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A note on values of the Dedekind zeta-function at odd positive integers

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For an algebraic number field K, let $\zeta_K(s)$ be the associated Dedekind zeta-function. It is conjectured that $\zeta_K(m)$ is transcendental for any positive integer $m > 1$. The only known case of this conjecture was proved independently by Siegel and Klingen, namely that, when K is a totally real number field, $\zeta_K(2n)$ is an algebraic multiple of $\pi^{2n[K:\mathbb{Q}]}$ and hence, is transcendental. If K is not totally real, the question of whether $\zeta_K(m)$ is irrational or not remains open. In this paper, we prove that for a fixed integer $n \geq 1$, at most one of $\zeta_K(2n+1)$ is rational, as K varies over all imaginary quadratic fields. We also discuss a generalization of this theorem to CM-extensions of number fields.

Keywords: Dedekind zeta-function of imaginary quadratic fields; irrationality of zeta-values.

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1. Introduction

The Riemann zeta-function, $\zeta(s)$ has occupied center stage in mathematics since its introduction in the phenomenal 1859 paper of Riemann. In this paper, Riemann

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proved that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\text{-prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

for $\Re(s) > 1$ has an analytic continuation to the entire complex plane, except for a simple pole at $s = 1$ with residue 1, and satisfies a functional equation relating the value at s to the value at $1-s$. Moreover, it was proved independently by Hadamard and de la Vallée Poussin that the distribution of primes (in particular, the prime number theorem) is a consequence of the non-vanishing of $\zeta(s)$ on the line $\Re(s) = 1$ together with the simple pole at $s = 1$. These ideas gave birth to the study of zeta and L-functions in other number theoretic contexts.

The focus of the current paper is another question about $\zeta(s)$, which has been baffling mathematicians since the 18th century. In 1735, Euler proved that for $k \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = -\frac{(2\pi i)^{2k} B_{2k}}{2(2k)!},$$

where B_k is the kth Bernoulli number given by the generating function

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k t^k}{k!}.$$

We recognize that the infinite series evaluated by Euler are nothing but the special values, $\zeta(2k)$. Whether such an "explicit evaluation" exists for the values $\zeta(2k+1)$ as well, is still an open question. Using the functional equation of $\zeta(s)$, Euler's theorem implies that

$$\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1} \quad \text{for } k \in \{0, 1, 2, \ldots\}.$$

Thus, the value of $\zeta(s)$ at negative integers is rational with $\zeta(-2n) = 0$ for $n \in \mathbb{N}$.

Furthermore, the transcendence of π due to Lindemann implies that $\zeta(2k) \in \pi^{2k}\,\mathbb{Q}^{\times}$, is also transcendental for every $k \in \mathbb{N}$. On the other hand, the algebraic/transcendental nature of $\zeta(2k+1)$ is shrouded in mystery. Spectacular breakthroughs have recently been made by Apéry [1] in 1978 who showed that $\zeta(3) \notin \mathbb{Q}$; by Rivoal [16] in 2000 and Ball and Rivoal [3] in 2001, who showed that for infinitely many k, $\zeta(2k+1) \notin \mathbb{Q}$; and Zudilin [24] who proved that at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$ and $\zeta(11)$ is irrational.

Let K be a number field with $[K : \mathbb{Q}] = n$ and $\mathcal{O}_K$ be its ring of integers. Then the Dedekind zeta-function attached to K is defined as

$$\zeta_K(s) := \prod_{\substack{\mathfrak{p} \subseteq \mathcal{O}_K, \\ \mathfrak{p} \neq 0}} \left(1 - \frac{1}{N\mathfrak{p}^s}\right)^{-1} = \sum_{\substack{\mathfrak{A} \subseteq \mathcal{O}_K, \\ \mathfrak{A} \neq 0}} \frac{1}{N\mathfrak{A}^s}, \quad \Re(s) > 1,$$

where the product is over non-zero prime ideals in $\mathcal{O}_K$ and N denotes the absolute norm. When $K = \mathbb{Q}$, the Dedekind zeta-function $\zeta_{\mathbb{Q}}(s)$ is simply the Riemann zeta-function $\zeta(s)$.

