**Hardy-Ramanujan Journal 48 (2025) Part A, 209-219** submitted 22/04/2025, accepted 26/01/2026, revised 26/01/2026

# Triple convolution of Ramanujan sums and applications

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**Abstract.** We give a combinatorial proof of the limit theorem for triple convolutions of Ramanujan sums that had been earlier proved by Chaubey, Goel and Murty using the theory of exponential sums. This result has applications to the study of triple convolutions of certain arithmetical functions.

**Keywords.** Ramanujan sums, triple convolutions.

**2010 Mathematics Subject Classification.** 11M06, 11L03, 11N37.

## 1. Introduction

The Ramanujan sum $c_r(n)$ is defined as

$$c_r(n) := \sum_{\substack{1\le j\le r \\ (j,r)=1}} e^{2\pi i j n/r}$$

and was formally introduced by Ramanujan [Ramanujan18] in his famous paper dealing with "Fourier expansions" of arithmetical functions. As Ramanujan states in his paper, these sums had been studied before in the context of the cyclotomic polynomial but were not given any special attention as they relate to number theory in general. Their ubiquity is made clear in [Ramanujan18], and now more than a century later, their importance has only been reinforced and amplified.

An easy exercise using the Möbius inversion formula gives us

$$c_r(n) = \sum_{d|(n,r)} d\mu(r/d), \tag{1.1}$$

where $\mu$ denotes the Möbius function. This formula was first discovered in 1906 and attributed to Kluyver (see page 343, section 21 of [Ramanujan]).

In 1932, Carmichael [Car32] derived an "orthogonality principle" for Ramanujan sums by showing that

$$\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} c_r(n)c_s(n+h) = \begin{cases} c_r(h) & \text{if } r=s, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

Motivated by this elegant result, Chaubey, Goel and Murty investigated in [CGM23], the limit involving triple convolutions of Ramanujan sums:

$$\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} c_r(n)c_s(n+h)c_t(n+j).$$

They showed, in particular, that if $r,s,t$ are mutually coprime, the limit is zero unless $r=s=t=1$, in which case the limit is 1. In the case that $r,s,t$ are not mutually coprime, the limit value is more complicated and involves higher dimensional variants of the classical Ramanujan sums. In the case that $r,s,t$ are squarefree, the limit can be succinctly stated using these higher variants and then applied to give a new heuristic derivation of the Hardy-Littlewood conjecture for the number of 3-tuples of primes below a given bound. This was the motivation and content of the paper [CGM23].

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A more direct and combinatorial approach to this limit, along with a generalization was obtained by the last two authors in [GoMu25]. Indeed, consider a fixed set $T = \{a_1, ..., a_k\}$ of integers and define the arithmetical function of several variables $g(d_1, ..., d_k)$ by setting

$$g(d_1, ..., d_k) = 1$$

if and only if the system of congruences

$$n \equiv a_1 \pmod{d_1}, \quad \cdots, \quad n \equiv a_k \pmod{d_k}$$

has a solution and $g(d_1, ..., d_k) = 0$ otherwise. Then, the last two authors proved in [GoMu25] that

**Theorem 1.1.** *For positive integers $q_1, \dots, q_k$, and integers $a_1, \dots, a_k$, we have*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} c_{q_1}(n+a_1)\cdots c_{q_k}(n+a_k) = \sum_{d_1|q_1,\cdots,d_k|q_k} d_1\mu\left(\frac{q_1}{d_1}\right)\cdots d_k\mu\left(\frac{q_k}{d_k}\right)\frac{g(d_1, ..., d_k)}{[d_1, ..., d_k]}.$$

It is easy to deduce Carmichael's limit theorem (1.2) from this by setting $k = 2$ as well as the results of Chaubey, Goel and Murty in [CGM23] by setting $k = 3$.

