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A remark on a conjecture of Chowla

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Abstract. We make some remarks on a special case of a conjecture of Chowla regarding the Möbius function $\mu(n)$.

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1. Introduction

The Möbius function $\mu(n)$ is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^r & \text{if } n = p_1 \cdots p_r, \text{ where } p_i \text{ are distinct primes,} \\ 0 & \text{if } p^2 | n \text{ for any prime } p. \end{cases}$$

In 1965, Sarvadaman Chowla [1] made the following conjecture.

Conjecture 1 (Chowla). Let $h_1 < \cdots < h_k$ be non-negative integers and $a_1, \ldots, a_k$ be positive integers with at least one of the a_i odd. Then, as $x \to \infty$, we have

$$\sum_{n \le x} \mu(n + h_1)^{a_1} \cdots \mu(n + h_k)^{a_k} = o(x).$$

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This conjecture is known only in the case $k = 1$, where the statement is equivalent to the prime number theorem. A more general conjecture along these lines was formulated by P. D. T. A. Elliott [2]. Recently, K. Matomäki, M. Radziwiłł and T. Tao [5,11] formulated a slightly modified, "corrected" form of Elliott's conjecture. They also succeeded in proving, for $k = 2$, the averaged and logarithmically averaged versions of the Chowla and Elliott conjectures. Variants of such conjectures are important in the context of the twin prime problem since they encode what is known as the parity principle in sieve theory. This is elaborated upon in more detail by the authors in [8]. It does not seem that sieve methods alone are enough to resolve the twin prime problem. Indeed, this led the second author to find an error in [3] (cf. [4]).

In this note, we consider the following special case of Chowla's conjecture:

Conjecture 2. Let $h_1, \ldots, h_k$ be distinct non-negative integers. As $x \to \infty$, we have

$$\sum_{n \le x} \mu^2(n + h_1) \cdots \mu^2(n + h_{k-1}) \mu(n + h_k) = o(x).$$

In his blog, Tao [10] remarks that this special case is "essentially the prime number theorem in arithmetic progressions after some sieving." For the benefit of non-experts, we indicate in section 2 how to deduce this. This is instructive since we are interested in sharper error terms and by applying unconditional results for the prime number theorem for arithmetic progressions, the elementary sieving argument would only yield an error term of

$$O\left(\frac{x}{(\log x)^{1/2} \log\log x}\right).$$

The purpose of this paper is to show that one can derive a stronger unconditional theorem. We prove:

Theorem 1. *Let $k \ge 2$ and $h_1, \ldots, h_k$ be distinct non-negative integers. We have for any $A > 0$,*

$$\sum_{n \le x} \mu^2(n + h_1) \cdots \mu^2(n + h_{k-1}) \mu(n + h_k) \ll \frac{x}{(\log x)^A},$$

as $x \to \infty$. The implied constant above depends upon k, A as well as $h_1, \ldots, h_k$.

If we invoke the generalized Riemann hypothesis in the case $k \ge 2$, we can deduce an error of $O(x^{\beta_k})$ for some $1 > \beta_k > 1/2$. More precisely, we obtain:

Theorem 2. *Let $k \geq 2$ and $h_1, \ldots, h_k$ be distinct non-negative integers. Assuming the generalized Riemann hypothesis for Dirichlet L-functions, we have*

$$\sum_{n \leq x} \mu^2(n+h_1) \cdots \mu^2(n+h_{k-1})\mu(n+h_k) \ll_k x^{\beta_k},$$

for any $\beta_k > 1 - (1/2k)$. The implied constant above may depend on $h_1, \ldots, h_k$.

We believe that it should be possible to improve the exponent in Theorem 2. We expect to investigate this in a future work. It seems reasonable to expect that any $\beta > 1/2$ is permissible and it would be of considerable interest to prove such a result on the generalized Riemann hypothesis. Such a result will (conditionally) establish a stronger version of Chowla's conjecture (formulated by Nathan Ng in [9]) for the special case considered here. The following predicts stronger upper bounds for the sum in Conjecture 1.

