0$ of the "twist" by ψ (see definition below) of $L_{K^H}(s)$ in the notation of section 1.

Set

$$\theta_H = \sum_{\psi} n(H,\psi)\psi$$

where the sum is over all irreducible characters ψ of H. Our first step is to show that

Proposition 1. $\theta_G|_H = \theta_H$.

Proof.

$$\theta_G|_H = \sum_\chi n(G,\chi)\chi|_H$$
$$= \sum_\chi n(G,\chi)\left(\sum_\psi (\chi|_H,\psi)\psi\right)$$

where the inner sum is over all irreducible characters of H and the outer sum is over all irreducible characters of G. By Frobenius reciprocity, $(\chi|_H,\psi) = (\chi, Ind_H^G\ \psi)$ and so

$$\theta_G|_H = \sum_\psi\left(\sum_\chi n(G,\chi)(\chi, Ind_H^G\psi)\right)\psi.$$

But now, by property (1), the inner sum is $n(G, Ind_H^G\psi)$ which equals $n(H,\psi)$ by property (2). Thus, $\theta_G|_H = \theta_H$.

This immediately implies:

Proposition 2. *Let reg denote the regular representation of G. Suppose for every cyclic subgroup H of G, we have $n(H,\psi)\geq 0$. Then $n(G,\chi)$ is real for every irreducible character χ of G and*

$$\sum_\chi n(G,\chi)^2 \leq n(G,reg)^2.$$

Proof. By Artin's theorem, every character can be written as a rational linear combination of characters induced from cyclic subgroups and so $n(G,\chi)$ is real. By the orthogonality relations,

$$(\theta_G,\theta_G) = \sum_\chi n(G,\chi)^2.$$

On the other hand,

$$(\theta_G,\theta_G) = \frac{1}{|G|}\sum_{g\in G}|\theta_G(g)|^2.$$

By Proposition 1,

$$\theta_G(g) = \theta_{\langle g\rangle}(g) = \sum_\psi n(\langle g\rangle,\psi)\psi(g)$$

which is bounded by $n(G, reg)$ in absolute value by our hypothesis and property (1). This completes the proof.

Similar reasoning implies

Proposition 3. *Let ρ be an arbitrary character of G. Suppose for every cyclic subgroup H of G, and irreducible character ψ of H, we have $n(H, \rho|_H \otimes \psi) \geq 0$, then $n(G, \rho \otimes \chi)$ is real for every irreducible character χ of G and*

$$\sum_{\chi} n(G, \rho \otimes \chi)^2 \leq n(G, \rho \otimes reg)^2.$$

3. Base change and automorphic induction.

Let K/k be a Galois extension and let $G = Gal(K/k)$. If ρ is an irreducible representation of G, let $L(s, \rho, K/k)$ denote the Artin L-series attached to ρ. We can extend the definition of Artin L-series to an arbitrary representation of G by additivity:

$$L(s, \rho_1 \oplus \rho_2, K/k) = L(s, \rho_1, K/k)L(s, \rho_2, K/k).$$

If now ψ is a representation of a subgroup H of G, then $L(s, \psi, K/K^H)$ is the Artin L-series belonging to the extension K/K^H where K^H denotes the field fixed by H. A simple calculation shows that Artin L-series are invariant under induction:

$$L(s, Ind_H^G \psi, K/k) = L(s, \psi, K/K^H).$$

Recall the Langlands' reciprocity conjecture [8]: for each ρ, there is $\pi(\rho) \in \mathfrak{A}(GL_n(\mathbb{A}_k))$ ($n = \deg \rho$) so that

$$L(s, \rho, K/k) = L(s, \pi).$$

It is easy to see that

$$L(s, \rho|_H, K/K^H) = L(s, \rho \otimes Ind_H^G 1, K/k).$$

But $Ind_H^G 1 = reg_H$ is the permutation representation on the cosets of H. This suggests that we make the following definition. Let $\pi \in \mathfrak{A}(GL_n(\mathbb{A}_k))$. For each unramified π_v, let $A_v \in GL_n(\mathbb{C})$ be the semi-simple conjugacy class attached by

Langlands [8]. If in addition, v is unramified in K, define

$$L_v(s, B(\pi)) = det(1 - A_v \otimes reg_H(\sigma_v) Nv^{-s})^{-1},$$

where σ_v is the Artin symbol of v. The idea is that there should be a $B(\pi) \in \mathfrak{A}(GL_n(\mathbb{A}_M))$, where $M = K^H$, so that the v-factor in the definition of $L(s, B(\pi))$ coincides with $L_v(s, B(\pi))$ as defined above for all but finitely many places v. The problem of base change is to determine when this map exists.