The purpose of this paper is two-fold. We first give a simpler derivation of the limit formula in [CGM23] using essentially (1.1). We will also get error terms and this will allow applications to the study of triple convolutions of arithmetical functions. The error terms for Theorem 1.1 will also be derived, thus refining some results in [GoMu25] in the case $k = 3$. Error terms for the case $k = 2$ were first supplied by Gadiyar, Murty and Padma in [GMP14]. These were derived using the method of exponential sums. By contrast, the combinatorial approach of this paper gives consonant error terms that are in some cases better. It may be possible to extend this method for values of $k$ greater than 3.

## 2. Notations and some preliminary lemmas

It is convenient to introduce some notation.

We define for any general proposition $\wp$, the symbolic function:

$$\mathbf{1}_\wp := \begin{cases} 1 & \text{if } \wp \text{ is true,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

The condition $n \equiv a(q)$ abbreviates $n \equiv a(\bmod\ q)$. Also $(a, b)$ denotes $\gcd(a, b)$ (the greatest common divisor of $a$ and $b$) throughout the paper. When $d$ and $v$ are coprime, we use the notation $\overline{d} \pmod{v}$ to indicate the inverse of $d \pmod{v}$.

We will also make frequent use of the following general Tauberian theorem from analytic number theory. Suppose we have a Dirichlet series of the form

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

with $a_n = O(n^\epsilon)$ and which can be factored as

$$\zeta(s)^k g(s),$$

with $g(s)$ a Dirichlet series absolutely convergent for $\Re(s) > \delta$ with $\delta < 1$ and $k$ is a natural number. Then

$$\sum_{n\le x} a_n \sim \frac{g(1)x(\log x)^{k-1}}{(k-1)!},$$

as $x$ tends to infinity. See for example Problem 4.4.17 on page 67 of [Mur08].

If we let $d_3(\ell)$ be the number of triples $(a, b, c)$ of positive integers such that $abc = \ell$, we have the generating series

$$\zeta^3(s) = \sum_{\ell=1}^{\infty} \frac{d_3(\ell)}{\ell^s}.$$

By the Tauberian theorem discussed in the previous paragraph, we have

$$\sum_{n \le x} d_3(n) \sim \frac{x(\log x)^2}{2}.$$

**Lemma 2.1.**

$$\sum_{\ell > x} \frac{d_3(\ell)}{\ell^{1+\delta}} \ll \frac{\log^2 x}{x^\delta}.$$

*Proof.* This follows easily by partial summation and the well-known fact that

$$\sum_{\ell \le x} d_3(\ell) \ll x(\log x)^2.$$

Indeed, we have

$$\sum_{\ell > x} \frac{d_3(\ell)}{\ell^{1+\delta}} \ll \int_x^\infty \frac{(\log t)^2 dt}{t^{1+\delta}} \ll \frac{\log^2 x}{x^\delta},$$

as claimed.

**Lemma 2.2.** *For fixed $h, k$, we have*

$$\sum_{n \le x} d(n)d(n+h)d(n+k) \ll x(\log x)^7$$

*Proof.* Applying a well-known extension of Hölder's inequality (see page 22 of [HLP88]), we have that the sum in question is

$$\ll \left( \sum_{n \le x} d(n)^3 \right)^{1/3} \left( \sum_{n \le x} d(n+h)^3 \right)^{1/3} \left( \sum_{n \le x} d(n+k)^3 \right)^{1/3} \ll x(\log x)^7,$$

using the standard estimate or by the Tauberian theorem discussed earlier,

$$\sum_{n \le x} d(n)^3 \ll x(\log x)^7,$$

proving the claim.

For $hk \ne 0$, it is conjectured by Browning [Bro11] that the exponent in the lemma is 3. In fact, he makes a precise conjecture on its asymptotic behaviour. This problem is addressed in the paper [MMS26]. When $h = k = 0$, the exponent 7 cannot be improved.

