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# Spectral estimates for Abelian Cayley graphs

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**Abstract**

We give two short proofs that for fixed $d$, a $d$-regular Cayley graph on an Abelian group of order $n$ has second eigenvalue bounded below by $d - O(dn^{-4/d})$, where the implied constant is absolute. We estimate the constant in the $O(dn^{-4/d})$ notation. We show that for any fixed $d$, then for a large odd prime, $n$, the $O(dn^{-4/d})$ cannot be improved; more precisely, most $d$-regular graphs on prime $n$ vertices have second eigenvalue at most $d - \Omega(dn^{-4/d})$ for an odd prime, $n$.


*Keywords:* Abelian Cayley graphs; Eigenvalue bounds

## 1. Introduction

In this paper we give two short, self-contained lower bounds on the second eigenvalue of a Cayley graph over an Abelian group of a given degree, $d$, and given number of vertices, $n$. We shall show that the second eigenvalue is at least $d - O(dn^{-4/d})$, and that for all $d$ and prime $n$ this result is achieved for most Abelian Cayley graphs. Such specific results on the second eigenvalue of Cayley graphs do not seem to be known, despite the well-

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known folklore result that Abelian Cayley graphs cannot be good expanders. A result of Alon and Roichman, [AR94, Proposition 3, Section 3], shows that second eigenvalue must tend to $d$ as $n \to \infty$; this result proves the folklore result; however, the second eigenvalue is not explicitly estimated. The work of Klawe, [Kla81,Kla84], deals with affine linear transformations, $x \mapsto a_i x + b_i$, of the integers modulo $n$, where the coefficients of the transformation (the $a_i, b_i$) are fixed independent of $n$ (the expansion bounds obtained depend on the $|b_i|$); thus Klawe's results do not apply to the subject of this paper.

Now we shall give precise definitions below.

**Definition 1.** Let $G$ be a group, and $g_1, \ldots, g_r, \widetilde{g}_1, \ldots, \widetilde{g}_t \in G$ with $\widetilde{g}_i^2 = 1$ for all $i$, where $1 \in G$ is the identity. We form the *$(r,t)$-Cayley graph*, $X = \mathrm{Cay}(G; \{g_i\}; \{\widetilde{g}_i\}) = \mathrm{Cay}(G; g_1, \ldots, g_r; \widetilde{g}_1, \ldots, \widetilde{g}_t)$, as follows. The vertex set of $X$ is $G$; $X$ has edges $(x, g_i x)$ and $(x, g_i^{-1} x)$ for each $x \in G$ and $i$, and an edge $(x, \widetilde{g}_i x)$ for each $x \in G$ and $i$. $X$ is therefore a directed graph but may be regarded as an undirected graph; $X$ is $d$-regular with $d = 2r + t$. $X$ is said to have *group* $G$ and *generators* $\{g_i\}$, $\{\widetilde{g}_i\}$.

Let $X = \mathrm{Cay}(G; \{g_i\}; \{\widetilde{g}_i\})$ be a Cayley graph as in the definition above. As usual, $X$ has an *adjacency matrix*, $A = A_X$, mapping functions on $G$ to themselves, via

$$(A_X f)(x) = \sum_{i=1}^{r} \Big( f(g_i x) + f(g_i^{-1} x) \Big) + \sum_{i=1}^{t} f(\widetilde{g}_i x).$$

If $G$ is Abelian, then it is well-known that the eigenvectors of $A_X$ are the characters, $\chi$, of $G$, with corresponding eigenvalue

$$\lambda = \lambda_\chi = \sum_{i=1}^{r} \Big( \chi(g_i) + \chi\big(g_i^{-1}\big) \Big) + \sum_{i=1}^{t} \chi(\widetilde{g}_i).$$

We denote by $\lambda_2 = \lambda_2(X)$ the second largest eigenvalue of $A_X$, which is equal to

$$\lambda_2 = \max_{\chi \neq 1} \lambda_\chi.$$

In Section 3 we shall use the Rayleigh quotient characterization of $\lambda_2$, namely that

$$\lambda_2 = \max_{v \perp \vec{1}} \mathscr{R}_A(v),$$

where $\vec{1}$ denotes the all 1's vector, and where

$$\mathscr{R}_B(v) = \frac{(Bv, v)}{(v, v)},$$

where $(u, w)$ denotes the inner product of $u$ and $w$.

