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General Section

Convolution of values of the Lerch zeta-function ☆

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ABSTRACT

We investigate generalizations arising from the identity

$$\zeta_2(n-1,1) = \frac{n-1}{2}\zeta(n) - \frac{1}{2}\sum_{j=2}^{n-2}\zeta(j)\,\zeta(n-j),$$

where $\zeta_2(k,1)$ denotes a double zeta value at $(k,1)$, or an Euler-Zagier sum. In particular, we prove analogues of the above identity for Lerch zeta-functions and Dirichlet L-functions. Such an attempt has met with limited success in the past. We highlight that this study naturally leads one into the realm of *multiple* L-values and multiple L^*-values.

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1. Introduction

In the early 18th century, Euler extensively studied infinite series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^k},$$

for any positive integer $k > 1$. After Riemann's introduction of the zeta-function,

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \Re(s) > 1,$$

we recognize the series studied by Euler as special values of $\zeta(s)$ at positive integers. In particular, Euler's resolution of the Basel problem leads to

$$\zeta(2k) \in \pi^{2k}\, \mathbb{Q}^*,$$

for any positive integer $k \geq 1$. Thus, the values $\zeta(2k)$ are all transcendental, thanks to Lindemann's theorem that π is transcendental. However, the arithmetic nature of $\zeta(2k+1)$ for an integer $k \geq 1$ remains shrouded in mystery.

Recently, significant progress was made in this direction when Apéry [2] proved that $\zeta(3)$ is irrational, T. Rivoal [20] proved that infinitely many of $\zeta(2k+1)$ are irrational and W. Zudilin [24] showed that at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational. The transcendence of the values $\zeta(2k+1)$ is not known, although they are expected to be so. Moreover, it is widely believed that

$$\pi,\ \zeta(3),\ \zeta(5),\ \zeta(7),\ \cdots$$

are algebraically independent.

In an attempt to understand the nature of the special values of the Riemann zeta-function, it seems fruitful to adopt a larger perspective. The values then seem intimately connected with special values of the multi-zeta functions. A multi-zeta value (MZV) of depth r and weight w is defined as the nested sum,

$$\zeta_r(k_1, k_2, \cdots, k_r) := \sum_{n_1 > n_2 > \cdots > n_r \geq 1} \frac{1}{{n_1}^{k_1}\, {n_2}^{k_2}\, \cdots\, {n_r}^{k_r}},$$

where k_i are positive integers, $k_1 \geq 2$ and $k_1 + k_2 + \cdots + k_r = w$. These values not only appear in several areas of mathematics but also in quantum physics. MZVs have been the focus of intense research in recent times. They satisfy a wide variety of relations. Recently, F. Brown [7] proved a remarkable theorem which states that all multiple zeta-values of weight n are $\mathbb{Q}$-linear combinations of

$$\left\{\zeta(a_1,\cdots,a_r)\,:\,a_i\in\{2,\,3\}\text{ for }1\le i\le r,\;a_1+\cdots+a_r=n\right\}.$$

The MZVs are also intricately related to the values of the Riemann zeta-function itself. Perhaps the most striking example of such a relation is that

$$\zeta_2(2,1)=\zeta(3),$$

or more generally,

$$\zeta_2(n-1,1)=\frac{n-1}{2}\zeta(n)-\frac{1}{2}\sum_{j=2}^{n-2}\zeta(j)\,\zeta(n-j), \tag{1}$$

for a positive integer $n\ge 3$, which was certainly known to Euler. Thus, it is expected that the study of MZVs will shed light upon the arithmetic nature of $\zeta(2k+1)$.

These convolution sum identities suggest that there must exist similar identities for other L-functions such as Dirichlet L-functions. Yet, to our knowledge, no one has derived such analogues. The closest we come to such an attempt revolves around a celebrated theorem of Ramanujan: let α, $\beta>0$ with $\alpha\beta=\pi^2$, and let k be any non-zero integer. Then

$$\begin{aligned}\alpha^{-k}&\left\{\frac{1}{2}\,\zeta(2k+1)+\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}(e^{2\alpha n}-1)}\right\}\\&=(-\beta)^{-k}\left\{\frac{1}{2}\,\zeta(2k+1)+\sum_{n=1}^{\infty}\frac{1}{n^{2k+1}\,(e^{2\beta n}-1)}\right\}\\&\quad-2^{2k}\sum_{j=0}^{k+1}(-1)^j\,\frac{B_{2j}}{(2j)!}\,\frac{B_{2k+2-2j}}{(2k+2-2j)!}\,\alpha^{k+1-j}\,\beta^j,\end{aligned}$$

where B_n denotes the n-th Bernoulli number (see [3]). The last term on the right hand side of the above identity can be viewed as a convolution sum of zeta values since

$$\zeta(2k)=\frac{(2\pi i)^{2k}\,B_{2k}}{2\,(2k)!}.$$

Attempts to generalize this identity to Dirichlet L-functions have met with limited success. For example, S. Chowla [8] derived an analog of this identity if $\zeta(s)$ is replaced by $L(s,\chi_4)$ where χ_4 is the non-trivial Dirichlet character modulo 4 (see [3, pg. 277]). It is the purpose of this note to initiate a systematic study of such convolution identities. As Dirichlet L-functions are linear combinations of Hurwitz zeta-functions, it seems appropriate to derive convolution sum identities for them, and more generally for the Lerch zeta-functions.

The Hurwitz zeta-function was isolated for independent study by A. Hurwitz [13] in 1882. For $0 < x \leq 1$, the Hurwitz zeta-function is defined as the series

$$\zeta(s;x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad \Re(s) > 1.$$

In 1887, Lerch [14] studied an exponential twist of the Hurwitz zeta-function. For $|z| \leq 1$, $\alpha \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}$, the Lerch zeta-function is defined as

$$\Phi(z;\alpha;s) := \sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^s},$$

which converges for $\Re(s) > 1$ if $z = 1$ and $\Re(s) > 0$ otherwise. The Riemann and Hurwitz zeta-functions are special cases of the Lerch zeta-function.

Moreover, this function generalizes another special function that makes an appearance in the theory of special values of zeta-functions, namely, the polylogarithm. For $|z| \leq 1$, the s-th polylogarithm is defined as

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\,\Phi(z,1,s).$$

This series converges for $s > 1$ when $z = 1$ and $s > 0$ when $|z| \leq 1$ and $z \neq 1$.

Fix a positive integer $q \geq 3$. A Dirichlet character χ modulo q is a group homomorphism, $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$, extended as a completely multiplicative, periodic function on the integers. The L-function associated to χ is defined as

$$L(s;\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

which converges absolutely for $\Re(s) > 1$. It can be shown that (see [16, Section 5] for details) $L(k;\chi) \in \pi^k\,\mathbb{Q}^*$ when k and χ have the same parity, i.e., both are either odd or even. However, when k and χ have the opposite parity, the nature of the values $L(k;\chi)$ is unknown. Since the function χ is periodic, for $\Re(s) > 1$,

$$L(s;\chi) = \frac{1}{q^s} \sum_{a=1}^{q} \chi(a)\,\zeta\left(s;\frac{a}{q}\right).$$

Thus, the Hurwitz zeta-functions are building blocks of the Dirichlet L-series.