## 3. Carmichael's theorem revisited

By way of motivation, we revisit the 1932 paper of Carmichael [Car32] and derive his theorem with an error term. Strictly speaking, the limit (1.2) is not explicitly given in [Car32]. As is often the case, the result is implicit and it is only through later refinements that an elegant formulation emerges, like the one given in [GMP14]. The idea of using Carmichael's limit theorem to study the twin prime problem first appears in the work of Gadiyar and Padma [GaPa99] in 1999. It is clear from a careful examination of their paper that further progress can only be made through detailed error term analysis.

**Theorem 3.1.** *For $h \neq 0$,*

$$\sum_{n\le x} c_q(n)c_r(n+h) = xc_q(h)\delta_{q,r} + O(\sigma(q)\sigma(r)),$$

*where $\delta_{q,r}$ is the Kronecker delta function (equal to 1 if $q = r$ and zero otherwise) and $\sigma(q)$ denotes the sum of the divisors of $q$.*

*Proof.* Using formula (1.1), we have that the sum in question equals

$$\sum_{n\le x}\sum_{d|q,d|n}\sum_{e|r,e|n+h} de\mu(q/d)\mu(r/e).$$

Interchanging summations and using the function $g$ (for $k = 2$), we get that this is

$$\sum_{d|q,e|r}\mu(q/d)\mu(r/e)de\left(\frac{xg(d,e)}{[d,e]} + O(1)\right),$$

where $[d, e]$ denotes the least common multiple of $d$ and $e$. Since $(d, e)[d, e] = de$, this simplifies to

$$x\sum_{d|q,e|r}\mu(q/d)\mu(r/e)g(d,e)(d,e) + O(\sigma(q)\sigma(r)).$$

Let us analyze $g(d, e)$. Writing $\delta = (d, e)$, we see that $g(d, e) = 0$ when $\delta \nmid h$. We may therefore suppose that $\delta|h$. Writing $d = \delta d_1, e = \delta e_1$, we require $(d_1, e_1) = 1$. In this setting, the system of congruences always has a solution so that $g(d, e) = 1$. The sum in question becomes

$$\sum_{\delta|h}\delta\sum_{\substack{\delta|d|q\\ \delta|e|r\\ (d/\delta,e/\delta)=1}}\mu(q/d)\mu(r/e) = \sum_{\delta|h}\delta\sum_{\substack{\delta|d|q\\ \delta|e|r}}\mu(q/d)\mu(r/e)\sum_{\substack{t|d/\delta\\ t|e/\delta}}\mu(t).$$

We can simplify this sum to be:

$$= \sum_{\delta|h}\delta\sum_{t}\mu(t)\left(\sum_{s|q/\delta t}\mu(q/\delta ts)\right)\left(\sum_{s'|r/\delta t}\mu(r/\delta ts')\right).$$

By the fundamental property of the Möbius function, the innermost sums are zero unless $\delta t = q$ and $\delta t = r$. In particular, we first need $q = r$, and in the innermost sums, there is only one term, namely that corresponding to $t = q/\delta$. This gives the result

$$\sum_{\delta|h}\delta\mu(q/\delta) = c_q(h).$$

Thus, the final result is

$$xc_q(h)\delta_{q,r} + O(\sigma(q)\sigma(r)),$$

as claimed.

The theorem should be compared to Lemma 2 in [GMP14] where the error term is $O(qr\log q\log r)$. Since $\sigma(q) = O(q\log q)$, our result here is sharper.

It is, in essence, the same strategy that we will use for triple convolutions. In theory, the method can be used for higher convolutions but the analysis is more cumbersome. The method of Goel and Murty [GoMu25] is streamlined and short.

## 4. Further lemmas

We now prepare the preliminary results needed to prove our main theorem.