In Section 2 we give a bound based on sphere packing to show that for any fixed $d$ there is a constant $C_d$ such that $\lambda_2 \geqslant d - C_d d n^{-4/d}$ for any $d$-regular Abelian Cayley graph on $n$ vertices. Furthermore we can show that

$$C_d \preccurlyeq \frac{4^{1-c_{\mathrm{KL}}} e}{\pi} = 2.015\ldots,$$

where $c_{\mathrm{KL}} = 0.5990\ldots$ is the constant in the Kabatiansky–Levenshtein sphere packing bound, and where we use the notation $g(d) \preccurlyeq h(d)$ or $h(d) \succcurlyeq g(d)$ (as $d \to \infty$) to mean that

$$\liminf_{d\to\infty}\bigl(h(d) - g(d)\bigr) \geqslant 0.$$

Admittedly, our use of the Kabatiansky–Levenshtein bound is not self-contained, but there is a (trivial) "naive sphere-packing bound" that gives

$$C_d \preccurlyeq \frac{4e}{\pi} = 3.4610\ldots$$

(that we describe in full).

In Section 3 we give an alternate method (arguably less simple) for bounding $\lambda_2$, which gives $\lambda_2 \geqslant d - C_d dn^{-4/d} + o(n^{-4/d})$ with

$$C_d \preccurlyeq \pi^2/2 = 4.93480\ldots.$$

The method is an eigenfunction "pushing" argument, as is done (for different graphs) in [FT02].

We mention that there are other possible approaches to finding lower bounds for $\lambda_2$ other than those given in Sections 2 and 3. Here is a rough description of another approach. In expanders, the size of a ball about a vertex grows exponentially with respect to the radius, at least until the ball becomes fairly large. But it is not hard to see that balls in Abelian Cayley of fixed degree graphs grow only polynomially. This could be made into a $\lambda_2$ lower bound, but we do not know if it would be as good as those given in this paper. Another possible approach might be based on the fact that an Abelian Cayley graph has a large trace. This approach is worked out in detail in [Ci05].

In Section 4 we show that for any fixed even $d$ and odd prime $n$ we have that most ($d$-regular) Abelian $(d/2, 0)$-Cayley graphs on $n$ vertices have $\lambda_2 \leqslant d - C'_d dn^{-4/d} + o(n^{-4/d})$, with

$$C'_d \succcurlyeq \pi/e \geqslant 1.15572\ldots.$$

Notice that a $(d/2, 0)$-Cayley graph on $G = (\mathbf{Z}/2\mathbf{Z})^m$ will be disconnected (and therefore have $\lambda_2 = d$) if the $d/2$ generators all have the value 0 on one of their $m$ components; hence such a random graph has $\lambda_2 < d$ with probability no more than $\bigl(1 - 2^{-d/2}\bigr)^m$. It follows that for fixed even $d$ there are some groups, $G$, for which most $(d/2, 0)$-Cayley graphs on $G$ have $\lambda_2 = d$. Hence any bound holding for most graphs as that in the previous paragraph requires some restriction on $n$ or the types of groups on which the Cayley graphs are based.

## 2. Sphere packing bound

In this section we use a sphere packing argument to bound $\lambda_2$ from below. For this purpose, consider an eigenvalue $\lambda_\chi$ which corresponds to a character which acts trivially on the $\widetilde{g}_i$'s[4]. We denote by $H$ the subgroup of characters which meet this property. Notice that there are at least $n' \stackrel{\text{def}}{=} \frac{n}{2^t}$ characters of this kind. For such a character $\chi$ we have

$$\lambda_\chi = t + \sum_{i=1}^{r} 2\cos(\theta_i),$$

where $e^{i\theta_j} = \chi(g_j)$.

Since $\cos x \geqslant 1 - x^2/2$, it follows that

$$\lambda_\chi \geqslant d - \sum_{i=1}^{r} \theta_i^2. \tag{1}$$

This leads us to define a distance between characters of $H$ as follows. Consider the map

$$U : \chi \mapsto \big(\chi(g_1), \ldots, \chi(g_r)\big).$$

This maps characters, $\chi$, to $T^r$, where $T$ is the complex unit circle. For two points, $e^{i\theta}$, $e^{iv}$, on $T$, define their distance, $\rho$, to be the minimum of $|\theta - v + 2\pi m|$ over all integers $m$. For two points on $T^r$ define their distance to be the $L^2$ distance, i.e.