The function $\zeta_K(s)$ was introduced by Dedekind, who also conjectured its analytic continuation, which was proved later by Hecke [8]. The function $\zeta_K(s)$ extends analytically to the entire complex plane except for a simple pole at $s = 1$. The residue at $s = 1$ is given by the analytic class number formula,

$$\lim_{s\to 1^+} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} hR}{\omega\sqrt{|d_K|}},$$

where r_1 is the number of real embeddings of K, $2r_2$ is the number of complex embeddings of K, h denotes the class number, R is the regulator, ω is the number of roots of unity in K and d_K is the discriminant of K (see [14, Chap. 1]).

Analogous to the Riemann zeta-function, the Dedekind zeta-function captures crucial information about the distribution of prime ideals in $\mathcal{O}_K$. For example, the non-vanishing of $\zeta_K(s)$ on the line $\Re(s) = 1$ along with its simple pole at $s = 1$, implies the prime ideal theorem. The prime ideal theorem asserts that if $\pi_K(x) := \#\{\mathfrak{p} \in \mathcal{O}_K : \mathfrak{p} \text{ is prime}, N\mathfrak{p} \le x\}$, then

$$\pi_K(x) \sim \frac{x}{\log x},$$

as $x \to \infty$. For a proof of this theorem, we refer the reader to the exposition in [14, Theorem 3.2].

The Dedekind zeta function satisfies a functional equation in the same spirit as the Riemann zeta-function, namely,

$$\xi_K(s) = \xi_K(1-s),$$

where

$$\xi_K(s) := \left(\frac{\sqrt{|d_K|}}{2^{r_2}\pi^{n/2}}\right)^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2}\zeta_K(s),$$

which is analytic in the entire complex plane except for simple poles at $s = 0$ and $s = 1$. Since the gamma function has poles at negative integers, we see that if $r_1 > 0$, then from the functional equation one can deduce that $\zeta_K(s)$ is always zero at all non-zero negative even integers. Additionally, if K is not totally real (i.e. $r_2 > 0$), then $\zeta_K(s)$ is zero at all odd negative integers as well. Thus, the only non-zero values of $\zeta_K(s)$, at negative integers $-m$, arise when K is totally real and $m > 0$ is odd. From the functional equation, these values correspond to $\zeta_K(2n)$ for an integer $n > 0$.

In 1940, Hecke [9] proved that $\zeta_K(2n)$ is an algebraic multiple of π^{4n} for a real quadratic field K. This led him to conjecture similar phenomena when K is any totally real field. Indeed, it was shown by Siegel and Klingen [11] independently, that when F is totally real, $\zeta_F(1-2n)$ is rational. This translates to $\zeta_F(2n)$ being an algebraic multiple of $\pi^{2n[F:\mathbb{Q}]}$, generalizing Euler's 1737 theorem for the Riemann zeta-function. The method utilized by them relied on the theory of Hilbert modular forms. An accessible exposition of the proof can be found in the appendix of Siegel's TIFR lecture notes [19]. In 1976, Shintani [17] provided an alternate proof of this theorem from a classical perspective, whereas geometric proofs have recently appeared in [4, 5].

When K is not totally real, nothing is known regarding the irrationality or transcendence of $\zeta_K(n)$. In 1990, Zagier [23] put forth a conjecture connecting these values to the polylogarithm function,

$$\mathrm{Li}_k(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^k}, \quad |z| < 1.$$

He conjectured that "$\zeta_K(n)$ is a simple multiple of the determinant of a matrix whose entries are linear combinations of polylogarithms evaluated at a certain number in K". The case $n = 3$ of Zagier's conjecture was settled by Goncharov [7]. However, we are still far from understanding the nature of these numbers.

If $K/\mathbb{Q}$ is an imaginary quadratic field, then $\zeta_K(s) = \zeta(s)L(s,\chi)$, where χ is an odd Dirichlet character. We know that $L(2m+1,\chi)$ is an algebraic multiple of π^{2m+1} (see [15, Proposition 2]). Thus, $\zeta_K(2m+1) = \zeta(2m+1)L(2m+1,\chi)$ is an algebraic multiple of $\pi^{2m+1}\zeta(2m+1)$. We would anticipate all of these numbers to be transcendental, however, we are far from establishing this.