**Lemma 4.1.** *Fix $d, v, l \in \mathbb{N}$ with $(l, v) = 1$ and take $x > 0$ large enough. Then, we have*

$$\sum_{\substack{n\le x\\ n\equiv 0(d)\\ n+h\equiv 0(v)\\ n+j\equiv 0(l)}} 1 = \mathbf{1}_{(d,v)|h}\cdot \mathbf{1}_{(d,l)|j}\cdot\left(\frac{x}{dvl}(d,v)(d,l)+O(1)\right),$$

*uniformly for all $h, j \in \mathbb{Z}$.*

*Proof.* Set $\tilde{v} := (d, v)$, $\tilde{l} := (d, l)$, and $v' := v/\tilde{v}$, $l' := l/\tilde{l}$, $d' := d/\tilde{v}$, and $d'' := d/\tilde{l}$. Consider the sums

$$\sum_{\substack{m\le x/d\\ dm\equiv -h(v)\\ dm\equiv -j(l)}} 1 = \mathbf{1}_{\tilde{v}|h}\cdot\mathbf{1}_{\tilde{l}|j}\cdot\sum_{\substack{m\le x/d\\ m\equiv -h'\overline{d'}(v')\\ m\equiv -j'\overline{d''}(l')}} 1,$$

where $\tilde{v}|h$ allows $h' := h/\tilde{v}$ and $j' := j/\tilde{\ell}$ if $\tilde{\ell}|j$. Now, assume that $(l, v) = 1$, this implies $(v', l') = 1$ because $(l, v) = 1$ implies $\left(\frac{l}{(d,l)}, \frac{v}{(d,v)}\right) = 1$ and by the Chinese remainder theorem

$$\sum_{\substack{m\le x/d\\ m\equiv -h'\overline{d'}(v')\\ m\equiv -j'\overline{d''}(l')}} 1 = \sum_{\substack{m\le x/d\\ m\equiv r(l'v')}} 1,$$

where $1 \le r < l'v'$. Therefore, we have

$$\sum_{\substack{m\le x/d\\ m\equiv r(l'v')}} 1 = \sum_{\substack{-r<m\le x/d-r\\ m\equiv 0(l'v')}} 1 = \sum_{-\frac{r}{l'v'}<m\le\frac{x}{dl'v'}-\frac{r}{l'v'}} 1$$
$$= \left[\frac{x}{dl'v'}-\frac{r}{l'v'}\right]-\left[-\frac{r}{l'v'}\right] = \frac{x}{dl'v'}+O(1).$$

This proves the lemma.

We apply this lemma to get the following, say, "three-times-Kluyver application", for the orthogonality of three Ramanujan sums.

**Theorem 4.2.** *Fix $r, s, t \in \mathbb{N}$ and $x \in \mathbb{R}$ such that $x > 0$. Assume that $(s, t) = 1$, then for all $h, j \in \mathbb{Z}$, we have*

$$\frac{1}{x}\sum_{n\le x} c_r(n)c_s(n+h)c_t(n+j) = \mathbf{1}_{s|h}\cdot\mathbf{1}_{t|j}\cdot\mathbf{1}_{r=st}\cdot\varphi(st)+O\left(\frac{\sigma(r)\sigma(s)\sigma(t)}{x}\right).$$

*Proof.* Applying Kluyver's Formula to the triple convolution of Ramanujan sums and from Lemma 4.1, we have

$$\frac{1}{x}\sum_{n\le x} c_r(n)c_s(n+h)c_t(n+j) =$$

$$\sum_{d|r}\mu\left(\frac{r}{d}\right)d\sum_{\substack{v|s\\(d,v)|h}}\mu\left(\frac{s}{v}\right)v\sum_{\substack{\ell|t\\(d,\ell)|j}}\mu\left(\frac{t}{\ell}\right)\ell\frac{(d,v)(d,\ell)}{dv\ell}+O\left(\frac{1}{x}\sum_{d|r}d\sum_{v|s}v\sum_{\ell|t}\ell\right)$$

$$= M_{r,s,t}(h,j)+O\left(\frac{\sigma(r)\sigma(s)\sigma(t)}{x}\right)$$

where

$$M_{r,s,t}(h,j) := \sum_{v|s}\mu\left(\frac{s}{v}\right)\sum_{\ell|t}\mu\left(\frac{t}{\ell}\right)\sum_{\substack{d|r\\(d,v)|h\\(d,\ell)|j}}\mu\left(\frac{r}{d}\right)\sum_{a|(d,v)}\varphi(a)\sum_{b|(d,\ell)}\varphi(b).$$