That the above functions are inter-related is immediate from the following observations.

$$\zeta\left(k;\frac{1}{2}\right)=\left(2^{k}-1\right)\zeta(k),\quad \mathrm{Li}_{k}(-1)=-\Phi(-1;\,1;\,k)=\left(1-\frac{2}{2^{k}}\right)\zeta(k)$$
$$\Phi\left(-1;\,\frac{1}{2};\,k\right)=2^{k}\,L(k;\,\chi_4), \tag{2}$$

where χ_4 is the non-trivial character modulo 4.

Multi-variable analogs of the above zeta-functions have been studied by various authors. The theory of meromorphic continuation of multiple Hurwitz zeta-function was studied by Akiyama and Ishikawa [1] and by the first author and Kaneenika Sinha [18]. Around the same time, the multiple Hurwitz zeta-functions were also studied in [15]. The *multiple Hurwitz zeta-function*,

$$\widetilde{\zeta}(s_1,\cdots,s_r\,;\,x_1,\cdots,x_r):=\sum_{n_1>n_2>\cdots>n_r\geq 1}\frac{1}{(n_1+x_1)^{s_1}\cdots(n_r+x_r)^{s_r}},$$

converges when $x_i\in(0,\infty)$ and

$$\Re(s_1)>1,\quad \Re(s_1+s_2)>2,\quad \cdots,\quad \Re(s_1+\cdots+s_r)>r. \tag{3}$$

The analytic continuation of these multiple Hurwitz zeta-functions was the center of interest in [18]. However, the arithmetic nature of special values of the multiple Hurwitz zeta-functions has not been studied previously in full generality. In the special case that $x_i=1/2$, the multiple Hurwitz zeta-values are called multiple t-values. A detailed study of these special values in the spirit of multiple zeta-values has been carried out by M. E. Hoffman in [12], who conjectured that the dimension of the $\mathbb{Q}$-vector space generated by the weight k multiple t-values is the kth Fibonacci number. A basis for this vector space was conjectured by B. Saha in [22].

In order to ensure elegance of our formulas, we modify the above definition slightly to include the indices equal to 0 and ensure that $x_i\notin\{0,-1,-2,\cdots\}$. Thus, throughout this paper, we will consider the multiple Hurwitz zeta-function to be

$$\zeta(s_1,\cdots,s_r\,;\,x_1,\cdots,x_r)=\sum_{n_1>n_2>\cdots>n_r\geq 0}\frac{1}{(n_1+x_1)^{s_1}\cdots(n_r+x_r)^{s_r}}. \tag{4}$$

Adopting this convention implies that

$$\zeta(s_1,\cdots,s_r\,;\,1,\cdots,1)=\zeta_r(s_1,\cdots,s_r),$$

the usual multi-zeta function. Note that

$$\zeta(s_1,\cdots,s_r\,;\,x_1,\cdots,x_r)=\widetilde{\zeta}(s_1,\cdots,s_r\,;\,x_1,\cdots,x_r)+\frac{1}{x_r^{s_r}}\,\widetilde{\zeta}(s_1,\cdots,s_{r-1}\,;\,x_1,\cdots,x_{r-1}).$$

In the same vein, the meromorphic continuation of *multiple Lerch zeta-function* was studied by S. Gun and B. Saha in [11]. We will define a multiple Lerch-zeta function as

$$\Phi(z_1,\cdots,z_r;\alpha_1,\cdots,\alpha_r;s_1,\cdots,s_r):=\sum_{n_1>n_2>\cdots>n_r\geq 0}\frac{z_1^{n_1}\ \cdots\ z_r^{n_r}}{(n_1+\alpha_1)^{s_1}\ \cdots\ (n_r+\alpha_r)^{s_r}},\quad (5)$$

for $\alpha_i\in(0,\infty)$, $s_1,\cdots\ s_r$ satisfying (3) and z_i such that $\prod_{i=1}^{j}|z_i|\leq 1$ for all $1\leq j\leq r$ (for a proof, see [21, Section 2.2]). Note that we include the term corresponding to $n_r=0$ to ensure clean identities, so our definition includes more terms than that used in [11]. It is then easy to see that $\Phi(1,\cdots,1;\alpha_1,\cdots,\alpha_r;s_1,\cdots,s_r)=\zeta(s_1,\cdots,s_r;\alpha_1,\cdots,\alpha_r)$ and the multiple polylogarithms (see [23]),

$$\begin{aligned}&\mathrm{Li}_{s_1,\cdots,s_r}(z_1,\cdots,z_r)\\&\quad=\sum_{n_1>\cdots>n_r\geq 1}\frac{{z_1}^{n_1}\ \cdots\ {z_r}^{n_r}}{{n_1}^{s_1}\ \cdots\ {n_r}^{s_r}}=z_1\cdots z_r\,\Phi(z_1,\cdots,z_r;1,\cdots,1;s_1,\cdots,s_r).\end{aligned}$$

When $z_i=\pm1$, the corresponding multiple polylogarithms are called *alternating Euler-Zagier sums*, which have been extensively studied in the literature (for example, see [5] and [6]). In order to maintain consistency of notation for depth 2 sums, we use the following convention.

$$\zeta_2(\overline{r},s):=\sum_{m=2}^{\infty}\frac{(-1)^m}{m^r}\sum_{n=1}^{m-1}\frac{1}{n^s},\qquad \zeta_2(\overline{r},\overline{s}):=\sum_{m=2}^{\infty}\frac{(-1)^m}{m^r}\sum_{n=1}^{m-1}\frac{(-1)^n}{n^s}.\quad (6)$$

Multiple Dirichlet L-functions were considered by Akiyama and Ishikawa [1] and also appear in the work of Goncharov [10]. Let $\chi_1,\chi_2,\cdots,\chi_r$ be primitive Dirichlet characters of the same modulus q. Then the associated multiple Dirichlet L-function is defined as

$$L(s_1,\cdots,s_r\,;\,\chi_1,\cdots,\chi_r):=\sum_{n_1>n_2>\cdots>n_r\geq 1}\frac{\chi_1(n_1)\ \cdots\chi_r(n_r)}{{n_1}^{s_1}\ \cdots\ {n_r}^{s_r}}.$$

The convolution of values of Dirichlet L-functions is considerably more involved. The multiple L-functions that appear are more general than the multiple Dirichlet L-functions above, namely, if $f_1,\ f_2,\ \cdots,\ f_r$ are functions on the integers, that are periodic modulo the same modulus q, then define the multiple L-function,

$$L(s_1,\cdots,s_r\,;\,f_1,\cdots,f_r):=\sum_{n_1>n_2>\cdots>n_r\geq 1}\frac{f_1(n_1)\ \cdots f_r(n_r)}{{n_1}^{s_1}\ \cdots\ {n_r}^{s_r}},\quad (7)$$

which converges for $s_1,\ \cdots,\ s_r$ satisfying (3).

Another multiple Dirichlet series allied to (7) are *quasi-multiple L-functions*, where the strict inequality in (7) is replaced by a possible equality. Analogously, one can also define the *quasi-multiple Hurwitz zeta-functions*. They are a special case of a general multiple zeta-function introduced by Matsumoto [15] and are discussed in [18, pg. 13]. In particular, the quasi-multiple L-functions that appear in our work will be

$$L^*(s_1, \cdots, s_r\,;\, f_1, \cdots, f_r) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_r \geq 1} \frac{f_1(n_1)\, \cdots f_r(n_r)}{{n_1}^{s_1}\, \cdots\, {n_r}^{s_r}}, \tag{8}$$

for s_i satisfying (3). These can be related to the multiple L-functions via a simple inclusion-exclusion principle.