Conjecture 3 (Ng, [9]). Let $h_1 < \cdots < h_k$ be non-negative integers and $a_1, \ldots, a_k$ be positive integers with at least one of the a_i odd. Then, there exists $\beta_0 \in (1/2, 1)$, independent of k, such that as $x \to \infty$, we have

$$\sum_{n \leq x} \mu(n+h_1)^{a_1} \cdots \mu(n+h_k)^{a_k} \ll x^{\beta_0},$$

uniformly for all $h_i \leq x$.

It would be of interest to determine how small β_0 can be taken to be in this conjecture, at least in the case considered in this paper. We note that the Riemann hypothesis implies the above conjecture for $k = 1$, with $\beta_0 = \frac{1}{2} + \epsilon$, for any $\epsilon > 0$.

2. An elementary sieving argument

The elementary sieving argument alluded to in the introduction proceeds as follows. We will outline the main ideas since the details can be easily filled in by the reader. Let p_n denote the n-th prime. Fix t (which will be chosen later) and define for each $1 \leq i \leq t$,

$$A_i = \{n \leq x : p_i^2 | n + h_j \text{ for some } j \text{ with } 1 \leq j \leq k-1\}.$$

Let us first consider the sum

$$\sum_{\substack{n \leq x \\ n \notin A_i \forall i}} \mu(n+h_k).$$

Let $A_\varnothing = \{n \le x\}$. If we let $S = \{1, 2, ..., t\}$ and for each $I \subseteq S$, we set

$$A_I = \cap_{i \in I} A_i,$$

then the inclusion-exclusion principle gives that the above sum is

$$\sum_{I \subseteq S} (-1)^{|I|} \sum_{\substack{n \le x \\ n \in A_I}} \mu(n + h_k).$$

It is now easy to see that the innermost sum is again a sum of sums each of the form

$$\sum_{\substack{n \le x \\ n \equiv a \pmod{q^2}}} \mu(n + h_k)$$

for some q which divides the product $p_1 \cdots p_t$. It is well-known (see for example page 385 of [6]) that for any $A > 0$, and $q \le (\log x)^A$, we have for some $c > 0$ (independent of q) that

$$\sum_{\substack{n \le x \\ n \equiv a \pmod{q}}} \mu(n) = O(x \exp(-c\sqrt{\log x})),$$

for all a. The sum under consideration in our main theorem, that is,

$$\sum_{n \le x} \mu^2(n + h_1) \cdots \mu^2(n + h_{k-1}) \mu(n + h_k),$$

is then

$$\sum_{\substack{n \le x \\ n \notin A_i \forall i}} \mu(n + h_k) - \sum_{\substack{n \le x \\ \exists p > p_t : p^2 | n + h_j \text{ for some } j < k}} \mu(n + h_k).$$

The latter sum is clearly $O(x/p_t)$. By choosing p_t arbitrarily large, we deduce the $o(x)$ result which was deemed elementary. To refine the error term, one can choose p_t as a function of x. Indeed, it is clear that the sum is

$$\ll 2^t x \exp(-c\sqrt{\log x}) + \frac{x}{p_t}.$$

To exploit the unconditional error term to its fullest, we need to choose $t = c_1\sqrt{\log x}$ for a suitably small constant $c_1 < c$ which gives a final error term of

$$O\left(\frac{x}{(\log x)^{1/2} \log\log x} \right),$$

since $p_t \sim t \log t$ as t goes to infinity. This proves that

$$\sum_{n \le x} \mu^2(n + h_1) \cdots \mu^2(n + h_{k-1}) \mu(n + h_k) = O\left(\frac{x}{(\log x)^{1/2} \log\log x} \right),$$

as claimed in the introduction.

3. Preliminary results

We will make use of the following result by Siebert and Wolke [12], which can be thought of as a Bombieri-Vinogradov type theorem for the Möbius funtion.

Proposition 3 (c.f. Corollary 1 of [12]). *Given any $A > 0$, there exists $B > 0$ (depending on A) such that*

$$\sum_{q \le x^{1/2}/(\log x)^B} \max_{a \pmod q} \max_{y \le x} \left| \sum_{\substack{n \le y \\ n \equiv a \pmod q}} \mu(n) \right| \ll \frac{x}{(\log x)^A}.$$

Recall that the square of the Möbius function $\mu^2(n)$ detects whether n is square-free. The following identity will be useful in our discussion.