Now,

$$M_{r,s,t}(h,j) = \sum_{v|s}\mu\left(\frac{s}{v}\right)\sum_{\ell|t}\mu\left(\frac{t}{\ell}\right)\sum_{\substack{a|h\\a|v\\a|r}}\varphi(a)\sum_{\substack{b|j\\b|\ell\\b|r}}\varphi(b)\sum_{n|\frac{r}{ab}}\mu\left(\frac{r}{abn}\right)$$

and the innermost sum is zero unless $ab=r$ in which case it is equal to 1. Note that $(s,t)=1$ implies $(a,b)=1$ leading to

$$\begin{aligned}
M_{r,s,t}(h,j) &= \sum_{v|s}\mu\left(\frac{s}{v}\right)\sum_{\ell|t}\mu\left(\frac{t}{\ell}\right)\sum_{a|h,\,a|v,\,a|r}\varphi(a)\sum_{b|j,\,b|\ell,\,b|r}\varphi(b)\sum_{n|\frac{r}{ab}}\mu\left(\frac{r}{abn}\right)\\
&= \varphi(r)\sum_{v|s}\mu\left(\frac{s}{v}\right)\sum_{\ell|t}\mu\left(\frac{t}{\ell}\right)\sum_{\substack{a|h\\a|v}}\sum_{\substack{b|j,b|\ell\\ab=r}}1 = \varphi(r)\sum_{\substack{a|h\\a|s}}\sum_{\substack{b|j\\b|t\\ab=r}}\sum_{\substack{v|s\\v\equiv 0(a)}}\mu\left(\frac{s}{v}\right)\sum_{\substack{\ell|t\\\ell\equiv 0(b)}}\mu\left(\frac{t}{\ell}\right)\\
&= \varphi(r)\sum_{\substack{a|h\\a|s}}\sum_{\substack{b|j\\b|t\\ab=r}}\sum_{v''|\frac{s}{a}}\mu\left(\frac{s}{av''}\right)\sum_{\ell''|\frac{t}{b}}\mu\left(\frac{t}{b\ell''}\right) = \varphi(r)\sum_{s|h}\sum_{\substack{t|j\\st=r}}1 = \mathbf{1}_{s|h}\cdot\mathbf{1}_{t|j}\cdot\mathbf{1}_{r=st}\cdot\varphi(st),
\end{aligned}$$

where in the penultimate step, we see $a=s$ and $b=t$ by the defining property of the Möbius function. This completes the proof.

## 5. Some applications

There are numerous applications of our theorems and we will only suggest some directions since the general method has been explained at length in earlier papers such as [CGM23], [CMS17], [CoMu18], [MuSa15], [Sah16a], [Sah16b]. However, for the sake of completeness, we will use this occasion to briefly illustrate the method and derive some new applications.

The general idea is easy to explain. Given an arithmetical function $f$, we assume it has a "Ramanujan expansion" in terms of Ramanujan sums:

$$f(n) = \sum_{r\le R}\widehat{f}(r)c_r(n) + E_f(R,n),$$

where $E_f(R,n)$ is the error given by the summatory approximation. In many cases, the "Ramanujan coefficients" $\widehat{f}(r)$ can be computed and estimated. So, such expansions of functions allow the study of convolutions sums such as

$$\sum_{n\le x} f_1(n)f_2(n+h)$$

and

$$\sum_{n\le x} f_1(n)f_2(n+h)f_3(n+k).$$

This idea was used in earlier papers such as [CGM23], [GaPa99],[GoMu25] and [GMP14] to give a heuristic derivation of the Hardy-Littlewood conjecture on the number of prime tuples. But the idea can also be used to derive **unconditional** asymptotic formulas for certain convolution sums. Here is an example.