The identities we obtain naturally also include the digamma function $\psi(x)$, which is defined as the logarithmic derivative of the gamma function. Owing to the infinite product of $\Gamma(z)$, one obtains a series expansion for $\psi(z)$, namely

$$\psi(z) := -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n} \right), \quad z \neq 0,\, -1,\, -2, \cdots .$$

Here γ denotes the Euler-Mascheroni constant. Thus, $\psi(1) = -\gamma$.

We first prove convolution sum identities for Lerch zeta-functions where the argument $z \neq 1$. From these identities, we derive the analogous expressions for Hurwitz zeta-functions by careful analysis of the effect of taking limit as $z \to 1^-$. Thus, our main theorem is

Theorem 1.1. *Let $k \geq 3$ be a positive integer, $\alpha \in \mathbb{C} \backslash \{0, -1, -2, \cdots\}$ and $z_1,\, z_2 \in \mathbb{C}$ with $0 < |z_1|, |z_2| \leq 1$, $z_1 \neq 1$ and $z_2 \neq 1$. Then*

$$\begin{aligned} &\sum_{j=1}^{k-1} \Phi(z_1; \alpha; j)\, \Phi(z_2; \alpha; k-j) \\ &= (k-1)\, \Phi(z_1\, z_2; \alpha; k) - \Big(\log(1-z_1) + \log(1-z_2) \Big)\, \Phi(z_1\, z_2; \alpha; k-1) \\ &\qquad - z_2^{-1}\, \Phi(z_1\, z_2,\; {z_2}^{-1}\,;\, \alpha, 1\,;\, k-1, 1) - z_1^{-1}\, \Phi(z_1\, z_2,\; {z_1}^{-1}\,;\, \alpha, 1\,;\, k-1, 1), \end{aligned}$$

where the last two terms are multiple Lerch zeta-functions as defined in (5).

As an easy corollary of this theorem using (2), we deduce the following identity for values of $L(s;\, \chi_4)$, where χ_4 denotes the non-trivial Dirichlet character modulo 4.

Corollary 1.1. *Let $k \geq 3$ be a positive integer and χ_4 denote the non-trivial Dirichlet character modulo* 4*. Then,*

$$\sum_{j=1}^{k-1} L(j;\, \chi_4)\, L(k-j;\, \chi_4) = (k-1)\left(1 - \frac{1}{2^k}\right)\zeta(k) - \log 2\left(1 - \frac{1}{2^{k-1}}\right)\zeta(k-1) + 2\,\Phi\left(1; -1;\, \frac{1}{2}, 1;\, k-1, 1\right)$$

Now, we fix z_2 and take the limit as $z_1 \to 1^-$ in Theorem 1.1. This gives the following theorem for values of the Lerch and the Hurwitz zeta-function.

Theorem 1.2. *Let $k \geq 3$ be a positive integer, $0 < |z| \leq 1$ and $z \neq 1$ and $\alpha \in \mathbb{C}\backslash\{0, -1, -2, \cdots\}$. Then*

$$\sum_{j=2}^{k-1} \zeta(j;\alpha)\, \Phi(z;\, \alpha;\, k-j) = (k-1)\,\Phi(z;\, \alpha;\, k) + \Big(\psi(\alpha) + \gamma - \log(1-z)\Big)\Phi(z;\, \alpha;\, k-1) - z^{-1}\,\Phi(z, z^{-1};\, \alpha, 1;\, k-1, 1) - \Phi(z, 1;\, \alpha, 1;\, k-1, 1)$$

where the last two terms are multiple Lerch zeta-functions as defined in (5).

Similarly, on taking the limit as $z \to 1^-$ in the above theorem, we get

Corollary 1.2. *Let $k \geq 4$ be a positive integer and $\alpha \in \mathbb{C}\backslash\{0, -1, -2, \cdots\}$. Then*

$$\sum_{j=2}^{k-2} \zeta(j;\alpha)\, \zeta(k-j;\alpha) = (k-1)\,\zeta(k;\alpha) + 2\Big(\psi(\alpha) + \gamma\Big)\zeta(k-1;\alpha) - 2\,\zeta(k-1, 1;\, \alpha, 1).$$

In particular, when $k = 3$ and $\alpha = 1$, the above corollary implies $\zeta(3) = \zeta_2(2, 1)$. Moreover, we can take $z = -1$ and $\alpha = 1$ in Theorem 1.2 to obtain

Corollary 1.3. *For any integer $k \geq 4$,*

$$\sum_{j=2}^{k-2} \left(1 - \frac{2}{2^{k-j}}\right)\zeta(j)\,\zeta(k-j) = (k-1)\,\zeta(k) - \left(1 - \frac{2}{2^{k-1}}\right)(\log 2)\,\zeta(k-1) + \zeta_2(\overline{k-1}, \overline{1}) - \zeta_2(\overline{k-1}, 1),$$

where the last terms are alternating Euler-Zagier sums, defined in (6).

This identity has been discussed in detail in [5, Section 4, (15)].

In [17], the first author emphasized that $\zeta(2) = \pi^2/6$ itself implies the more general fact that $\zeta(2k) \in \pi^{2k}\mathbb{Q}$, simply because of the neat identity

$$\left(k + \frac{1}{2}\right)\zeta(2k) = \sum_{j=1}^{k-1} \zeta(2j)\,\zeta(2k-2j). \tag{9}$$

This is another relation among the zeta-values that Euler was familiar with. It is natural to inquire if convolutions of values of the Lerch zeta-functions at *even* positive integers lead to new identities, different from the ones described previously. Towards this question, we prove the following.

Theorem 1.3. *Let $k \geq 2$ be a positive integer and complex numbers z_1 and z_2 such that $0 < |z_1| = |z_2| \leq 1$ and $z_1,\ z_2 \neq 1$. Then*

$$\begin{aligned}
&\sum_{j=1}^{k-1} \Phi(z_1;\, \alpha; 2j)\, \Phi(z_2;\, \alpha; 2k-2j) \\
&= \left(k - \frac{1}{2}\right) \Phi(z_1 z_2;\, \alpha;\, 2k) - \frac{1}{2}\Phi(z_1 z_2;\, \alpha;\, 2k-1) \Big(\log(1-z_1) + \log(1-z_2) \Big) \\
&- \frac{1}{2}\Phi(z_1^{-1} z_2;\, \alpha;\, 2k-1)\, \Phi(z_1;\, 2\alpha;\, 1) - \frac{1}{2}\Phi(z_1 z_2^{-1};\, \alpha;\, 2k-1)\, \Phi(z_2;\, 2\alpha;\, 1) \\
&- \frac{z_2^{-1}}{2}\Phi(z_1 z_2, z_2^{-1};\, \alpha, 1;\, 2k-1, 1) + \frac{1}{2}\Phi(z_1 z_2^{-1}, z_2;\, \alpha, 2\alpha;\, 2k-1, 1) \\
&- \frac{z_1^{-1}}{2}\Phi(z_1 z_2, z_1^{-1};\, \alpha, 1;\, 2k-1, 1) + \frac{1}{2}\Phi(z_1^{-1} z_2, z_1;\, \alpha, 2\alpha;\, 2k-1, 1).
\end{aligned}$$

Taking $z_1 = z_2 = -1$ and $\alpha = 1/2$ in the above theorem, we deduce that

Corollary 1.4. *Let χ_4 denote the non-trivial character modulo 4 and $k \geq 2$ be an integer. Then*

$$\begin{aligned}
&\sum_{j=1}^{k-1} L(2j;\, \chi_4)\, L(2k-2j;\, \chi_4) \\
&\quad = \left(1 - \frac{1}{2^{2k}}\right) \left(k - \frac{1}{2}\right) \zeta(2k) - \left(1 - \frac{1}{2^{2k-1}}\right) (\log 2)\, \zeta(2k-1) \\
&\quad\quad + \frac{1}{2^{2k-1}}\, \Phi\left(1, -1;\, \frac{1}{2}, 1;\, 2k-1, 1\right).
\end{aligned}$$

On the other hand, considering the equation in Theorem 1.3 at $z = z_1 = z_2$, $|z| < 1$ and taking the limit as $z \to 1^-$, we deduce the following identity for the values of Hurwitz zeta-functions.