For $n = 2$, this was done by Langlands [9]. He then used these ideas to deal with the tetrahedral case of Artin's conjecture. The case $n = 3$ was considered by Flicker [2]. For arbitrary n, it is a recent breakthrough achieved by Arthur and Clozel [1]. Again, the situation is for M/k cyclic.

Now suppose that ψ is a representation of H. Corresponding to ψ there should be a $\pi \in \mathfrak{A}(GL_n(\mathbb{A}_M))$ where $n = deg\,\psi$. But the invariance of Artin L-series under induction implies that there should be an $I(\pi) \in \mathfrak{A}(GL_{nr}(\mathbb{A}_k))$ ($r = [G : H]$) so that

$$L(s, I(\pi)) = L(s, Ind_H^G \psi, K/k).$$

This map $\pi \mapsto I(\pi)$, called the automorphic induction map, is conjectured to exist. Again, Arthur and Clozel [1] showed this exists when M/k is cyclic and n is arbitrary. Thus, if M/k is a solvable extension of k, the base change and automorphic induction maps exist.

We have used the theory of Artin L-series to motivate the discussion of base change and automorphic induction. We will now use the existence of these maps to prove Theorems 1 and 2. We state the theorem of Arthur and Clozel [1] for future reference. We first need a few definitions.

Denote by M_r a generic r by r matrix and by I_r the $r \times r$ identity matrix. The standard parabolic subgroups of GL_n are in one-one correspondence with the partitions of $n = n_1 + \cdots + n_r$. The standard parabolic subgroup corresponding to a partition $n = n_1 + \cdots + n_r$ consists of matrices of the form

$$\begin{pmatrix} M_{n_1} & * & * & * \\ & . & * & * \\ & & . & * \\ & & & M_{n_r} \end{pmatrix}$$

and any parabolic subgroup is a GL_n conjugate of a standard parabolic subgroup.

Any parabolic subgroup P has a decomposition (called the Levi decomposition) of the form $P = MN$ where N is the unipotent radical. In the case of a standard parabolic, M and N can be described as consisting of matrices of the form

$$\begin{pmatrix} M_{n_1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & M_{n_r} \end{pmatrix}, \quad \begin{pmatrix} I_{n_1} & * & * & * \\ & \cdot & * & * \\ & & \cdot & * \\ & & & I_{n_r} \end{pmatrix}$$

respectively. An automorphic representation π of $GL_n(\mathbb{A}_k)$ is said to be **induced from cuspidal** if there is a cuspidal unitary representation σ of $M(\mathbb{A}_k)$ where $P = MN$ is a k-parabolic subgroup of GL_n such that

$$\pi = Ind_{M(\mathbb{A}_k)N(\mathbb{A}_k)}^{GL_n(\mathbb{A}_k)} (\sigma \otimes 1).$$

If M is a standard parabolic subgroup corresponding to the partition $n_1 + \cdots + n_r$ of n, then σ is of the form $\sigma_1 \otimes \cdots \otimes \sigma_r$ with $\sigma_i \in \mathfrak{A}(GL_{n_i}(\mathbb{A}_k))$. In this case, we write

$$\pi = \sigma_1 \times \cdots \times \sigma_r.$$

Of course, a cuspidal automorphic representation of $GL_n(\mathbb{A}_k)$ is (trivially, by definition) induced from cuspidal.