Proposition 4. *For any positive integer n, we have*

$$\mu^2(n) = \sum_{d^2 | n} \mu(d).$$

Proof. See Exercise 1.1.7 of Murty and Esmonde [7]. □

For $k \ge 1$, let $t_k(n)$ denote the number of ordered tuples $(d_1, \dots, d_k)$ of positive integers such that the lcm $[d_1, d_2, \dots, d_k] = n$. The function $t_k(n)$ is very close to the divisor function $\tau(n)$, which counts the number of positive divisors of n. Indeed, we have the following bound.

Proposition 5. *For $k \ge 1$, we have $t_k(n) \le k\tau(n)^{k-1}$.*

Proof. Fix $k \in \mathbb{N}$. It can be checked that $t_k(n)$ is a multiplicative function of n and hence it is enough to bound $t_k(p^a)$ for a prime p and $a \in \mathbb{N}$. From the equation

$$[d_1, \dots, d_k] = p^a,$$

we see that all the d_i are powers p^b with $0 \le b \le a$. Moreover, at least one of the d_i must be exactly p^a and there are k ways to make this choice. This gives $t_k(p^a) \le k(a+1)^{k-1} = k\tau(p^a)^{k-1}$. This completes the proof. □

4. Proof of Theorem 1 for $k = 2$

If $k = 2$, then by a suitable translation, we may consider the sum

$$S(x) := \sum_{n \le x} \mu^2(n)\mu(n+h),$$

where h is a fixed positive integer.

Using Proposition 4 and interchanging summation, we obtain

$$\begin{aligned} S(x) &= \sum_{d \le \sqrt{x}} \mu(d) \sum_{\substack{n \le x \\ n \equiv 0 \ (\mathrm{mod}\ d^2)}} \mu(n+h) = \sum_{d \le \sqrt{x}} \mu(d) \sum_{\substack{m \le x+h \\ m \equiv h \ (\mathrm{mod}\ d^2)}} \mu(m) \\ &= \sum_{d \le x^{1/4-\epsilon}} \mu(d) \sum_{\substack{m \le x+h \\ m \equiv h \ (\mathrm{mod}\ d^2)}} \mu(m) \\ &\quad + \sum_{x^{1/4-\epsilon} < d \le \sqrt{x}} \mu(d) \sum_{\substack{m \le x+h \\ m \equiv h \ (\mathrm{mod}\ d^2)}} \mu(m) \\ &= S_1(x) + S_2(x) \qquad \text{(say)}, \end{aligned}$$

for some fixed small $\epsilon > 0$.

The following lemmas give us estimates for the sums $S_1(x)$ and $S_2(x)$.

Lemma 6. *We have, for any $A > 0$, as $x \to \infty$,*

$$S_1(x) \ll_A \frac{x}{(\log x)^A}.$$

Proof. Writing d^2 as q, we see that

$$S_1(x) \ll \sum_{\substack{q \le x^{1/2-\epsilon} \\ q = d^2}} |\mu(d)| \left| \sum_{\substack{m \le x+h \\ m \equiv h \pmod q}} \mu(m) \right| \ll \sum_{q \le x^{1/2-\epsilon}} \left| \sum_{\substack{m \le x+h \\ m \equiv h \pmod q}} \mu(m) \right|.$$

By Proposition 3, the above expression is of the order $x/(\log x)^A$ for any $A > 0$. □

Lemma 7. *For any $\epsilon > 0$, we have, as $x \to \infty$,*

$$S_2(x) \ll_\epsilon x^{3/4+\epsilon}.$$

Proof. We have the trivial estimate

$$\sum_{\substack{m \le x+h \\ m \equiv h \ (\mathrm{mod}\ d^2)}} \mu(m) \ll \sum_{\substack{m \le x+h \\ m \equiv h \ (\mathrm{mod}\ d^2)}} 1 \ll \frac{x}{d^2} + 1.$$

This gives

$$S_2(x) \ll \sum_{x^{1/4-\epsilon} < d \le \sqrt{x}} \frac{x}{d^2} + 1 \ll_\epsilon x^{3/4+\epsilon},$$

using known elementary estimates. □

Combining Lemmas 6 and 7 gives $S(x) \ll_A x/(\log x)^A$, for any $A > 0$, as required.