Corollary 1.5. *Let $k \geq 2$ be an integer and $\alpha \in \mathbb{C} \backslash \{0, -1, -2, \cdots\}$. Then*

$$\begin{aligned}
\sum_{j=1}^{k-1} \zeta(2j; \alpha)\, \zeta(2k-2j; \alpha) = &\left(k - \frac{1}{2}\right) \zeta(2k; \alpha) + \Big(\psi(2\alpha) + \gamma\Big)\, \zeta(2k-1; \alpha) \\
&- \zeta(2k-1, 1\,;\, \alpha, 1) + \zeta(2k-1, 1\,;\, \alpha, 2\alpha),
\end{aligned}$$

where the last two terms are multiple Hurwitz zeta-functions as in (4)*.*

For Dirichlet L-functions associated to *primitive* Dirichlet characters, we have the following theorem.

Theorem 1.4. *Let* χ_1*,* χ_2 *be primitive Dirichlet characters modulo* $q \geq 3$ *and* $k \geq 3$ *be an integer. For a primitive Dirichlet character* $\chi \bmod q$*, define two allied periodic functions mod* q *by*

$$T_{q,a}(n) := \zeta_q^{an} \text{ and } T_{q,a,\chi}(n) := \chi(n)\, \zeta_q^{an},$$

for any $a \in \mathbb{Z}$*. Also, let* $\tau(\chi) = \sum_{a=1}^{q} \chi(a)\zeta_q^a$ *be the Gauss sum associated to* χ*. Then,*

$$\begin{aligned}
&\sum_{j=1}^{k-1} L(j;\chi_1)\, L(k-j\,;\chi_2) \\
&= (k-1)L(k\,;\,\chi_1\chi_2) - \frac{1}{\tau(\overline{\chi_2})} \sum_{a=1}^{q-1} \Big(\overline{\chi_2}(a)\, \log(1-\zeta_q^a)\, L(k-1; T_{q,a,\chi_1})\Big) \\
&\quad - \frac{1}{\tau(\overline{\chi_1})} \sum_{a=1}^{q-1} \Big(\overline{\chi_1}(a)\, \log(1-\zeta_q^a)\, L(k-1; T_{q,a,\chi_2})\Big) \\
&\quad - \frac{1}{\tau(\overline{\chi_2})} \sum_{a=1}^{q-1} \overline{\chi_2}(a)\, \zeta_q^{-a}\, L^*(k-1,1\,;\, T_{q,a,\chi_1}, T_{q,-a}) \\
&\quad\quad - \frac{1}{\tau(\overline{\chi_1})} \sum_{a=1}^{q-1} \overline{\chi_1}(a)\, \zeta_q^{-a}\, L^*(k-1,1\,;\, T_{q,a\chi_2}, T_{q,-a})
\end{aligned}$$

where the last terms involve multiple L^**-function as defined in* (8)*.*

This is a generalization of Corollary 1.1 and gives an idea of the various combinations of special values involved. It is not difficult to see that for $r, s \in \mathbb{N}$ with $1 < r$ and $1 \leq s$, and a primitive character $\chi \bmod q$,

$$\begin{aligned}
L^*(r,s\,;\,T_{q,a,\chi}, T_{q,-a}) &= \sum_{m=1}^{\infty} \frac{\chi(m)\,\zeta_q^{am}}{m^r} \cdot \frac{\zeta_q^{-am}}{m^s} + \sum_{m=1}^{\infty} \frac{\chi(m)\,\zeta_q^{am}}{m^r} \sum_{j=1}^{m-1} \frac{\zeta_q^{aj}}{j^s} \\
&= L(r+s,\chi) + L(r,s\,;\,T_{q,a,\chi}, T_{q,a}).
\end{aligned}$$

Using this in the above theorem, together with the fact that for a primitive Dirichlet character $\chi \bmod q$,

$$\sum_{a=1}^{q} \chi(a)\, \zeta_q^{-a} = \chi(-1)\tau(\chi),$$

simplifies the identity as follows:

$$\sum_{j=1}^{k-1} L(j\,;\chi_1)\,L(k-j\,;\chi_2) = (k-1)L(k\,;\chi_1\chi_2) - \chi_2(-1)\,L(k\,;\chi_1) - \chi_1(-1)\,L(k\,;\chi_2)$$
$$- \frac{1}{\tau(\overline{\chi_2})}\sum_{a=1}^{q-1}\Big(\overline{\chi_2}(a)\,\log(1-\zeta_q^a)\,L(k-1\,;T_{q,a,\chi_1})\Big)$$
$$- \frac{1}{\tau(\overline{\chi_1})}\sum_{a=1}^{q-1}\Big(\overline{\chi_1}(a)\,\log(1-\zeta_q^a)\,L(k-1\,;T_{q,a,\chi_2})\Big)$$
$$- \frac{1}{\tau(\overline{\chi_2})}\sum_{a=1}^{q-1}\overline{\chi_2}(a)\,\zeta_q^{-a}\,L(k-1,1\,;T_{q,a,\chi_1},T_{q,-a})$$
$$- \frac{1}{\tau(\overline{\chi_1})}\sum_{a=1}^{q-1}\overline{\chi_1}(a)\,\zeta_q^{-a}\,L(k-1,1\,;T_{q,a\chi_2},T_{q,-a}).$$

Remark. *It is evident from the above theorem that in order to study the special values of Dirichlet L-function, one must investigate the allied functions*

$$\mathrm{Li}_k(z;\chi) := \sum_{n=1}^{\infty}\frac{\chi(n)\,z^n}{n^k}, \quad |z|\le 1,$$

for a Dirichlet character χ modulo q. By the duality between Dirichlet characters and arithmetic progressions, these sums will be naturally related to the function,

$$\sum_{\substack{n=1,\\ n\equiv a \bmod q}}^{\infty}\frac{z^n}{n^k},$$

which is essentially the Lerch zeta-function $\Phi(z;\,a/q;\,k)$.