Proposition 4. (Arthur and Clozel [1]) *Let E/F be a cyclic extension and π, Π denote representations which are induced from cuspidal of $GL_n(\mathbb{A}_F)$ and $GL_n(\mathbb{A}_E)$ respectively. Then $B(\pi)$ and $I(\Pi)$ exist.*

We will also need results concerning the Rankin-Selberg convolution of two L-functions. Let π and σ be two cuspidal, unitary automorphic representations of $GL_n(\mathbb{A}_k)$ and $GL_m(\mathbb{A}_k)$ respectively. Let S be a finite set of primes such that π and σ are unramified outside of S. Form the L-function

$$L(s, \pi \otimes \sigma) = \prod_{v \notin S} det(1 - A_{\sigma,v} \otimes A_{\pi,v}\ Nv^{-s})^{-1}$$

where $A_{\pi,v}$ and $A_{\sigma,v}$ are the semi-simple conjugacy classes of $GL_n(\mathbb{C})$ and $GL_m(\mathbb{C})$ attached to π_v and σ_v respectively.

The following proposition follows, in principle, from the work of Jacquet, Piatetski-Shapiro and Shalika [7], but a proof can be derived also from the theory of Eisenstein series as in Arthur and Clozel [1]. (An introduction to the work of [7] can be found in Jacquet [6].)

Proposition 5. *The L-function $L(s, \pi \otimes \sigma)$ extends to a meromorphic function of s.*

The following is a simple consequence of the definitions and was used by Arthur and Clozel to construct $B(\pi)$ and $I(\sigma)$ in Proposition 4.

Proposition 6. *Let E/F be a cyclic extension and suppose $\pi \in \mathfrak{A}(GL_n(\mathbb{A}_F))$ and $\sigma \in \mathfrak{A}(GL_m(\mathbb{A}_E))$. Then, the Rankin-Selberg L-function satisfies the formal identity:*

$$L(s, B(\pi) \otimes \sigma) = L(s, \pi \otimes I(\sigma)).$$

4. Lemmas.

We record in this section the necessary facts from the theory of elliptic curves that will be needed in the proofs of Theorems 1 and 2.

Lemma 1. (Deuring [2]) *Let E be an elliptic curve defined over k. Suppose that E has CM by an order in an imaginary quadratic field F. If $k \supseteq F$, then $L_k(s)$ is the product of two Hecke L-series of k. If $k \not\supseteq F$, then $L_k(s)$ is equal to a Hecke L-series of the quadratic extension kF of k.*

Lemma 2. *The generalised Taniyama conjecture is true for CM elliptic curves.*

Remark. This was proved by Shimura [11, 12] for elliptic curves over $\mathbb{Q}$ using Weil's converse theorem.

Proof. Suppose that E is defined over k and has CM by an order in F. If $k \supseteq F$, then by Lemma 1,

$$L_k(s) \;=\; L(s, \psi_1)L(s, \psi_2).$$

A Hecke character of k is an automorphic form of $GL_1(\mathbb{A}_k)$. Thus, the the pair of Hecke characters ψ_1, ψ_2 corresponds to the induced from cuspidal representation

$$\pi \;=\; \psi_1 \;\times \psi_2$$

of $GL_2(\mathbb{A}_k)$. If $k \not\supseteq F$, then by Lemma 1, $L_k(s)$ equals a Hecke L-series $L(s, \phi)$ of kF, where $\phi \in \mathfrak{A}(GL_1(\mathbb{A}_{kF}))$. We can apply proposition 4 to the quadratic extension kF/k to obtain $I(\phi) \in \mathfrak{A}(GL_2(\mathbb{A}_k))$ with

$$L(s, \phi) = L(s, I(\phi)).$$

This completes the proof.

Remark. This lemma could have been established by using the earlier work of Jacquet and Langlands as we are dealing with only quadratic extensions and the theory of GL_2 applies.

We now recall the definition of Artin-Hecke L-series [16]. Let K/k be a Galois extension with group G. Let $L(s, \psi)$ be a Hecke L-series of k and ρ a representation of G. Weil [16] defined the Artin-Hecke L-series $L(s, \psi \otimes \rho, K/k)$ as an Euler product over all places v of k, which at the finite unramified places v of k is

$$det(1 - \psi(v)\rho(\sigma_v)Nv^{-s})^{-1}.$$

Using Brauer induction and class field theory, he proved [16] the following.