5. Proof of Theorem 1 for $k > 2$

5.1 A special case

Let us begin with the case $k = 3$. For the sake of simplicity, consider the sum

$$S(x) := \sum_{n \le x} \mu^2(n)\mu^2(n+1)\mu(n+2)$$

As done before, using Proposition 4, we can write

$$\begin{aligned} S(x) &= \sum_{n \le x} \mu(n+2) \sum_{d^2 | n} \mu(d) \sum_{e^2 | n+1} \mu(e) \\ &= \sum_{d \le \sqrt{x}} \mu(d) \sum_{\substack{n \le x \\ n \equiv 0 \ (\mathrm{mod}\ d^2)}} \mu(n+2) \sum_{e^2 | n+1} \mu(e). \end{aligned}$$

Let $\eta \in (0, 1/2)$ (to be chosen later). We can split the sum $S(x)$ into $S_1(x) + S_2(x)$, where

$$S_1(x) = \sum_{d \le x^\eta} \mu(d) \sum_{\substack{n \le x \\ n \equiv 0 \ (\mathrm{mod}\ d^2)}} \mu(n+2) \sum_{e^2 | n+1} \mu(e),$$

$$S_2(x) = \sum_{x^\eta < d \le \sqrt{x}} \mu(d) \sum_{\substack{n \le x \\ n \equiv 0 \ (\mathrm{mod}\ d^2)}} \mu(n+2) \sum_{e^2 | n+1} \mu(e), \tag{1}$$

We first estimate $S_2(x)$ as follows.

Lemma 8. *As $x \to \infty$, we have $S_2(x) \ll_\epsilon x^{1-\eta+\epsilon}$, for any $\epsilon > 0$.*

Proof. Clearly,

$$\begin{aligned} |S_2(x)| &\le \sum_{x^\eta < d \le \sqrt{x}} |\mu(d)| \left| \sum_{\substack{n \le x \\ n \equiv 0 \ (\mathrm{mod}\ d^2)}} \mu(n+2) \sum_{e^2 | n+1} \mu(e) \right| \\ &\le \sum_{x^\eta < d \le \sqrt{x}} \sum_{\substack{n \le x \\ n \equiv 0 \ (\mathrm{mod}\ d^2)}} \sum_{e^2 | n+1} 1. \end{aligned}$$

The inner most sum is bounded by the number of divisors of $n+1$. As $n \le x$, this is $O_\epsilon(x^\epsilon)$ for any $\epsilon > 0$. This gives

$$S_2(x) \ll_\epsilon x^\epsilon \sum_{x^\eta < d \le \sqrt{x}} \sum_{\substack{n \le x \\ n \equiv 0 \ (\mathrm{mod}\ d^2)}} 1 \ll_\epsilon x^\epsilon \sum_{x^\eta < d \le \sqrt{x}} \frac{x}{d^2} \ll_\epsilon \frac{x^{1+\epsilon}}{x^\eta},$$

as required. □

Since the above lemma shows that $S_2(x) \ll_A x/(\log x)^A$ for any $A > 0$, we can now focus our attention on $S_1(x)$. By interchanging summation in (1), we obtain

$$S_1(x) = \sum_{e \leq \sqrt{x+1}} \mu(e) \sum_{\substack{n \leq x \\ n \equiv -1 \pmod{e^2}}} \mu(n+2) \sum_{\substack{d \leq x^\eta \\ d^2 | n}} \mu(d).$$

Letting $\delta \in (0, 1/2)$ (to be chosen later), we can split the sum $S_1(x)$ into $S_{11}(x) + S_{12}(x)$, where

$$S_{11}(x) = \sum_{e \leq x^\delta} \mu(e) \sum_{\substack{n \leq x \\ n \equiv -1 \pmod{e^2}}} \mu(n+2) \sum_{\substack{d \leq x^\eta \\ d^2 | n}} \mu(d),$$

$$S_{12}(x) = \sum_{x^\delta < e \leq \sqrt{x+1}} \mu(e) \sum_{\substack{n \leq x \\ n \equiv -1 \pmod{e^2}}} \mu(n+2) \sum_{\substack{d \leq x^\eta \\ d^2 | n}} \mu(d). \tag{2}$$

We first estimate $S_{12}(x)$ as follows.