2. Proof of main theorems

The method of summation in evaluating the sums that arise in our theorems is based on the same general principle, which we outline below. Fix a positive integer $r\ge 1$ and a positive integer $k\ge 3$. For complex numbers z_1, z_2 with $|z_i|\le 1$ and $z_i\ne 1$, $i=1,\,2$, $\alpha\in\mathbb{C}\backslash\{0,-1,-2,\cdots\}$, let the r-level convolution be defined as

$$\mathcal{C}_r(z_1,z_2\,;\alpha) := \sum_{j=1}^{k-1}\Phi(z_1;\,\alpha;\,rj)\;\Phi(z_2\,;\alpha\,;r(k-j)).$$

Then we expand the right hand side as

$$\begin{aligned}\mathcal{C}_r(z_1, z_2\,;\alpha) &= \sum_{j=1}^{k-1} \sum_{n,m=0}^{\infty} \frac{z_1^m}{(m+\alpha)^{rj}} \cdot \frac{z_2^n}{(n+\alpha)^{r(k-j)}} \\ &= (k-1)\sum_{n=0}^{\infty} \frac{(z_1\,z_2)^n}{(n+\alpha)^{rk}} + \sum_{\substack{n,m=0,\\ n\neq m}}^{\infty} z_1^m\, z_2^n \sum_{j=1}^{k-1} \frac{1}{(m+\alpha)^{rj}} \cdot \frac{1}{(n+\alpha)^{r(k-j)}} \\ &= (k-1)\Phi(z_1\,z_2;\,\alpha;\,rk) + \sum_{\substack{n,m=0,\\ n\neq m}}^{\infty} \frac{z_1^m\, z_2^n}{(n+\alpha)^{rk}} \sum_{j=1}^{k-1} \left(\frac{n+\alpha}{m+\alpha}\right)^{rj}.\end{aligned}$$

Now, the inner sum can be evaluated as a geometric series,

$$\begin{aligned}&\frac{1}{(n+\alpha)^{rk}} \sum_{j=1}^{k-1} \left(\frac{n+\alpha}{m+\alpha}\right)^{rj} \\ &= \frac{1}{(m+\alpha)^{r(k-1)}(n+\alpha)^{r(k-1)}} \left(\frac{(n+\alpha)^{r(k-1)} - (m+\alpha)^{r(k-1)}}{(n+\alpha)^r - (m+\alpha)^r}\right).\end{aligned}$$

Thus we get

$$\begin{aligned}\mathcal{C}_r(z_1, z_2\,;\alpha) = (k-1)\Phi(z_1\,z_2;\,\alpha;\,rk) &+ \sum_{m=0}^{\infty} \frac{z_1^m}{(m+\alpha)^{r(k-1)}} \sum_{\substack{n=0,\\ n\neq m}}^{\infty} \frac{z_2^n}{(n+\alpha)^r - (m+\alpha)^r} \\ &+ \sum_{n=0}^{\infty} \frac{z_2^n}{(n+\alpha)^{r(k-1)}} \sum_{\substack{m=0,\\ m\neq n}}^{\infty} \frac{z_1^m}{(m+\alpha)^r - (n+\alpha)^r}.\end{aligned} \tag{10}$$

Therefore, the above computations naturally lead one into the study of the auxiliary sums

$$\mathcal{S}_{r,m}(z,\alpha) := \sum_{\substack{n=0,\\ n\neq m}}^{\infty} \frac{z^n}{(n+\alpha)^r - (m+\alpha)^r}, \tag{11}$$

where $z \in \mathbb{C}$ with $|z| \leq 1$, $z \neq 1$ and $\alpha \in \mathbb{C}\backslash\{0, -1, -2, \cdots\}$. Our focus will mostly be on the cases $r = 1$ and $r = 2$. We will also later indicate the difficulties in obtaining neat formulas for $r \geq 3$ using the above method.

2.1. Evaluation of auxiliary sums

For a non-negative integer m, let H_m denote the mth harmonic number, that is,

$$H_m := \sum_{j=1}^{m} \frac{1}{j},$$

if m is a strictly positive integer and $H_0 := 0$. It is not difficult to see that

$$H_N = \log N + \gamma + O\left(\frac{1}{N}\right). \tag{12}$$

Analogous to the harmonic numbers, we introduce the *generalized harmonic numbers*, defined as

$$H_k(z,\alpha) := \begin{cases} \sum_{j=0}^{k} \frac{z^j}{(j+\alpha)}, & \text{if } k \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

for $|z| \leq 1$ and $\alpha \in \mathbb{C}\backslash\{0,\, -1,\, -2,\, \cdots\}$. Let $H_k(\alpha) := H_k(1,\alpha)$, so that $H_m = H_{m-1}(1)$. The asymptotic behavior of these numbers is evident from the following lemma.

Lemma 2.1. *Let $|z| \leq 1$, $\alpha \in \mathbb{C}\backslash\{0, -1, -2, \cdots\}$ and k be a non-negative integer. Then,*

$$H_k(\alpha) = \log k - \psi(\alpha) + O\left(\frac{1}{k}\right),$$

as $k \to \infty$. If $z \neq 1$, then

$$\lim_{k\to\infty} H_k(z,\alpha) = \Phi(z;\alpha;1).$$

Proof. When $z \neq 1$, the series

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)}$$

can be shown to converge using Abel's theorem. When $z = 1$, the asymptotics follow from (12) and the series representation of the digamma function,

$$\frac{1}{\alpha} + \sum_{n=1}^{\infty} \left(\frac{1}{n+\alpha} - \frac{1}{n}\right) = -\gamma - \psi(\alpha),$$

for $\alpha \neq 0,\, -1,\, -2,\, \cdots$. $\square$

Also note that for $z \neq 1$ and $0 < |z| \leq 1$,

$$\Phi(z;1;1) = z^{-1}\,\log(1-z).$$

With this background, the auxiliary sum in the case $r = 1$ can be expressed as follows.

Lemma 2.2. *Let $z \in \mathbb{C}$ with $|z| \leq 1$, $z \neq 1$ and m be a non-negative integer. Then*

$$\mathcal{S}_m(z) := \sum_{\substack{n=0,\\ n\neq m}}^{\infty} \frac{z^n}{n-m} = -z^m\,\log(1-z) - z^{m-1}\,H_{m-1}(z^{-1},1),$$

where the last term involves a generalized harmonic number.

Proof. Separating the sum into two parts gives

$$\begin{aligned}\mathcal{S}_m(z) &= \sum_{m<n} \frac{z^n}{n-m} + \sum_{0\leq n<m} \frac{z^n}{n-m} = \sum_{j=1}^{\infty} \frac{z^{j+m}}{j} - \sum_{j=1}^{m} \frac{z^{m-j}}{j} \\ &= -z^m\,\log(1-z) - \sum_{j=0}^{m-1} \frac{z^{(m-j-1)}}{j+1} \\ &= -z^m\,\log(1-z) - z^{m-1}\,H_{m-1}(z^{-1},1). \quad \square\end{aligned}$$

In the case $r = 2$, we have

Lemma 2.3. *Let $z \in \mathbb{C}$ with $|z| \leq 1$, $z \neq 1$ and m be a non-negative integer. Then*

$$\begin{aligned}\mathcal{S}_{2,m}(z,\alpha) &:= \sum_{\substack{n=0,\\ n\neq m}}^{\infty} \frac{z^n}{(n+\alpha)^2 - (m+\alpha)^2} \\ &= \frac{z^m}{4(m+\alpha)^2} - \frac{1}{2(m+\alpha)}\left\{z^m\,\log(1-z) + z^{m-1}\,H_{m-1}(z^{-1},1)\right\} \\ &- \frac{1}{2(m+\alpha)}\left\{z^{-m}\Phi(z;2\alpha;1) - z^{-m}H_{m-1}(z,2\alpha)\right\}\end{aligned}$$