Lemma 3. *Each of these L-series $L(s, \psi \otimes \rho, K/k)$ extends to a meromorphic function and satisfies the identity: if ϕ is a representation of a subgroup H of G, then*

$$L(s, \psi \otimes Ind_H^G\ \phi, K/k) = L(s, (\psi \circ N_{K^H/k}) \otimes \phi, K/K^H).$$

5. Proof of Theorem 1.

Fix $s_0 \in \mathbb{C}$. Now suppose that K/k is a Galois extension with group G. Suppose first that $F \subseteq k$. Then

$$L_k(s) \;=\; L(s, \psi_1)L(s, \psi_2)$$

where ψ_1, ψ_2 are two Hecke characters of k. For each subgroup H of G, and character ϕ of H, the Artin-Hecke L-series

$$L(s, (\psi_1 \circ N_{K^H/k}) \otimes \phi)L(s, (\psi_2 \circ N_{K^H/k}) \otimes \phi)$$

is meromorphic (Lemma 3). Let us define $n(H, \phi)$ to be its order at $s = s_0$. If $\phi(1) = 1$, then $n(H, \phi) \geq 0$. Moreover, the numbers $\{n(H, \phi)\}_{(H,\phi)}$ clearly have the properties (1) and (2). By proposition 2, we deduce that

$$n(G, 1) \leq n(G, reg).$$

Since $L_k(s, reg) = L_K(s)$, we deduce that $L_K(s)/L_k(s)$ is entire.

Now we consider the case $F \not\subseteq k$. Let us set $M = Fk$. If $K \supseteq M$, then we know that $L_K(s)/L_M(s)$ is entire and so, it suffices to prove that $L_M(s)/L_k(s)$

is entire. But this is immediate since

$$L_M(s) = L(s,\psi)L(s,\psi^{(\rho)}) = L_k(s)L(s,\psi^{(\rho)}).$$

Here, ρ is the nontrivial element of $Gal(M/k)$ and $\psi^{(\rho)}$ is the conjugate Hecke character.

If $K \not\supseteq M$, let us set $M' = KM$. Then M'/M is Galois and

$$L_K(s) = L(s, \psi \circ N_{M'/M}).$$

Now, for each subgroup H of $\mathrm{Gal}(M'/M)$ and character ϕ of H, define

$$n(H,\phi) = ord_{s=s_0} L(s, (\psi \circ N_{R/M}) \otimes \phi)$$

where we have written R for the fixed field of H. The definition makes sense because Lemma 3 ensures that the $L-$ function is meromorphic. Once again, it is easy to verify that these numbers have the properties (1) and (2), and that $n(H,\phi) \geq 0$ if H is cyclic. We apply Proposition 2 and deduce as before that

$$\frac{L_K(s)}{L_k(s)} = \frac{L(s,\psi \circ N_{M'/M})}{L(s,\psi)}$$

is entire. This completes the proof in the CM case when K/k is Galois.

If now, K is contained in a solvable extension of k, then the normal closure of K is solvable, and we proceed as follows. We prove the theorem by induction on $[K:k]$. Let $\tilde{K}$ denote the Galois closure of K in k. Set $G = Gal(\tilde{K}/k)$. Let H be the subgroup of $+G$ corresponding to K. It suffices to prove the result when H is a maximal subgroup of G because if $G \supseteq I \supseteq H$ and M is the subfield fixed by I, then

$$L_K(s)/L_k(s) = (L_K(s)/L_M(s)) \cdot (L_M(s)/L_k(s)).$$

The first factor is entire by induction and the second factor is entire since we can take M to correspond to a maximal subgroup of G.

Let A be a minimal normal subgroup. Then, it is elementary abelian and in particular $A \neq 1$. Thus, $HA = H$ or G. If $HA = H$, then $A \subseteq H$. But then the subfield M fixed by A is normal over k and contains K and so $M = \tilde{K}$, contrary to $A \neq 1$. Thus, $HA = G$. Also, $H \cap A$ is a normal subgroup of G and also abelian, as it is contained in A. Hence, $H \cap A = 1$ as A is minimal.