In particular, we expect all of the numbers $\zeta_K(2n+1)$ (when K ranges over imaginary quadratic fields) to be irrational. We can prove the following.

Theorem 1.1. *Let $m \geq 1$ be a fixed integer. Then the numbers*

$$\{\zeta_K(2m+1) : K/\mathbb{Q} \text{ is an imaginary quadratic extension}\}$$

are irrational with at most one exception.

This immediately implies the following.

Corollary 1.2. *For a positive real number D, let*

$$\mathfrak{F}_D := \{K : K/\mathbb{Q} \text{ is an imaginary quadratic extension}, |d_K| \leq D\},$$

where $|d_K|$ denotes the absolute discriminant of the field K. Then, Theorem 1.1 implies that

$$\frac{\#\{\zeta_K(2m+1) : \zeta_K(2m+1) \in \mathbb{Q},\ K \in \mathfrak{F}_D \text{ and } m \in \mathbb{N},\ m \leq x\}}{\#\{(m,K)\ :\ K \in \mathfrak{F}_D,\ m \in \mathbb{N}, m \leq x\}} \leq \frac{1}{|\mathfrak{F}_D|}$$

and the right-hand side tends to zero as $D \to \infty$.

Additionally, as a consequence of Proposition 2.2 used in the proof of Theorem 1.1, we obtain the following interesting corollaries, which will be proved in Sec. 3.

Corollary 1.3. *Let m be a fixed positive integer. Then, either all the numbers*

$$\{\zeta_K(2m+1)\ :\ K/\mathbb{Q} \text{ is an imaginary quadratic extension}\}$$

are transcendental or all the above numbers are algebraic.

Corollary 1.4. *Let m be a fixed positive integer. Then the numbers*

$$\{\zeta_K(2m+1) \,:\, K/\mathbb{Q} \text{ is an imaginary quadratic extension}\}$$

are $\mathbb{Q}$-linearly independent.

A number field E is said to have complex multiplication (CM) if there exists a subfield F of E, such that F is totally real, and E is a totally imaginary quadratic extension of F. The aim of this paper is to highlight that an irrationality result for the values of the Dedekind zeta-function of CM-number fields can be deduced from our current knowledge. In particular, we obtain the following.

Theorem 1.5. *Let $m \geq 1$ be a fixed integer. Fix a totally real field F. Consider any family $\mathfrak{F}$ of CM-extensions E/F satisfying the following: for E_1, $E_2 \in \mathfrak{F}$ with $E_1 \neq E_2$, the square-free parts of d_{E_1} and d_{E_2} are co-prime. Then the numbers*

$$\{\zeta_E(2m+1) \,:\, E \in \mathfrak{F}\}$$

are irrational with at most one exception.

Our work will use the theory of Artin L-series and a central theorem of Coates and Lichtenbaum [6] regarding special values of certain Artin L-series.

2. Preliminaries

In this section, we review parts of algebraic number theory that are relevant to our discussion.

2.1. *Artin L-functions*

We summarize relevant facts regarding Artin L-functions in what follows. A gentle introduction to Artin L-functions can be found in Snyder's senior thesis, titled "Artin L-functions: A Historical Approach" [20]. A more concise account is included in the monograph [14, Chap. 2].

Let E/F be a Galois extension of number fields with Galois group G. Let $\rho : G \to GL(V)$ be a representation of G with character χ. Then the Artin L-function associated to the extension E/F and the representation ρ is defined as

$$L(s,\chi,E/F) = \prod_{\substack{\mathfrak{p}\subseteq\mathcal{O}_F,\\ \mathfrak{p}\text{ prime}}} L_{\mathfrak{p}}(s,\chi,E/F),$$

where the local factors at each prime ideal $\mathfrak{p}$ of $\mathcal{O}_F$ are as follows. Suppose first that $\mathfrak{p}$ is unramified in E. Let $\sigma_{\mathfrak{p}}$ denote the conjugacy class corresponding to the Frobenius at $\mathfrak{p}$. The local factor at $\mathfrak{p}$ is defined as

$$L_{\mathfrak{p}}(s,\chi,E/F) = \det(I - \rho(\sigma_{\mathfrak{p}})N\mathfrak{p}^{-s})^{-1}.$$