$$\text{distance}\big(\big(e^{i\theta_1}, \ldots, e^{i\theta_r}\big), \big(e^{iv_1}, \ldots, e^{iv_r}\big)\big) \stackrel{\text{def}}{=} \sqrt{\sum_{j=1}^{r} \rho^2\big(e^{i\theta_j}, e^{iv_j}\big)}.$$

The distance between characters is the distance induced by $U$, that is

$$\text{dist}(\chi, \chi') = \text{distance}(U(\chi), U(\chi')).$$

Eq. (1) now becomes

$$\lambda_\chi \geqslant d - \text{dist}^2(\chi, 1),$$

where 1 denotes the trivial character. Notice now that for any $\chi, \chi' \in H$ :

$$\text{dist}(\chi, \chi') = \text{dist}(\chi\chi'^{-1}, 1).$$

From this we deduce that

$$\lambda_2 \geqslant d - \min\Big(\text{dist}^2(\chi, \chi'), \chi \in H, \chi' \in H, \chi \neq \chi'\Big).$$

By bringing in $\varepsilon(k, r)$, the supremum of the minimum distance of $k$ points of $T^r$ (the minimum distance of a finite set of points being defined by the smallest distance between two points of the set), we obtain

$$\lambda_2 \geqslant d - \varepsilon^2(n', r). \tag{2}$$

In brief, we have proven the following theorem.

[4] We say that a character $\chi$ acts trivially on an element $g$ of the group iff $\chi(g) = 1$.

**Theorem 2.** *Let $G$ be an Abelian Cayley graph on $n$ vertices on $r$ generators and $t$ involutions. Then*

$$\lambda_2 \geqslant d - \varepsilon\left(\frac{n}{2^t}, r\right)^2.$$

There is a naive upper bound for $\varepsilon$ (which is nothing but a sphere packing bound) which follows by arguing that if there are $k$ points which are at least at distance $\varepsilon$ of each other in $T^r$, then the set of open balls of radius $\varepsilon/2$ centered about these points must be disjoint. From this remark we deduce that

$$k\omega_r\left(\frac{\varepsilon(k,r)}{2}\right)^r \leqslant (2\pi)^r,$$

where $\omega_r$ is the volume of the $r$-dimensional unit ball of radius 1. Recall that

$$\omega_r = \pi^{r/2}/\Gamma(r/2+1),$$

where

$$\Gamma(z) \stackrel{\text{def}}{=} \int_0^\infty t^{z-1} e^{-t}\, dt.$$

This implies that

$$\varepsilon(k,r) \leqslant \frac{4\pi}{(k\omega_r)^{\frac{1}{r}}}. \tag{3}$$

An even better bound was obtained by Kabatiansky and Levenshtein (see [CS99]) which gives that asymptotically (as $r$ tends to infinity)

$$\varepsilon(k,r) \leqslant \frac{4^{1-c_{\mathrm{KL}}/2}\pi}{(k\omega_r)^{\frac{1}{r}}}(1+o(1)). \tag{4}$$

This last upper bound can be used to obtain the following corollary of Theorem 2.

**Corollary 3.** *For each even $d$ there is a constant $C_d$ such that any Abelian $(d/2, 0)$-Cayley graph on $n$ vertices has*

$$\lambda_2 \geqslant d - dC_d n^{-4/d}.$$

*Furthermore we may take $C_d \precsim \frac{4^{1-c_{\mathrm{KL}}}\pi}{e}$.*

**Proof.** By using Theorem 2 with the Kabatiansky–Levenshtein bound we obtain

$$\lambda_2 \geqslant d - \frac{(4\pi)^2(1+o(1))}{\left(2^{c_{\mathrm{KL}}} n^{2/d} \omega_{d/2}^{2/d}\right)^2}.$$

Notice now that

$$\omega_{d/2}^{-4/d} = \left(\frac{\Gamma(d/4+1)}{\pi^{d/4}}\right)^{4/d} = \frac{d}{4e\pi}(1+o(1)),$$

as $d \to \infty$. This implies that

$$\lambda_2 \geqslant d - \frac{4\pi}{e4^{c_{\mathrm{KL}}}} dn^{-4/d}(1+o(1)),$$

as $d \to \infty$. $\square$

**Remarks.**

1. Using the sphere packing bound (3) instead of the Kabatiansky–Levenshtein bound yields the slightly worse lower bound

$$\lambda_2 \geqslant d - \frac{4\pi}{e} dn^{-4/d}(1+o(1)).$$

2. Despite that the sphere packing bound is asymptotically worse than the Kabatianski–Levensthein bound, it is sharp when $r = 1$. In this case the sphere packing-bound gives that

$$\varepsilon(n, 1) \leqslant \frac{2\pi}{n}$$

which implies that in the case $d = 2, r = 1, t = 0$:

$$\lambda_2 \geqslant 2 - \frac{4\pi^2}{n^2}.$$

This bound is tight when $n$ tends to infinity, since in this case the Abelian Cayley graph with the lowest $\lambda_2$ is nothing but a single cycle of length $n$, for which $\lambda_2 = 2 - 2\cos(2\pi/n)$.