**Theorem 5.1.** *Let $f_1, f_2,$ and $f_3$ be arithmetical functions with absolutely convergent Ramanujan expansion*

$$f_i(n) = \sum_{r=1}^{\infty} \widehat{f_i}(r)c_r(n)$$

*for $i = 1, 2, 3$ with*

$$|\widehat{f_i}(r)| \ll \frac{1}{r^{1+\delta}} \tag{5.4}$$

*for some $\delta > 0$. Then for $\delta > 1$, we have*

$$\sum_{n\le x} f_1(n)f_2(n+h)f_3(n+k) = x\mathfrak{S}(h,k) + O(1),$$

*and for $0 < \delta \le 1$, we have*

$$\sum_{n\le x} f_1(n)f_2(n+h)f_3(n+k) = x\mathfrak{S}(h,k) + O\left(x^{1-\delta}(\log x)^8\right).$$

*where*

$$\mathfrak{S}(h,k) = \sum_{r,s,t=1}^{\infty} \widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)\mathbf{1}_{s|h}\cdot\mathbf{1}_{t|k}\cdot\mathbf{1}_{r=st}\cdot\varphi(st) = \sum_{s|h,t|k} \widehat{f_1}(st)\widehat{f_2}(s)\widehat{f_3}(t)\varphi(st)$$

**Remark 1.** *In the above theorem, constraints $\mathbf{1}_{s|h}$, $\mathbf{1}_{t|k}$, and $\mathbf{1}_{r=st}$ imply that the main term is a finite sum.*

*Proof.* First, assume that $\delta > 1$. From Theorem 4.2, we have

$$\begin{aligned}\sum_{n\le x} f_1(n)f_2(n+h)f_3(n+k) =& \sum_{n\le x}\sum_{r,s,t=1}^{\infty} \widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)c_r(n)c_s(n+h)c_t(n+k)\\ =& x\sum_{r,s,t=1}^{\infty} \widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)\left\{\mathbf{1}_{s|h}\cdot\mathbf{1}_{t|k}\cdot\mathbf{1}_{r=st}\cdot\varphi(st) + O\left(\frac{\sigma(r)\sigma(s)\sigma(t)}{x}\right)\right\}.\end{aligned}$$

From the bounds on the coefficients of Ramanujan expansion (5.4), we get

$$\begin{aligned}&\sum_{n\le x} f_1(n)f_2(n+h)f_3(n+k)\\ &= x\sum_{r,s,t=1}^{\infty} \widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)\mathbf{1}_{s|h}\cdot\mathbf{1}_{t|k}\cdot\mathbf{1}_{r=st}\cdot\varphi(st) + O\left(\sum_{r,s,t=1}^{\infty}\frac{\sigma(r)\sigma(s)\sigma(t)}{(rst)^{1+\delta}}\right).\end{aligned}$$

Finally, using the fact that for $\Re(s) > 2$,

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n^s} = \zeta(s)\zeta(s-1)$$

converges, gives the result for $\delta > 1$.

Next, we divide the sum into two parts for $0 < \delta \leq 1$ as,

$$\begin{aligned}\sum_{n\leq x} f_1(n)f_2(n+h)f_3(n+k) &= \sum_{n\leq x}\sum_{r,s,t=1}^{\infty} \widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)c_r(n)c_s(n+h)c_t(n+k) \\ &= \sum_{n\leq x}\sum_{\substack{r,s,t\\ rst\leq R}} \widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)c_r(n)c_s(n+h)c_t(n+k) \\ &+ \sum_{n\leq x}\sum_{\substack{r,s,t\\ rst> R}} \widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)c_r(n)c_s(n+h)c_t(n+k) \\ &= S_1 + S_2,\end{aligned}$$

where

$$S_1 := \sum_{n\leq x}\sum_{\substack{r,s,t\\ rst\leq R}} \widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)c_r(n)c_s(n+h)c_t(n+k),$$

and

$$S_2 := \sum_{n\leq x}\sum_{\substack{r,s,t\\ rst> R}} \widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)c_r(n)c_s(n+h)c_t(n+k). \tag{5.5}$$