Now suppose that $\mathfrak{p}$ is ramified in E and fix a prime $\mathfrak{P}$ above $\mathfrak{p}$. Let $V^{I_{\mathfrak{P}}}$ be the subspace of vectors fixed by the inertia group $I_{\mathfrak{P}}$, pointwise. That is,

$$V^{I_{\mathfrak{P}}} = \{v \in V : \rho(\iota)\cdot v = v, \text{ for all } \iota \in I_{\mathfrak{P}}\}.$$

Since $I_{\mathfrak{P}}$ is a normal subgroup of $G_{\mathfrak{P}}$, one can see that $V^{I_{\mathfrak{P}}}$ is $G_{\mathfrak{P}}$-invariant. Let $\sigma_{\mathfrak{P}}$ be any Frobenius element at $\mathfrak{P}$. Then,

$$L_{\mathfrak{p}}(s, \chi, E/F) = \det(I - \rho(\sigma_{\mathfrak{P}})|_{V^{I_{\mathfrak{P}}}} N\mathfrak{p}^{-s})^{-1},$$

where $\sigma|_{V^{I_{\mathfrak{P}}}}$ denotes σ restricted to the invariant subspace $V^{I_{\mathfrak{P}}}$ for $\sigma \in G_{\mathfrak{P}}$. Note that the above definition is independent of the choice of the Frobenius element. The infinite product consisting of all these local factors converges absolutely for $\Re(s) > 1$ and defines the Artin L-function associated to ρ and the extension E/F.

These L-functions take more familiar shape in certain scenarios. For example, suppose E/F is Galois with Galois group G. Then, the Artin L-function obtained by considering the trivial representation of G is nothing but the Dedekind zeta-function attached to the ground field, $\zeta_F(s)$. On the other hand, the Artin L-functions associated to characters of $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ are precisely the Dirichlet L-functions.

Artin conjectured that any Artin L-function $L(s, \chi, E/F)$ associated to a character χ of $\mathrm{Gal}(\overline{F}/F)$ extends to an analytic function to the entire complex plane except for a possible pole at $s = 1$, of order equal to the multiplicity of the trivial representation in χ. This is one of the classical conjectures in number theory and remains open in general. It is known in the special case when $\mathrm{Gal}(E/F)$ is abelian. In this case, by Artin's reciprocity law, the Artin L-function of an irreducible character corresponds to a Hecke L-function, which is known to be entire (see [12, Chap. 9] for further details). There are also some recent results in the two-dimensional case due to Langlands [13], Tunnell [21] and Khare and Wintenberger [10].

Artin L-functions satisfy a functional equation in the same spirit as the Riemann zeta-function. At the infinite primes, i.e. the Archimedean places, the corresponding Euler factors are defined as follows. Let ν be an Archimedean place of F. Then,

$$L_\nu(s, \chi, E/F) = \begin{cases} \left((2\pi)^{-s}\,\Gamma(s)\right)^{\dim(\rho)} & \text{if } \nu \text{ is complex,} \\ \left(\pi^{-s/2}\,\Gamma(s/2)\right)^a \left(\pi^{-(s+1)/2}\,\Gamma((s+1)/2)\right)^b & \text{if } \nu \text{ is real.} \end{cases}$$

Here a is the dimension of the $+1$ eigenspace of complex conjugation and b is the dimension of -1 eigenspace of complex conjugation. Hence,

$$a + b = \dim(\rho).$$

Therefore, the gamma factors for $L(s, \rho, E/F)$ are

$$\gamma(s, \chi, E/F) = \prod_{\nu \text{ - Archimedean place of } F} L_\nu(s, \chi, E/F).$$