Let us write

$$Ind_H^G 1_H - 1_G = \sum_\chi m_\chi \chi$$

where m_χ are non-negative integers and χ ranges over irreducible characters of G. For any irreducible character ϕ of A, let T_ϕ be the inertia group of ϕ:

$$T_\phi = \{\sigma \in G : \phi^\sigma = \phi\}.$$

Of course T_ϕ contains A. The main result of [14] and [15] states that for each χ with $m_\chi \neq 0$, there exists an irreducible character ϕ of A and an abelian character ψ_χ of T_ϕ such that

$$(*) \qquad \chi = \mathrm{Ind}_{T_\phi}^G \psi_\chi.$$

If $F \subseteq k$, then

$$L_K(s)/L_k(s) = \prod_\chi (L(s, \psi_1 \otimes \chi) L(s, \psi_2 \otimes \chi))^{m_\chi},$$

By (*) and Lemma 3,

$$L(s, \psi \otimes \chi) = L(s, \psi \otimes Ind_{T_\phi}^G \psi_\chi) = L(s, \psi \circ N_{\tilde{K}^{T_\phi}/k} \otimes \psi_\chi)$$

and the latter object is entire, being a Hecke L-series of a non-trivial character.

If $F \not\subseteq k$, then we proceed as follows. Consider the Galois extension $\tilde{K}F/k$ and χ as in (*). Let $G' = \mathrm{Gal}(\tilde{K}F/k)$ and

$H' = \mathrm{Gal}(\tilde{K}F/F)$. Let R_ϕ be the fixed field of T_ϕ. Then, we may view ψ_χ as a character of $H_\phi = Gal(\tilde{K}F/R_\phi)$ and we have the identity

$$\mathrm{Ind}_{H_\phi}^{G'} \psi_\chi = \mathrm{Ind}_{T_\phi}^G \psi_\chi.$$

Moreover,

$$(\mathrm{Ind}_{H_\phi}^{G'} \psi_\chi)|_{H'} = \mathrm{Ind}_{H' \cap H_\phi}^{H'} (\psi_\chi|_{H' \cap H_\phi}).$$

Now, we have the identity

$$L_K(s)/L_k(s) = \prod_\chi L(s, \psi \otimes (\mathrm{Ind}_{H_\phi}^{G'} \psi_\chi)|_{H'})^{m_\chi}.$$

It follows from the above that

$$L_K(s)/L_k(s) = \prod_\chi L(s, \psi \otimes \mathrm{Ind}_{H' \cap H_\phi}^{H'} (\psi_\chi|_{H' \cap H_\phi}))^{m_\chi}$$

and by the argument given at the end of the previous paragraph, we see that each of the factors is entire. This completes the proof.

6. Proof of Theorem 2.

If Taniyama's conjecture is true for E over k, then $L_k(s)$ equals $L(s,\pi)$ for some $\pi \in \mathfrak{A}(GL_2(\mathbb{A}_k))$. We will first handle the situation when K/k is a cyclic extension with group G. Since the base change $B(\pi)$ of π to K exists, $L_K(s)$ is an entire function. Moreover, we know that

$$L_K(s) = \prod_\rho L(s, \pi \otimes \rho)$$

where the product is over irreducible characters of $\mathrm{Gal}(K/k)$. (Indeed, if E is an elliptic curve over k and K/k is a finite extension, the L-function of E over K is given by the family of ℓ-adic representations

$$\rho_\ell : Gal(\bar{k}/K) \longrightarrow Aut(T_\ell(E_{/K}))$$

where T_ℓ denotes the Tate module. Since $T_\ell(E_{/K}) = T_\ell(E_{/k})$ as $Gal(\bar{k}/K)$-modules, it follows that ρ_ℓ is the restriction of the representation

$$\psi_\ell : Gal(\bar{k}/k) \longrightarrow Aut(T_\ell(E_{/k})).$$

Thus,

$$L(s, \rho_\ell) = L(s, \psi_\ell|_{Gal(\bar{k}/K)}) = L(s, \psi_\ell \otimes reg) = \prod_\rho L(s, \psi_\ell \otimes \rho).$$

By Artin reciprocity, ρ corresponds to a Hecke character of finite order of k. Thus $\pi \otimes \rho \in \mathfrak{A}(\mathrm{GL}_2(\mathbb{A}_k))$ and is cuspidal, and so each $L(s, \pi \otimes \rho)$ is entire. Thus $L_K(s)/L_k(s)$ is entire. Now, if K/k is a solvable extension, we proceed in cyclic stages with the help of the base change map.