Proof. By partial fractions, we know that

$$\frac{1}{(n+\alpha)^2 - (m+\alpha)^2} = \frac{1}{2(m+\alpha)}\left(\frac{1}{n-m} - \frac{1}{n+m+2\alpha}\right).$$

The required sum can then be re-written as

$$\mathcal{S}_{2,m}(z,\alpha) = \frac{1}{2(m+\alpha)}\,\mathcal{S}_m(z) + \frac{z^m}{4(m+\alpha)^2} - \frac{1}{2(m+\alpha)}\sum_{n=0}^{\infty} \frac{z^n}{n+m+2\alpha}.$$

The last sum can be determined as follows

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{z^n}{n+m+2\alpha} &= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{z^n}{n+m+2\alpha} = \lim_{N\to\infty} \sum_{j=m}^{N+m} \frac{z^{j-m}}{j+2\alpha} \\ &= z^{-m} \lim_{N\to\infty} \left(\sum_{j=0}^{N+m} \frac{z^j}{j+2\alpha} - \sum_{j=0}^{m-1} \frac{z^j}{j+2\alpha} \right) \\ &= z^{-m} \lim_{N\to\infty} \Big(H_{N+m}(z, 2\alpha) - H_{m-1}(z, 2\alpha) \Big) \\ &= z^{-m}\, \Phi(z;\, 2\alpha;\, 1) - z^{-m}\, H_{m-1}(z, 2\alpha).\end{aligned}$$

The evaluation of $\mathcal{S}_{2,m}(z,\alpha)$ is now evident from Lemma 2.2. □

Remark. *Using partial fractions, it is possible to obtain that for $|z| \le 1$ and $z \ne 1$,*

$$\mathcal{S}_{r,m}(z,\alpha) = \frac{1}{r(m+\alpha)^{r-1}} \sum_{k=1}^{r} \zeta_r^k \sum_{\substack{n=0,\\ n\ne m}}^{\infty} \frac{z^n}{(n+\alpha) - \zeta_r^k\,(m+\alpha)},$$

where ζ_r denotes a primitive r-th root of unity. However, for $r \ge 3$, since the roots of unity are complex, the evaluation of inner sums is not immediate. Moreover, when $r = 2^s$, the sums arising above have the special form

$$\sum_{n=0}^{\infty} \frac{z^n}{(n+\alpha)^{2^t} + (m+\alpha)^{2^t}}, \qquad 0 \le t \le s-1.$$

When $t = 1$, $\alpha = 1$ and $z = 1$, the resulting sum can be evaluated using [19, Theorem 2]. This highlights the importance of the study of the series

$$\sum_{n=0}^{\infty} \frac{A(n)}{B(n)}\, z^n,$$

where $A(X)$ and $B(X)$ are suitable polynomials with rational coefficients and $|z| \le 1$.

2.2. Proof of Theorem 1.1

Let $r = 1$ in (10). Then, we have

$$\begin{aligned}\mathcal{C}_1(z_1, z_2\,; \alpha) &= (k-1)\Phi(z_1, z_2;\, \alpha;\, k) \\ &\quad + \sum_{m=0}^{\infty} \left(\frac{z_1^m}{(m+\alpha)^{k-1}} \cdot \mathcal{S}_m(z_2) \right) + \sum_{n=0}^{\infty} \left(\frac{z_2^n}{(n+\alpha)^{k-1}} \cdot \mathcal{S}_n(z_1) \right).\end{aligned}$$

The above two sums can be simplified using the expressions for $\mathcal{S}_m(z)$ obtained in Lemma 2.2. For instance,

$$\begin{aligned}&\sum_{m=0}^{\infty}\left(\frac{z_1^m}{(m+\alpha)^{k-1}}\cdot\mathcal{S}_m(z_2)\right)\\&=-\log(1-z_2)\sum_{m=0}^{\infty}\frac{(z_1\,z_2)^m}{(m+\alpha)^{k-1}}-z_2^{-1}\sum_{m=0}^{\infty}\frac{(z_1\,z_2)^m}{(m+\alpha)^{k-1}}\,H_{m-1}(z_2^{-1},1)\\&=-\log(1-z_2)\,\Phi(z_1z_2;\,\alpha;\,k-1)-z_2^{-1}\,\Phi(z_1z_2,z_2^{-1};\,\alpha,1;\,k-1,1),\end{aligned}$$

where the last term is a multiple Lerch zeta-function as defined in (5). The remaining sum can also be evaluated similarly. This proves Theorem 1.1.

2.3. Proof of Theorem 1.2

The idea of the proof is that for a fixed integer $k > 1$, $\alpha \in \mathbb{C}\backslash\{0,-1,-2,\cdots\}$, the function

$$\Phi(z;\,\alpha;\,k) := \sum_{j=0}^{\infty}\frac{z^j}{(j+\alpha)^k}$$

is a continuous function of z on the disk $\{z \in \mathbb{C} \,:\, |z| \le 1\}$. However, when $k = 1$, the limit $\lim_{z\to 1^-}\Phi(z;\alpha;1)$ does not exist because of the pole of the Hurwitz zeta-function at $s = 1$. Therefore, we re-write the identity obtained in Theorem 1.1 as follows.

$$\begin{aligned}&\sum_{j=2}^{k-1}\Phi(z_1;\alpha;j)\,\Phi(z_2;\alpha;k-j)\\&=(k-1)\,\Phi(z_1\,z_2;\alpha;k)-\Big(\log(1-z_2)\Big)\,\Phi(z_1\,z_2;\alpha;k-1)\\&\quad-z_2^{-1}\,\Phi(z_1\,z_2,\;{z_2}^{-1};\alpha,1\,;\,k-1,1)-z_1^{-1}\,\Phi(z_1\,z_2,\;{z_1}^{-1};\alpha,1\,;\,k-1,1)\\&\quad-\Big\{\log(1-z_1)\,\Phi(z_1z_2;\,\alpha;\,k-1)+\Phi(z_1;\,\alpha;\,1)\,\Phi(z_2;\,\alpha;\,k-1)\Big\}\end{aligned}$$

For a fixed $z_2 \ne 1$, we would like to consider the limit $z_1 \to 1^-$. That is, we let $z_1 \in \mathbb{R}$ with $0 < z_1 < 1$ and then take the limit as $z_1 \to 1$. For all the terms in the above identity except the ones in curly brackets, the limit as $z_1 \to 1^-$ exists. Hence, we concentrate on just those two terms. Observe that

$$\begin{aligned}&\lim_{z_1\to 1^-}\log(1-z_1)\,\Phi(z_1z_2;\,\alpha;\,k-1)+\Phi(z_1;\,\alpha;\,1)\,\Phi(z_2;\,\alpha;\,k-1)\\&=\lim_{z_1\to 1^-}\lim_{N\to\infty}\left[\Phi(z_2;\,\alpha;\,k-1)\left(\sum_{j=0}^{N}\frac{z_1^j}{j+\alpha}\right)-\Phi(z_1z_2;\,\alpha;\,k-1)\left(\sum_{j=1}^{N}\frac{z_1^j}{j}\right)\right].\end{aligned}$$