An important invariant that makes an appearance in the functional equation is the Artin conductor, $\mathfrak{f}_\chi$. The Artin conductor is an ideal in the ring $\mathcal{O}_F$ and is defined by the restriction of χ to the inertia group and its various subgroups. We refrain from giving the technical definition here and refer the reader to [14, p. 28] for the precise version. However, we note one of the useful connections of the Artin conductor to the relative discriminants of number fields. In 1931, Artin [2] proved the conductor-discriminant formula for any Galois extension of number fields E/F. This formula states that

$$\mathfrak{D}_{E/F} = \prod_{\chi \in \widehat{G}} \mathfrak{f}_{\chi}^{\chi(1)}, \tag{2.1}$$

where $\mathfrak{D}_{E/F}$ denotes the relative discriminant of E/F, $\widehat{G}$ denotes the set of all irreducible characters of G and $\chi(1)$ is the dimension of the irreducible representation corresponding to χ.

Let

$$A_{\chi} = |d_F|^{\chi(1)} N\mathfrak{f}_{\chi} \in \mathbb{Q},$$

where d_F denotes the discriminant of the field F. The completed Artin L-function can then be defined as

$$\Lambda(s, \chi, E/F) := A_{\chi}^{s/2} \gamma(s, \chi, E/F) L(s, \chi, E/F).$$

This completed Artin L-function satisfies the functional equation

$$\Lambda(s, \chi, E/F) = W(\chi) \Lambda(1 - s, \overline{\chi}, E/F), \tag{2.2}$$

for all $s \in \mathbb{C}$. The number $W(\chi)$ is called the Artin root number and is a complex number of absolute value 1, carrying deep arithmetic meaning. One important observation here is that if χ is real-valued, then $W(\chi) = \pm 1$. This can be seen by comparing the above functional equation with its complex conjugate.

Using basic functorial properties of Artin L-functions, one can translate the group theoretic identity,

$$\mathrm{reg}_G = \sum_{\chi \in \widehat{G}} \chi(1) \quad \chi,$$

to a factorization identity, namely,

$$\zeta_E(s) = \zeta_F(s) \prod_{\substack{\chi \in \widehat{G}, \\ \chi \neq 1}} L(s, \chi, E/F)^{\chi(1)}, \tag{2.3}$$

where reg_G denotes the regular representation of G.

2.2. *Values of zeta-functions at negative integers*

Let $\zeta(s)$ denote the Riemann zeta-function. As a by-product of Riemann's proof of analytic continuation and functional equation of $\zeta(s)$, one can obtain the evaluation of $\zeta(-n)$ for a positive integer n in terms of Bernoulli numbers. Analogously, for a positive integer q, let $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ be a Dirichlet character mod q and let $L(s, \chi)$ be the Dirichlet L-series attached to χ. It can be shown that $L(s, \chi)$ is entire when χ is non-principal. Furthermore, for any integer $n \geq 0$,

$$L(-n, \chi) = -\frac{q^n}{n+1} \sum_{a=1}^{q} \chi(a) B_{n+1}\left(\frac{a}{q}\right),$$

where $B_n(X) \in \mathbb{Q}[X]$ is the nth Bernoulli polynomial. We refer the reader to [22, Chap. 4] for a proof. This implies that if χ is a quadratic character, then $L(-n,\chi) \in \mathbb{Q}$.

Similarly, the Siegel–Klingen theorem proves that the values of Dedekind zeta-functions attached to totally real fields at odd negative integers are rational. Moreover, Siegel [18] proved an analogue of this theorem for Hecke L-series associated to ray class characters. It was further suggested by Serre that Siegel's work itself implies a similar result for *all* Artin L-functions. This appears in the paper of Coates and Lichtenbaum [6, Theorem 1.2]. In particular, they show the following.

Theorem 2.1 (Coates–Lichtenbaum). *Let F be a totally real number field and E/F be a Galois extension with Galois group G. Let ρ be a representation of G with character χ and $L(s,\chi,E/F)$ be the associated Artin L-function. Let $\mathbb{Q}(\chi) = \mathbb{Q}(\{\chi(g) : g \in G\})$ and n be a positive integer such that $L(-n,\chi,E/F) \neq 0$. Then $L(-n,\chi,E/F)$ is an algebraic number lying in the field $\mathbb{Q}(\chi)$.*

It is evident from the functional equation (2.2) that there exist positive integers n such that $L(-n,\chi,E/F)$ is not zero if and only if F is totally real and either (a) E is totally real and n is odd or (b) E is totally imaginary and n is even.