Lemma 9. *As $x \to \infty$, we have $S_{12}(x) \ll_\epsilon x^{1-\delta+\epsilon}$, for any $\epsilon > 0$.*

Proof. We have,

$$S_{12}(x) \ll \sum_{x^\delta < e \leq \sqrt{x+1}} \sum_{\substack{n \leq x \\ n \equiv -1 \pmod{e^2}}} \sum_{\substack{d \leq x^\eta \\ d^2 | n}} 1.$$

The inner most sum is at most the number of divisors of n, which is $O_\epsilon(x^\epsilon)$ for any $\epsilon > 0$. Hence,

$$S_{12}(x) \ll_\epsilon x^\epsilon \sum_{x^\delta < e \leq \sqrt{x+1}} \sum_{\substack{n \leq x \\ n \equiv -1 \pmod{e^2}}} 1.$$

Since the inner sum counts the number of integers $n + 1 \leq x + 1$ which are divisible by e^2, we have

$$S_{12}(x) \ll_\epsilon x^\epsilon \sum_{x^\delta < e \leq \sqrt{x+1}} \frac{x}{e^2} \ll_\epsilon \frac{x^{1+\epsilon}}{x^\delta},$$

as needed. □

We are now left to estimate $S_{11}(x)$, which we do as follows.

Lemma 10. *Choose $\eta, \delta \in (0, 1/2)$ such that $2(\delta + \eta) < 1/2$. Then, for any $A > 0$, we have*

$$S_{11}(x) \ll_A \frac{x}{(\log x)^A},$$

as $x \to \infty$.

Proof. Interchanging the order of summation in (2), we have

$$
\begin{aligned}
S_{11}(x) &= \sum_{d \le x^{\eta}} \mu(d) \sum_{e \le x^{\delta}} \mu(e) \sum_{\substack{n \le x \\ n \equiv 0 \ (\mathrm{mod}\ d^2) \\ n \equiv -1 \ (\mathrm{mod}\ e^2)}} \mu(n+2) \\
&\ll \sum_{d \le x^{\eta}} \sum_{e \le x^{\delta}} |\mu(d)\mu(e)| \left| \sum_{\substack{m \le x+2 \\ m \equiv a \ (\mathrm{mod}\ d^2 e^2)}} \mu(m) \right|, \qquad (3)
\end{aligned}
$$

where we have used the Chinese remainder theorem to combine the congruence conditions for the co-prime moduli d^2 and e^2 into a single congruence condition a modulo d^2e^2. We denote d^2e^2 by q. Then the above sum over d and e can be thought of as a sum over $q \le x^{2(\eta+\delta)}$, with each q appearing $\le \tau(q)$ times, since

$$
\#\{(d, e) : d^2e^2 = q, d \le x^{\eta}, e \le x^{\delta}\} \le \tau(q),
$$

where $\tau(q)$ denotes the number of divisors of q. Thus, we have

$$
\begin{aligned}
S_{11}(x) &\ll \sum_{q \le x^{2(\eta+\delta)}} \tau(q) \left| \sum_{\substack{m \le x+2 \\ m \equiv a \ (\mathrm{mod}\ q)}} \mu(m) \right| \\
&\ll \left(\sum_{q \le x^{2(\eta+\delta)}} \tau(q)^2 \left| \sum_{\substack{m \le x+2 \\ m \equiv a \ (\mathrm{mod}\ q)}} \mu(m) \right| \right)^{1/2} \\
&\quad \times \left(\sum_{q \le x^{2(\eta+\delta)}} \left| \sum_{\substack{m \le x+2 \\ m \equiv a \ (\mathrm{mod}\ q)}} \mu(m) \right| \right)^{1/2},
\end{aligned}
$$