7. Concluding remarks.

The method is capable of further generalization and application. First, an application of Proposition 3 to the Artin L-series $L(s, \rho, K/k)$ corresponding to a Galois representation yields the following result which can be thought of as a generalization of the Aramata-Brauer theorem:

Theorem 3. *Let K/k be a Galois extension with group G. Let ρ be an irreducible representation of G and $L(s, \rho, K/k)$ the corresponding Artin L-series.*

Then,

$$L(s, \rho \otimes reg, K/k)/L(s, \rho, K/k)$$

is entire.

An argument similar to that used to prove Theorem 2 also yields the following.

Theorem 4. *Let π be an automorphic cuspidal representation of $GL_n(\mathbb{A}_k)$. If K is a solvable extension of k and $B(\pi)$ denotes the base change of π to K, then,*

$$L(s, B(\pi))/L(s, \pi)$$

is entire.

If $n = 1$ and K/k is a Galois extension, then $L(s, B(\pi))/L(s, \pi)$ is entire by an application of Lemma 3 . This has arithmetic consequences. For instance, the zeta function of a CM abelian variety over an arbitrary number field is given in terms of Hecke L-series (see Yoshida [17]) and therefore the analogue of Theorem 1 applies to such varieties. There are other varieties for which the zeta functions have been identified. For instance, the Jacobians of the modular curves have zeta function equal to products of L-functions attached to modular forms by a theorem of Shimura. Recent work of Kottwitz identifies the zeta functions of other classes of Shimura varieties. In both instances, the appropriate generalization of Theorem 2 applies.

Acknowledgement. The authors would like to thank L. Clozel and H. Kisilevsky for their suggestions and comments.

REFERENCES

[1] J. Arthur and L. Clozel, Simple algebras, base change, and the advanced theory of the trace formula, *Annals of Math. Studies 120*, Princeton University Press (1990).

[2] M. Deuring, Die zetafunktion einer algebraischen Kurve vom Geschlechte Eins. I, II, III, IV, *Nachr. Akad. Wiss., Göttingen* (1953) 85-94; (1955) 13-42; (1956) 37-76; (1957) 55-80.

[3] Y. Flicker, The trace formula and base change for $GL(3)$, *Springer Lecture Notes* **927** (1982).

[4] R. Foote and V. Kumar Murty, Zeros and poles of Artin L-series, *Math. Proc. Camb. Phil. Soc.*, **105** , 5 (1989) 5 -11.

[5] H. Heilbronn, On real simple zeros of Dedekind ζ-functions, *Canadian Journal of Mathematics,* **25** (1973) 870-873.

[6] H. Jacquet, On the base change problem: after J. Arthur and L. Clozel, in Number Theory, Trace Formulas and Discrete Groups, Academic Press (1989).

[7] H. Jacquet, I. Piatetski-Shapiro, J. Shalika, Rankin-Selberg convolutions, *American Journal of Mathematics,* **105**(1983) 367-464.

[8] R. Langlands, Problems in the theory of automorphic forms, in Lectures in Modern Analysis and Applications, *Springer Lecture Notes* **170** (1970) 18-86.

[9] R. Langlands, Base change for GL_2, Princeton University Press (1980).

[10] V. Kumar Murty, Holomorphy of Artin L-functions, in Proceedings of the Ramanujan centennial international conference, (1988) 55-66.

[11] G. Shimura, On the zeta function of an abelian variety with complex multiplication, *Annals of Math.,* **94** (1971) 504-533.

[12] G. Shimura, On elliptic curves with complex multiplication as factors of the Jacobian of modular function fields, *Nagoya Math. J.* **43** (1971) 199-208.

[13] H. Stark, Some effective cases of the Brauer-Siegel theorem, *Inventiones Math.,* **23** (1974) 135-152.

[14] K. Uchida, On Artin L-functions, *Tôhuku Math. J.,* **27** (1975) 75-81.

[15] R. van der Waall, On a conjecture of Dedekind on zeta functions, *Indag. Math.,* **37** (1975) 83-86.

[16] A. Weil, Dirichlet series and automorphic forms, *Springer Lecture Notes,* **189** (1971).

[17] H. Yoshida, Abelian varieties with complex multiplication and representations of the Weil groups, *Annals of Math.,* **114** (1981) 87-102.

Department of Mathematics, McGill University, Montréal, Canada

Department of Mathematics, University of Toronto, Toronto, Canada