First, we estimate the sum $S_1$. From Theorem 4.2, we have

$$\begin{aligned}S_1 &= x\sum_{\substack{r,s,t\\ rst\leq R}} \widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)\left\{\mathbf{1}_{s|h}\cdot\mathbf{1}_{t|k}\cdot\mathbf{1}_{r=st}\cdot\varphi(st) + O\left(\frac{\sigma(r)\sigma(s)\sigma(t)}{x}\right)\right\} \\ &= x\sum_{\substack{r=st\\ r^2\leq R}} \widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)\mathbf{1}_{s|h}\cdot\mathbf{1}_{t|k}\cdot\varphi(r) + O\left(\sum_{\substack{r,s,t\\ rst\leq R}} |\widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)|\sigma(r)\sigma(s)\sigma(t)\right).\end{aligned}$$

We write $S_1$ as follows:

$$S_1 = xS_{11} - xS_{12} + O\left(E_1\right),$$

where

$$S_{11} := \sum_{\substack{r=st\\ r\geq 1}} \widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)\mathbf{1}_{s|h}\cdot\mathbf{1}_{t|k}\cdot\varphi(r),$$

$$S_{12} := \sum_{\substack{r=st\\ r^2\geq R}} \widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)\mathbf{1}_{s|h}\cdot\mathbf{1}_{t|k}\cdot\varphi(r),$$

and

$$E_1 = \sum_{\substack{r,s,t\\ rst\leq R}} |\widehat{f_1}(r)\widehat{f_2}(s)\widehat{f_3}(t)|\sigma(r)\sigma(s)\sigma(t).$$

From (5.4), we have

$$S_{12} \ll \sum_{r^2\geq R}\frac{\varphi(r)}{r^{2+2\delta}} \ll \sum_{r^2\geq R}\frac{1}{r^{1+2\delta}} \ll \frac{1}{R^{\delta}}.$$

Next, the average of the sum of divisors function is given as

$$\sum_{n\le x}\sigma(n)=\frac{\zeta(2)}{2}x^2+O(x\log x).$$

Therefore, from (5.4), partial summation, and the average of sum of divisors function, we have

$$\begin{aligned}E_1 &\ll \sum_{\substack{r,s,t\\ rst\le R}}\frac{\sigma(r)\sigma(s)\sigma(t)}{r^{1+\delta}s^{1+\delta}t^{1+\delta}} \ll \sum_{r\le R}\frac{\sigma(r)}{r^{1+\delta}}\sum_{s\le R/r}\frac{\sigma(s)}{s^{1+\delta}}\sum_{t\le R/rs}\frac{\sigma(t)}{t^{1+\delta}}\\ &\ll R^{1-\delta}\sum_{r\le R}\frac{\sigma(r)}{r^2}\sum_{s\le R/r}\frac{\sigma(s)}{s^2}\ll R^{1-\delta}(\log R)^2.\end{aligned}$$

Finally, we estimate $S_2$. From (1.1) and (5.4), we have

$$\begin{aligned}S_2 &\le \sum_{n\le x}\sum_{\substack{r,s,t\\ rst>R}}|\widehat{f}_1(r)\widehat{f}_2(s)\widehat{f}_3(t)c_r(n)c_s(n+h)c_t(n+k)|\\ &\ll \sum_{\substack{r,s,t\\ rst>R}}\frac{1}{(rst)^{1+\delta}}\sum_{r'|r\,,\,s'|s\,,\,t'|t}r's't'\sum_{\substack{n\le x\\ r'|n\,,\,s'|n+h\,,\,t'|n+k}}1.\end{aligned}$$