Now, note that for a fixed z_2, $\Phi(z\, z_2;\, \alpha;\, k-1)$ is a continuous function of z. Thus, we have that the limit equals

$$\Phi(z_2;\, \alpha;\, k-1) \lim_{z_1 \to 1^-} \lim_{N \to \infty} \left[\left(\sum_{j=0}^{N} \frac{z_1^j}{j+\alpha} - \sum_{j=1}^{N} \frac{z_1^j}{j} \right) \right]$$

$$= \Phi(z_2;\, \alpha;\, k-1) \lim_{z_1 \to 1^-} \lim_{N \to \infty} \left[\frac{1}{\alpha} + \sum_{j=1}^{N} \frac{\alpha\, z_1^j}{j(j+\alpha)} \right].$$

Since $\sum_{j=1}^{\infty} 1/(j\,(j+\alpha)) < \infty$, one can interchange the limits thanks to the dominated convergence theorem, to get that the above limit is in fact

$$\Phi(z_2;\, \alpha;\, k-1) \left[\frac{1}{\alpha} + \lim_{N \to \infty} \sum_{j=1}^{N} \left(\frac{1}{j+\alpha} - \frac{1}{j} \right) \right] = -\Big(\psi(\alpha) + \gamma \Big) \Phi(z_2;\, \alpha;\, k-1).$$

This implies Theorem 1.2.

2.4. Proof of Theorem 1.3

We take $r = 2$ in (10). Therefore, we have

$$\mathcal{C}_2(z_1, z_2;\, \alpha) = (k-1)\, \Phi(z_1 z_2;\, \alpha;\, 2k) + \sum_{m=0}^{\infty} \frac{z_1^m}{(m+\alpha)^{2(k-1)}}\, \mathcal{S}_{2,m}(z_2, \alpha)$$
$$+ \sum_{n=0}^{\infty} \frac{z_2^n}{(n+\alpha)^{2(k-1)}}\, \mathcal{S}_{2,n}(z_1, \alpha).$$

Using the evaluation of $\mathcal{S}_{2,m}(z, \alpha)$ from Lemma 2.3, we get

$$\sum_{m=0}^{\infty} \frac{z_1^m}{(m+\alpha)^{2(k-1)}}\, \mathcal{S}_{2,m}(z_2, \alpha)$$
$$= \frac{1}{4} \Phi(z_1 z_2;\, \alpha; 2k) - \frac{1}{2} \log(1-z_2) \Phi(z_1 z_2;\, \alpha; 2k-1)$$
$$- \frac{1}{2} \Phi(z_2;\, 2\alpha;\, 1)\, \Phi(z_1 z_2^{-1};\, \alpha;\, 2k-1)$$
$$- \frac{z_2^{-1}}{2} \Phi(z_1 z_2, z_2^{-1};\, \alpha, 1;\, 2k-1, 1) + \frac{1}{2} \Phi(z_1 z_2^{-1},\, z_2;\, \alpha, 2\alpha;\, 2k-1, 1),$$

where the last two terms are multiple Lerch zeta-functions. The theorem now follows since the remaining sum can be computed by symmetry.

2.5. Dirichlet L-functions: Proof of Theorem 1.4

Recall that for a primitive Dirichlet character χ,

$$\sum_{a=1}^{q} \overline{\chi}(a)\, \zeta_q^{an} = \chi(n)\, \tau(\overline{\chi}), \tag{13}$$

where ζ_q is a primitive q-th root of unity and $\tau(\chi) = \sum_{a=1}^{q} \chi(a)\zeta_q^a$ is the Gauss sum associated to χ. Since χ is primitive, $\tau(\chi) \neq 0$. Thus, we have the following lemma.

Lemma 2.4. *Let χ be a primitive Dirichlet character mod q and m be a fixed positive integer. Then,*

$$\sum_{\substack{n=1,\\ n\neq m}}^{\infty} \frac{\chi(n)}{n-m} = -\ \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \left(\overline{\chi}(a)\, \zeta_q^{am}\ \log(1-\zeta_q^a)\right)$$
$$- \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \left(\overline{\chi}(a)\, \zeta_q^{a(m-1)}\, H_{m-1}(\zeta_q^{-am}, 1)\right).$$

Proof. Substituting the value of $\chi(n)$ from (13), we have

$$\begin{aligned}
\sum_{\substack{n=1,\\ n\neq m}}^{\infty} \frac{\chi(n)}{n-m} &= \frac{1}{\tau(\overline{\chi})} \sum_{\substack{n=1,\\ n\neq m}}^{\infty} \frac{1}{n-m} \sum_{a=1}^{q} \overline{\chi}(a)\zeta_q^{an} \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \sum_{\substack{n=1,\\ n\neq m}}^{\infty} \frac{\zeta_q^{an}}{n-m} \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \left(\mathcal{S}_m(\zeta_q^a) + \frac{1}{m}\right) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)\, \mathcal{S}_m(\zeta_q^a) + \frac{1}{m\, \tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) \\
&= \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)\, \mathcal{S}_m(\zeta_q^a).
\end{aligned}$$

The value of $\mathcal{S}_m(\zeta_q^a)$ can be calculated from Lemma 2.2. This proves the lemma. □

Applying the above lemma, one can prove Theorem 1.4 as follows. For simplicity of notation, let

$$C_k(\chi_1, \chi_2) := \sum_{j=1}^{k-1} L(j; \chi_1)\, L(k-j; \chi_2).$$

Using the definition of the Dirichlet L-functions, we have

$$
\begin{aligned}
C_k(\chi_1,\chi_2) &= \sum_{m,n=1}^{\infty}\sum_{j=1}^{k-1}\frac{\chi_1(m)}{m^j}\cdot\frac{\chi_2(n)}{n^{k-j}} \\
&= (k-1)\sum_{m=1}^{\infty}\frac{(\chi_1\,\chi_2)(m)}{m^k} + \sum_{\substack{m,n=1,\\ m\neq n}}^{\infty}\sum_{j=1}^{k-1}\frac{\chi_1(m)}{m^j}\cdot\frac{\chi_2(n)}{n^{k-j}} \\
&= (k-1)L(k\,;\,\chi_1\chi_2) + \sum_{\substack{m,n=1,\\ m\neq 1}}^{\infty}\frac{\chi_1(m)\,\chi_2(n)}{n^k}\sum_{j=1}^{k-1}\left(\frac{n}{m}\right)^j.
\end{aligned}
$$

Since $m \neq n$ in the second sum, the inner sum can be simplified as a geometric sum,

$$
\frac{1}{n^k}\sum_{j=1}^{k-1}\left(\frac{n}{m}\right)^j = \frac{1}{(n-m)}\left(\frac{1}{m^{k-1}}-\frac{1}{n^{k-1}}\right).
$$

Therefore, the convolution sum becomes

$$
C_k(\chi_1,\chi_2) = (k-1)L(k\,;\,\chi_1\chi_2) + \sum_{m=1}^{\infty}\frac{\chi_1(m)}{m^{k-1}}\sum_{\substack{n=1,\\ n\neq m}}^{\infty}\frac{\chi_2(n)}{n-m} + \sum_{n=1}^{\infty}\frac{\chi_2(n)}{n^{k-1}}\sum_{\substack{m=1,\\ m\neq n}}^{\infty}\frac{\chi_1(m)}{m-n}.
$$