by applying the Cauchy-Schwarz inequality. As the average order of the function $\tau(x)^2$ is $(\log x)^c$ for some fixed constant c, using crude estimates, the first term in parenthesis is $\ll x^{1/2}(\log x)^c$. To bound the second term, since $2(\eta + \delta) < 1/2$, we use Proposition 3 to obtain

$$
\left(\sum_{q \le x^{2(\eta+\delta)}} \left| \sum_{\substack{m \le x+2 \\ m \equiv a \ (\mathrm{mod}\ q)}} \mu(m) \right| \right)^{1/2} \ll_A \frac{x^{1/2}}{(\log x)^A},
$$

for any $A > 0$. This completes the proof. □

5.2 The general case

For the case $k = 3$, we may consider in general, the sum

$$S(x) := \sum_{n \le x} \mu^2(n + h_1)\mu^2(n + h_2)\mu(n + h_3),$$

for some fixed non-negative integers $h_1 < h_2 < h_3$. Then $S(x)$ can be written as $S_{11} + S_{12} + S_2$ as in Section 5. The previous argument can be repeated with Lemmas 8 and 9 going through as before. In the proof of Lemma 10, the Chinese remainder theorem now gives us a single congruence condition modulo $[d^2, e^2]$ instead of d^2e^2 in (3), as d, e may no longer be co-prime. Writing $[d^2, e^2] = q$, the sum over d and e can be thought of as a sum over $q \le x^{2(\eta+\delta)}$, with each q now appearing $T(q)$ times, where

$$T(q) = \#\{\text{ordered pairs } (d^2, e^2) : [d^2, e^2] = q, d \le x^{\eta}, e \le x^{\delta}\}.$$

We obtain

$$S_{11}(x) \ll \sum_{q \le x^{2(\eta+\delta)}} T(q) \Bigg| \sum_{\substack{m \le x+2 \\ m \equiv a \pmod q}} \mu(m) \Bigg|.$$

Clearly, $T(q) \le t_2(q)$ where $t_2(q)$ is as defined prior to Proposition 5. Using Proposition 5, we see that $T(q)$ has average order $\ll (\log x)^c$ for some fixed constant c, so that the remainder of the proof of Lemma 10 goes through. This gives $S(x) \ll_A x/(\log x)^A$ for any $A > 0$.

It is also evident that this argument would work for any $k \ge 3$. Indeed, for fixed non-negative integers $h_1 < \cdots < h_k$, consider the sum

$$S(x) = \sum_{n \le x} \mu^2(n + h_1)\mu^2(n + h_2) \cdots \mu^2(n + h_{k-1})\mu(n + h_k).$$

We write

$$\mu^2(n + h_j) = \sum_{d_j^2 | n+h_j} \mu(d_j) \qquad (j = 1, \ldots, k-1),$$

and take $\eta_1, \ldots, \eta_{k-1} \in (0, 1/2)$ to be chosen later. We interchange the sums over d_1 and n, and then split the range of d_1 into $d_1 < x^{\eta_1}$ and $x^{\eta_1} < d_1 \le x + h_1$. Then the tail sum

$$\sum_{x^{\eta_1} < d_1 \le \sqrt{x+h_1}} \mu(d_1) \sum_{\substack{n \le x \\ d_1^2 | n+h_1}} \mu(n + h_k) \sum_{d_2^2 | n+h_2} \mu(d_2) \cdots \sum_{d_{k-1}^2 | n+h_{k-1}} \mu(d_{k-1}) \tag{4}$$

can be shown to be $O_\epsilon(x^{1-\eta_j+\epsilon})$ for any $\epsilon > 0$, by absolutely bounding the sums over $d_2, \dots d_{k-1}$ by divisor functions, as done in Lemma 8. The same argument works in turn for each of the other tails $x^{\eta_j} < d_j \le x + h_j$, $2 \le j \le k-1$.