## 3. A covering argument

The basic idea in this section, that is also used in [FT02], is to take a Dirichlet eigenfunction (see [Fri93]), $f$, of the universal cover of an Abelian Cayley graph, $G$, and to "push it forward" (via the covering map, $\pi$) to a function, $\pi_* f$, on $G$. These notions will be made precise below. First we prove a lemma that will be applied to $\pi_* f$ to get a $\lambda_2$ bound; a very similar lemma appears in [FT02].

**Lemma 4.** *Let $G = (V, E)$ be a $d$-regular graph with adjacency matrix $A$, and let $v$ be a vector indexed on the vertices, $V$. Let $\nu$ be the number of vertices where $v$ is not* 0; *i.e.*,

*$\nu$ is the size of the support of $v$. Then*

$$\lambda_2 = \lambda_2(A) \geqslant \frac{n\mathscr{R}(v) - d\nu}{n - \nu}$$

*(where $\mathscr{R}(v) \stackrel{\text{def}}{=} \frac{(Av,v)}{(v,v)}$ denotes the Rayleigh quotient of $v$).*

**Proof.** Let $e = \vec{1}/\sqrt{n}$. Then $v' = v - (v,e)e$ is perpendicular to $e$, and so $\lambda_2 \geqslant \mathscr{R}(v')$. On the other hand we easily see that

$$(Av', v') = (Av, v) - (v,e)^2 d \quad \text{and} \quad (v', v') = (v, v) - (v, e)^2,$$

and so

$$\mathscr{R}(v') = \frac{(Av, v) - (v,e)^2 d}{(v,v) - (v,e)^2} = \frac{(v,v)\mathscr{R}(v) - (v,e)^2 d}{(v,v) - (v,e)^2}. \tag{5}$$

Since $\mathscr{R}(v) \leqslant d$, we see that for fixed positive $\alpha$ we have

$$\frac{\alpha\mathscr{R}(v) - \beta d}{\alpha - \beta} = d + \frac{\alpha\bigl(\mathscr{R}(v) - d\bigr)}{\alpha - \beta}$$

is decreasing in $\beta$ (for $0 < \beta < \alpha$). So to bound the rightmost side of Eq. (5) (which is the case where $\alpha = (v,v)$ and $\beta = (v,e)^2$) it suffices to give an upper bound on $(v,e)^2$. But if the support of $v$ is $U$, and $1_U$ is the characteristic function of $U$ (i.e., the function that is 1 on $U$ and 0 elsewhere), then

$$(v,e)^2 = (v, 1_U)^2/n \leqslant (v,v)(1_U, 1_U)/n = (v,v)\nu/n.$$

It follows that

$$\lambda_2 \geqslant \mathscr{R}(v') \geqslant \frac{(v,v)\mathscr{R}(v) - (v,v)\nu d/n}{(v,v) - (v,v)\nu/n},$$

which proves the lemma. □

If $H$ is a normal subgroup of $G$, we can form $Y = \mathrm{Cay}(G/H; \{g_i\}; \{\widetilde{g_i}\})$; the natural map from $G$ to $G/H$ gives rise to a graph homomorphism [5] $\pi: X \to Y$ (this $\pi$ is a *covering map* in the usual sense, as in [Fri93]). We define a map, $\pi_*$, from functions on $G$ to functions on $G/H$ via

$$(\pi_* f)(y) = \sum_{x \in \pi^{-1}(\{y\})} f(x).$$

**Proposition 5.** *Let $\pi: X \to Y$ be as above. Let $f$ be a finitely supported non-negative function on the vertices of $X$ such that $A_X f \geqslant \lambda f$ (at each vertex) for some $\lambda \geqslant 0$. Then $A_Y \pi_* f \geqslant \lambda \pi_* f$; hence $\mathscr{R}_{A_Y}(\pi_* f) \geqslant \lambda$.*

[5] A graph homomorphism is a map from vertices and edges of one graph to another that maps incident vertices and edges from the first graph to those of the second.