2.3. *Quotients of special values of zeta-functions*

The proof of our theorem is based on the following proposition. In order to state the proposition, we define the notion of rational equivalence. Two complex numbers α and β are said to be rationally equivalent, i.e. $\alpha \sim_{\mathbb{Q}} \beta$ if $\beta = u\,\alpha$ for some $u \in \mathbb{Q}^{\times}$. With this definition, we show the following.

Proposition 2.2. *Fix a totally real number field F. Let E_1 and E_2 be two CM-extensions of F and d_{E_1} and d_{E_2} be their respective discriminants. Then, for any fixed integer $m > 0$,*

$$\frac{\zeta_{E_1}(2m+1)}{\zeta_{E_2}(2m+1)} \sim_{\mathbb{Q}} \left(\frac{|d_{E_2}|}{|d_{E_1}|}\right)^{1/2}.$$

Proof. Let $G_j := \mathrm{Gal}(E_j/F)$ for $j = 1, 2$. Then we have, $G_j = \{1, c_j\}$, where c_j denote complex conjugation. Let the characters corresponding to c_j be $\chi_j : G_j \to \{\pm 1\}$ where $\chi_j(c_j) = -1$. By the factorization (2.3),

$$\zeta_{E_j}(s) = \zeta_F(s) L(s,\chi_j,E_j/F), \quad j = 1, 2.$$

Thus,

$$\frac{\zeta_{E_1}(2m+1)}{\zeta_{E_2}(2m+1)} = \frac{L(2m+1,\chi_1,E_1/F)}{L(2m+1,\chi_2,E_2/F)}.$$

The functional equation of Artin L-functions (2.2) relate the value at $2m+1$ with the value at $-2m$. Since F is totally real, all archimedean places of F are real and hence, none of the gamma factors appearing in the functional equation have poles at these integers. Thus, for $j = 1, 2$, we have

$$A_{\chi_j}^{(2m+1)/2}\gamma(2m+1, \chi_j, E_j/F)L(2m+1, \chi_j, E_j/F)$$
$$= W(\chi_j)A_{\chi_j}^{-m}\gamma(-2m, \chi_j, E_j/F)L(-2m, \chi_j, E_j/F).$$

This implies that

$$L(2m+1, \chi_j, E_j/F)$$
$$= W(\chi_j)(A_{\chi_j})^{-2m-1/2}\frac{\gamma(-2m, \chi_j, E_j/F)}{\gamma(2m+1, \chi_j, E_j/F)}L(-2m, \chi_j, E_j/F).$$

Since E_j is a totally imaginary extension of F, Theorem 2.1 and (2.2) imply that

$$0 \neq L(-2m, \chi_j, E_j/F) \in \mathbb{Q}(\chi_j) = \mathbb{Q}.$$

We would like to remark here that we do not need the full generality of Theorem 2.1. Since an L-function associated to a one-dimensional character is a product of Hecke L-functions, Siegel's result would suffice for the above conclusion.

Moreover, $W(\chi_j) = \pm 1$ because χ_j are real valued for $j = 1, 2$. Hence,

$$L(2m+1, \chi_j, E_j/F) \sim_{\mathbb{Q}} A_{\chi_j}^{-1/2}\frac{\gamma(-2m, \chi_j, E_j/F)}{\gamma(2m+1, \chi_j, E_j/F)}. \tag{2.4}$$

On taking the ratio of $L(2m+1, \chi_j, E_j/F)$ for $j = 1, 2$, the gamma factors cancel as they are same for both the Artin L-functions under consideration. Thus,

$$\frac{\zeta_{E_1}(2m+1)}{\zeta_{E_2}(2m+1)} \sim_{\mathbb{Q}} \left(\frac{A_{\chi_2}}{A_{\chi_1}}\right)^{1/2}.$$