Write $r=r'a$, $s=s'b$ and $t=t'c$ and using Lemma 2.1, we get

$$\begin{aligned}S_2 &\ll \sum_{\substack{r'\le x\\ s'\le x+h\\ t'\le x+k}}\frac{1}{(r's't')^{\delta}}\sum_{\substack{a\,,\,b\,,\,c\\ abc>R/r's't'}}\frac{1}{(abc)^{1+\delta}}\sum_{\substack{n\le x\\ r'|n\,,\,s'|n+h\,,\,t'|n+k}}1\\ &\ll \sum_{\substack{r'\le x\\ s'\le x+h\\ t'\le x+k}}\frac{1}{(r's't')^{\delta}}\sum_{\ell>R/r's't'}\frac{d_3(\ell)}{\ell^{1+\delta}}\sum_{\substack{n\le x\\ r'|n\,,\,s'|n+h\,,\,t'|n+k}}1\\ &\ll \sum_{\substack{r'\le x\\ s'\le x+h\\ t'\le x+k}}\frac{1}{(r's't')^{\delta}}\frac{\log^2(R/r's't')}{(R/r's't')^{\delta}}\sum_{\substack{n\le x\\ r'|n\,,\,s'|n+h\,,\,t'|n+k}}1\\ &\ll \frac{\log^2 R}{R^{\delta}}\sum_{n\le x}d(n)d(n+h)d(n+k).\end{aligned}$$

Using Lemma 2.2, we find that

$$S_2\ll\frac{x(\log x)^7\log R}{R^{\delta}}.$$

Combining the above estimates, we obtain

$$\begin{aligned}\sum_{n\le x}f_1(n)f_2(n+h)f_3(n+k) &= x\sum_{\substack{r=st\\ r\ge 1}}\widehat{f}_1(r)\widehat{f}_2(s)\widehat{f}_3(t)\mathbf{1}_{s|h}\cdot\mathbf{1}_{t|k}\cdot\varphi(r)\\ &\quad+O\left(\frac{x}{R^{2\delta}}+R^{1-\delta}(\log R)^2+\frac{x(\log x)^7\log R}{R^{\delta}}\right).\end{aligned}$$

To optimize the error terms, we choose $R=x\log x$ and obtain the result for $0<\delta\le 1$.

## 6. Concluding remarks

If $\Lambda(n)$ denotes the usual von Mangoldt function, then Hardy [Har21] proved that

$$\frac{\varphi(n)}{n}\Lambda(n) = \sum_{r=1}^{\infty} \frac{\mu(r)}{\varphi(r)} c_r(n).$$

The condition for applying our Theorem 5.1 to the problem of enumerating prime triples just fails since in this case $\delta = 0$.

The asymptotic behaviour of the sum

$$\sum_{n\le x} d(n)d(n+h)d(n+k)$$

appearing in Lemma 2.2 is discussed in a paper by Misra, Murty and Saha [MMS26]. For $hk \neq 0$, it is conjectured by Browning [Bro11] that as $x$ tends to infinity, the sum is asymptotic to

$$cx(\log x)^3$$

for some non-zero constant $c$. The authors in [MMS26] show that

$$x(\log x)^3 \ll \sum_{n\le x} d(n)d(n+h)d(n+k) \ll x(\log x)^3.$$

Thus, the exponent 8 in Theorem 5.1 can certainly be reduced to 4.

It is clear from our discussion that the method will have further applications. For instance, as was done in [CGM23], one can obtain a new heuristic derivation of the Hardy-Littlewood conjecture on prime triples. Further applications can be given but, in the interest of brevity, we have only included an illustrative example.

**Acknowledgement.** We thank Bikram Misra, Sunil Naik and Biswajyoti Saha for their comments on an earlier version of this paper. We sincerely thank the referee for helpful comments.

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