The inner sums were computed in Lemma 2.4. For any Dirichlet character χ mod q and $1 \leq a < q$, let $T_{q,a}(m) := \zeta_q^{am}$ and $T_{q,a,\chi}(m) := \chi(m)\,\zeta_q^{am}$. Thus, $T_{q,a}$ and $T_{q,a,\chi}$ define periodic functions on the integers, periodic modulo q. With this notation, the convolution becomes,

$$
\begin{aligned}
C_k(\chi_1,\chi_2) = (k-1)L(k\,;\,\chi_1\chi_2) &- \frac{1}{\tau(\overline{\chi_2})}\sum_{a=1}^{q-1}\overline{\chi_2}(a)\,\log(1-\zeta_q^a)\sum_{m=1}^{\infty}\frac{T_{q,a,\chi_1}(m)}{m^{k-1}} \\
&- \frac{1}{\tau(\overline{\chi_1})}\sum_{a=1}^{q-1}\overline{\chi_1}(a)\,\log(1-\zeta_q^a)\sum_{n=1}^{\infty}\frac{T_{q,a,\chi_2}(n)}{n^{k-1}} \\
&- \frac{1}{\tau(\overline{\chi_2})}\sum_{a=1}^{q-1}\overline{\chi_2}(a)\zeta_q^{-a}\sum_{m=1}^{\infty}\frac{T_{q,a,\chi_1}(m)}{m^{k-1}}\sum_{j=1}^{m}\frac{\zeta_q^{-aj}}{j} \\
&- \frac{1}{\tau(\overline{\chi_1})}\sum_{a=1}^{q-1}\overline{\chi_1}(a)\zeta_q^{-a}\sum_{n=1}^{\infty}\frac{T_{q,a,\chi_2}(n)}{n^{k-1}}\sum_{j=1}^{n}\frac{\zeta_q^{-aj}}{j} \\
= (k-1)L(k\,;\,\chi_1\chi_2) &- \frac{1}{\tau(\overline{\chi_2})}\sum_{a=1}^{q-1}\Big(\overline{\chi_2}(a)\,\log(1-\zeta_q^a)\,L(k-1;T_{q,a,\chi_1})\Big)
\end{aligned}
$$

$$- \frac{1}{\tau(\overline{\chi_1})} \sum_{a=1}^{q-1} \left(\overline{\chi_1}(a) \log(1 - \zeta_q^a)\, L(k-1; T_{q,a,\chi_2}) \right)$$

$$- \frac{1}{\tau(\overline{\chi_2})} \sum_{a=1}^{q-1} \overline{\chi_2}(a)\, \zeta_q^{-a}\, L^*(k-1, 1\,;\, T_{q,a,\chi_1}, T_{q,-a})$$

$$- \frac{1}{\tau(\overline{\chi_1})} \sum_{a=1}^{q-1} \overline{\chi_1}(a)\, \zeta_q^{-a}\, L^*(k-1, 1\,;\, T_{q,a\chi_2}, T_{q,-a})$$

where

$$T_{q,a,\chi}(n) = \chi(n)\, \zeta_q^{an} \qquad \text{and} \qquad T_{q,-a}(n) = \zeta_q^{-an}.$$

This proves Theorem 1.4.

Remark. *It is clear from the above proof that in order to understand r-level convolution of values of Dirichlet L-functions, one needs to understand sums of the form*

$$\sum_{\substack{n=1,\\ n \neq m}}^{\infty} \frac{\chi(n)}{n^r - m^r},$$

for a primitive Dirichlet character χ mod q. These sums are interesting in their own right and we relegate their investigation to future research.

3. Concluding remarks

The theorems included here are only the opening themes of a larger symphony of ideas. It is now clear that to understand the nature of $\zeta(2k+1)$, it is necessary to study the multi-zeta values. Our paper shows that a similar approach is needed to understand $L(k;\chi)$ when k and χ have opposite parity.

In Theorems 1.1, 1.2 and 1.3, one can consider the more general case when the corresponding Lerch and Hurwitz zeta-functions have different parameters. For example, one can compute the convolution of values of $\Phi(z_1;\alpha_1;s)$ and values of $\Phi(z_2;\alpha_2;s)$ with $\alpha_1 \neq \alpha_2$. The method outlined in this paper would also go through in these cases. However, the identities in these scenarios are not as elegant as the ones mentioned here.

Let G denote the Catalan's constant, that is,

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = L(2;\chi_4) = 4\,\Phi\left(-1; \frac{1}{2}; 2\right).$$

Then $k = 3$ and $k = 2$ cases of Corollaries 1.1 and 1.4 furnish interesting relations among G, $L(1,\chi_4)$, π^2, $\zeta(3)$ and values of multiple zeta-functions.

A curious observation emerges from the identity stated in Corollary 1.2. For $k = 3$, the left-hand side of the formula in Corollary 1.2 is empty and hence, zero. Substituting $\alpha = 1/2$ and simplifying the right-hand side leads to the identity

$$\zeta(3) = \frac{6}{7}\,(\log 2)\,\zeta(2) + \frac{4}{7}\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^2}.$$

Furthermore, taking $k = 3$ in Corollary 1.3, we also get

$$\zeta(3) = \frac{1}{4}\,(\log 2)\,\zeta(2) + \frac{1}{2}\zeta(\overline{2},1) - \frac{1}{2}\zeta(\overline{2},\overline{1}).$$

This is interesting since (it seems) Euler conjectured that

$$\zeta(3) = \alpha\,\pi^2\,\log 2\, +\, \beta\,(\log 2)^2$$

for certain rational numbers α and β (see for example, [9, pg. 60]). This observation leads us to inquire whether

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^2}\ \text{or}\ \zeta(\overline{2},1) - \zeta(\overline{2},\overline{1})$$

can be explicitly evaluated in terms of $\pi^2\log 2$ and $(\log 2)^2$. Perhaps not. To date, no one has disproved Euler's conjecture.

In this vein, we would like to highlight a conjecture by D. Bailey, J. Borwein and R. Girgensohn [4, Section 7, pg. 27] based on numerical evidence. To each (alternating) Euler-Zagier sum, $\Phi(\epsilon_1,\cdots,\epsilon_r;1,\cdots,1;k_1,\cdots k_r)$, $\epsilon_j\in\{\pm 1\}$, one can associate the weight $w = k_1+\cdots+k_r$. Moreover, the weight of the product $\Phi(\epsilon_1,\cdots,\epsilon_r;1,\cdots,1;k_1,\cdots,k_r)\cdot\Phi(\delta_1,\cdots,\delta_r;1,\cdots,1;m_1,\cdots,m_s)$ is given by the sum $k_1+\cdots+k_r+m_1+\cdots+m_s$. Then, the conjecture of Bailey, Borwein and Girgensohn can be stated as follows.

Conjecture 1 *(Bailey, J. Borwein, Girgensohn). Alternating Euler-Zagier sums of different weights are $\mathbb{Q}$-linearly independent.*

Now, $\zeta(3)$ and $\pi^2\log 2$ have weight 3 each. However, $(\log 2)^2 = \Phi(-1;1;1)^2 = 2\zeta(\overline{1},1)$ (see [5, pg. 291]) and hence, has weight 2. Therefore, Conjecture 1 would imply that $\zeta(\overline{2},1)-\zeta(\overline{2},\overline{1})$ is a rational multiple of $\pi^2\log(2)$. This is not expected (see [5, pg. 291]) and thus, Euler's conjecture seems to be false.

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