The factor of $|d_F|$ will be common to both the values, and disappears in the ratio. Thus, the contributing factor reduces to

$$\frac{N_{F/\mathbb{Q}}\mathfrak{f}_{\chi_1}}{N_{F/\mathbb{Q}}\mathfrak{f}_{\chi_2}}.$$

Since the conductor of the trivial representation is the unit ideal, the conductor-discriminant formula (2.1) implies

$$\mathfrak{f}_{\chi_j} = \mathfrak{D}_{E_j/F}.$$

The relative discriminant of E_j/F is related to the absolute discriminant of E_j by the formula

$$d_{E_j} = d_F^2.N_{F/\mathbb{Q}}\mathfrak{D}_{E_j/F}.$$

The statement of the proposition now follows. □

3. Proof of the Main Theorems

3.1. *Proof of Theorem* 1.1

By Proposition 2.2, we know that if K_1 and K_2 are two imaginary quadratic extensions of $\mathbb{Q}$, then

$$\frac{\zeta_{K_1}(2m+1)}{\zeta_{K_2}(2m+1)} \sim_{\mathbb{Q}} \left(\frac{|d_{K_2}|}{|d_{K_1}|}\right)^{1/2}.$$

Since K_1 and K_2 are distinct quadratic extensions of $\mathbb{Q}$, and $K_j = \mathbb{Q}(\sqrt{|d_{K_j}|})$, the numbers $\sqrt{|d_{K_j}|}$ are not rational multiples of each other. Hence, the above quotient is irrational, proving the theorem.

3.2. *Proof of Corollary* 1.3

The statement of Proposition 2.2 implies that for two distinct imaginary quadratic fields, K_1 and K_2, the quotient

$$\frac{\zeta_{K_1}(2m+1)}{\zeta_{K_2}(2m+1)} \in \overline{\mathbb{Q}}.$$

Therefore, if $\zeta_K(2m+1) \in \overline{\mathbb{Q}}$ for some imaginary quadratic field K, then the same will be true for all imaginary quadratic fields.

3.3. *Proof of Corollary* 1.4

Suppose that there exist imaginary quadratic fields $K_j/\mathbb{Q}$, $1 \le j \le r$ such that

$$\sum_{j=1}^{r} c_j \zeta_{K_j}(2m+1) = 0, \quad c_j \in \mathbb{Q}, \quad 1 \le j \le r.$$

By the factorization (2.3), we have $\zeta_{K_j}(2m+1) = \zeta(2m+1)L(2m+1, \chi_j)$. Hence, the above relation reduces to a relation among the values of Dirichlet L-functions. Now take $F = \mathbb{Q}$, $E_j = K_j$ in (2.4). Note that the corresponding ratio of gamma factors is independent of χ_j and simplifies to

$$\frac{\gamma(-2m, \chi_j, K_j/\mathbb{Q})}{\gamma(2m+1, \chi_j, K_j/\mathbb{Q})} = \frac{\pi^{\left(m-\frac{1}{2}\right)}\Gamma\left(-m+\frac{1}{2}\right)}{\pi^{-(m+1)}\Gamma(m+1)} \sim_{\mathbb{Q}} \pi^{\left(2m+\frac{1}{2}\right)}\Gamma\left(-m+\frac{1}{2}\right) \sim_{\mathbb{Q}} \pi^{2m+1},$$

as $\Gamma(1/2) = \sqrt{\pi}$. Thus, the above relation becomes

$$\sum_{j=1}^{r} \frac{R_j}{\sqrt{|d_{K_j}|}} = 0,$$

for certain rational numbers R_j. However, the numbers $\sqrt{|d_{K_j}|}$ are $\mathbb{Q}$-linearly independent. This proves the corollary.

3.4. *Proof of Theorem* 1.5

The conditions on the family $\mathfrak{F}$ ensure that for any E_1 and E_2 in $\mathfrak{F}$, $|d_{E_1}|/|d_{E_2}|$ is not a perfect square in $\mathbb{Q}$. The corollary is now immediate from Proposition 2.2.

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