**Proof.** We easily verify that $A_Y\pi_* = \pi_* A_X$ (this holds for any covering map, $\pi$, of graphs). Clearly the inequality $A_X f \geqslant \lambda f$ is preserved upon applying $\pi_*$, and we can conclude

$$A_Y\pi_* f = \pi_* A_X f \geqslant \pi_*(\lambda f) = \lambda\pi_* f. \qquad \square$$

Let $\mathbf{Z}$ be the integers, and let $C = \mathbf{Z}/2\mathbf{Z}$ be the cyclic group of order 2. Let $U = U(r,t)$ be the Cayley graph $\mathrm{Cay}(G_{\mathrm{univ}}, \{g_i\}, \{\widetilde{g}_i\})$ where $G_{\mathrm{univ}} = \mathbf{Z}^r \times C^t$, and $g_1, \ldots, g_r$ and $\widetilde{g}_1, \ldots, \widetilde{g}_t$ are 1 in the appropriate component and 0 in all other components. Then any $(r,t)$-Cayley graph, $X$, over an Abelian group, $G$, admits a map $\pi\colon U \to X$ (via the realization of $G$ as a quotient of $G_{\mathrm{univ}}/R$, where $R$ consists of the "relations satisfied by $X$'s generators").

Given an even integer $m \geqslant 1$ let $f$ be defined on $G_{\mathrm{univ}}$ via

$$f(a_1, \ldots, a_r; \widetilde{a}_1, \ldots, \widetilde{a}_t) = \begin{cases} \sin(\pi a_i/m) \cdots \sin(\pi a_r/m) & \text{if } 1 \leqslant a_i \leqslant m-1 \text{ for all } i, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f$ is non-negative and satisfies $A_U f \geqslant \lambda f$ where

$$\lambda = t + 2r\cos(\pi/m). \tag{6}$$

**Theorem 6.** *Let $d$ be a fixed positive integer, and let $\mu \in [0,1]$. Then we have that any $d$-regular Abelian Cayley graph, $X$, on $n$ vertices has*

$$\lambda_2(X) \geqslant d - C_d d n^{-4/d} + o(n^{-4/d})$$

*with*

$$C_d \leqslant (1-\mu)^{-1}\mu^{-4/d}\pi^2/2.$$

*In particular, as $d \to \infty$ we can take*

$$C_d \precsim \pi^2/2.$$

**Proof.** Choose the largest even $m$ with $(m-1)^{d/2} \leqslant n\mu$. Consider a $\pi_* f$ as above. Its support is on a set of size

$$v = (m-1)^r 2^t \leqslant (m-1)^{d/2} \leqslant n\mu,$$

provided that $m \geqslant 6$. Also $\mathscr{R}(\pi_* f) \geqslant t + 2r\cos(\pi/m)$ according to Eq. (6). We therefore have, using Lemma 4,

$$\lambda_2 \geqslant \frac{t + 2r\cos(\pi/m) - \mu d}{1-\mu}.$$

We use the following lower bound for the numerator

$$t + 2r\cos(\pi/m) - \mu d \geqslant d(1-\mu) - r(\pi/m)^2.$$

Furthermore

$$r(\pi/m)^2 \leqslant d(\pi^2/2)m^{-2}$$

and

$$m^{-2} = \bigl(1+o(1)\bigr)(n\mu)^{-4/d}.$$

This proves the first claim about $C_d$. The second follows from taking, say, $\mu = 1/d$. $\square$

## 4. An upper bound

It is easy to see that there are graphs with $\lambda_2 \leqslant d - cn^{-4/d}$, by considering $(\mathbf{Z}/m\mathbf{Z})^{d/2}$ and standard generators (that have 0's in all components except one, in which they have a 1), in which case $\lambda_2 = d-2+2\cos(2\pi/m)$. The goal of this section is to prove the stronger statement that there are Abelian Cayley graphs for which $\lambda_2 \leqslant d - \Omega(dn^{-4/d})$.