In this way, the range of each d_j has effectively been shortened to $d_j < x^{\eta_j}$ and after using the Chinese remainder theorem, we are left to estimate the sum

$$\sum_{\substack{d_j < x^{\eta_j}, \\ 1 \le j \le k-1}} \mu(d_1) \cdots \mu(d_{k-1}) \sum_{\substack{n \le x \\ n \equiv a \ (\mathrm{mod}\ [d_1^2, \dots, d_{k-1}^2])}} \mu(n + h_k). \tag{5}$$

Again, writing $[d_1^2, \dots, d_{k-1}^2] = q$, the sums over d_i can be thought of as a sum over $q \le x^{2(\sum_{j=1}^{k-1} \eta_j)}$, with each q now appearing $T_{k-1}(q)$ times, where

$$T_{k-1}(q) = \#\{ \text{ ordered pairs } (d_1^2, \dots, d_{k-1}^2) : [d_1^2, \dots, d_{k-1}^2] = q, \\ d_i \le x^{\eta_i}, i = 1, \dots, k\}.$$

We obtain

$$S_{11}(x) \ll \sum_{q \le x^{2(\sum_{j=1}^{k-1} \eta_j)}} T_{k-1}(q) \left| \sum_{\substack{m \le x+2 \\ m \equiv a \ (\mathrm{mod}\ q)}} \mu(m) \right|.$$

Since $T_{k-1}(q) \le t_{k-1}(q)$, we can invoke Proposition 5. We see that so long as η_j are chosen to satisfy $2 \sum_{j=1}^{k-1} \eta_j < 1/2$, the proof of Lemma 10 follows mutatis mutandis. This proves Theorem 1.

6. Proof of Theorem 2

Assuming a generalized Riemann hypothesis (for Dirichlet L-functions), one has the bound

$$\sum_{\substack{n \le x \\ n \equiv a \ (\mathrm{mod}\ q)}} \mu(n) = O_\epsilon(x^{1/2} q^\epsilon), \tag{6}$$

for any $\epsilon > 0$, where the implied O-constant does not depend on the modulus q.

Using this instead of Proposition 3 in our analysis for the case $k = 2$ would give

$$\sum_{n \le x} \mu^2(n)\mu(n+h) \ll_\epsilon x^{3/4+\epsilon},$$

for any $\epsilon > 0$, conditional upon GRH.

Similarly, we can use (6) in place of Proposition 3 to analyze the sum

$$S(x) = \sum_{n \le x} \mu^2(n + h_1) \cdots \mu^2(n + h_{k-1}) \mu(n + h_k),$$

with $k \ge 3$. Recall the argument of Section 5.2. Choosing $\eta_j = \eta$ (say), for all $1 \le j \le k-1$, we see (as done in Lemmas 8 and 9) that all the tail sums of the type (4) are unconditionally $O_\epsilon(x^{1-\eta+\epsilon})$ for any $\epsilon > 0$. Assuming GRH, the remaining sum (5) can be seen to be

$$\ll_\epsilon x^{1/2} \sum_{\substack{d_j < x^\eta, \\ 1 \le j \le k-1}} d_1^\epsilon \cdots d_{k-1}^\epsilon \ll_\epsilon x^{1/2+(k-1)\eta(1+\epsilon)}$$

for any $\epsilon > 0$, after applying (6). Choosing $\epsilon = \epsilon(k)$ sufficiently small and putting everything together, we have under GRH,

$$S(x) \ll_{\epsilon,k} x^{\frac{1}{2}+(k-1)\eta+\epsilon} + (k-1)x^{1-\eta+\epsilon},$$

for any $\epsilon > 0$. The implied constant above may also depend upon $h_1, \ldots, h_k$.

Optimizing this estimate yields the choice $\eta = 1/2k$. Thus, for $k \ge 3$, under GRH we have the stronger result

$$\sum_{n \le x} \mu^2(n + h_1) \cdots \mu^2(n + h_{k-1}) \mu(n + h_k) \ll_k x^{\beta_k},$$

for any $\beta_k > 1 - \frac{1}{2k}$.

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