Consider random $(d/2, 0)$-Cayley graphs over $\mathbf{Z}/n\mathbf{Z}$, where we assume that $n$ is a prime number congruent to 1 modulo 4 (the case of 3 modulo 4 is handled similarly[6]). Let $X$ be the Cayley graph over $\mathbf{Z}/n\mathbf{Z}$, with generators $g_1, g_2, \ldots, g_r$, where the $g_i$'s are chosen uniformly at random in $\mathbf{Z}/n\mathbf{Z} \setminus \{0\}$. The eigenvalues of $A_X$ are simply the quantities

$$\lambda^u = 2\sum_{i=1}^{r} \cos\frac{2\pi g_i u}{n},$$

where $u$ ranges over $\{0, 1, \ldots, n-1\}$. To prove that with high probability the $\lambda^u$'s for $u \neq 0$ are bounded away from $d = 2r$ we use straightforward large deviation techniques, that is, we use the Markov exponential inequality which states that for any reals $t > 0$, $\varepsilon > 0$, and any $u \in \{1, \ldots, n-1\}$ we have

$$\operatorname{Prob}\bigl\{\lambda^u \geqslant d(1-\varepsilon)\bigr\} \leqslant \frac{\mathbf{E}(e^{t\lambda^u})}{e^{td(1-\varepsilon)}}. \tag{7}$$

One reason for using an exponential function in Markov's inequality is that $\mathbf{E}(e^{t\lambda^u}) = (g(t))^r$, where $g(t) = \mathbf{E}(e^{2t\cos\frac{2\pi g_1 u}{n}}) = \frac{1}{n-1}\sum_{i=1}^{n-1} e^{2t\cos\frac{2\pi i}{n}}$. We bound this function by an integral as follows

$$g(t) = \frac{1}{n-1}\left[\sum_{i=1}^{\frac{n-1}{4}} e^{2t\cos\frac{2\pi i}{n}} + \sum_{i=\frac{n+3}{4}}^{\frac{3n-3}{4}} e^{2t\cos\frac{2\pi i}{n}} + \sum_{i=\frac{3n+1}{4}}^{n-1} e^{2t\cos\frac{2\pi i}{n}}\right] \tag{8}$$

$$\leqslant 1 + \frac{2}{n-1}\sum_{i=1}^{\frac{n-1}{4}} e^{2t\cos\frac{2\pi i}{n}} \tag{9}$$

$$\leqslant 1 + \frac{n}{n-1}\int_{-\frac{1}{4}}^{\frac{1}{4}} e^{2t\cos(2\pi x)}\,dx. \tag{10}$$

[6] We only need modify the limits of summation in Eq. (8) and make minor modifications that result.

Hence

$$\operatorname{Prob}\left\{\lambda^u \geqslant d(1-\varepsilon)\right\} \leqslant \frac{\left(1+\frac{n}{n-1}\int_{-\frac{1}{4}}^{\frac{1}{4}} e^{2t\cos(2\pi x)}\,dx\right)^r}{e^{td(1-\varepsilon)}}. \tag{11}$$

We choose $t = \frac{1}{4\varepsilon}$ in this inequality and estimate the integral in the right-hand term by using Laplace's method; recall that Laplace's method (see for instance Section 5.1 in [BH86]) gives the following asymptotic expansion for an integral of type $\int_a^b e^{t\phi(x)}\,dx$ where $\phi(t)$ is a sufficiently smooth function which attains its unique maximum in $a < t_0 < b$ with $\phi''(t_0) \neq 0$:

$$\int_a^b e^{t\phi(x)}\,dx = e^{t\phi(t_0)}\sqrt{-\frac{2\pi}{t\phi''(t_0)}}\big(1+O(1/t)\big),$$

when $t$ tends to infinity. In our case we obtain

$$\int_{-\frac{1}{4}}^{\frac{1}{4}} e^{2t\cos(2\pi x)}\,dx = e^{2t}\sqrt{\frac{1}{4\pi t}}\big(1+O(1/t)\big).$$

Plugging this asymptotic expansion in (11) yields that for fixed $d$

$$\begin{aligned}\operatorname{Prob}\left\{\lambda^u \geqslant d(1-\varepsilon)\right\} &\leqslant \frac{\left(\frac{n}{n-1}\right)^{\frac{d}{2}} e^{td}(4\pi t)^{-d/4}\big(1+O(1/t)\big)^{d/2}}{e^{td\left(1-1/(4t)\right)}}\\ &= \big(1+O(\varepsilon)\big)\left(\frac{\varepsilon n^2 e}{(n-1)^2\pi}\right)^{d/4}.\end{aligned} \tag{12}$$

We notice now that by the union bound

$$\begin{aligned}\operatorname{Prob}\{\lambda_2(X) \geqslant d(1-\varepsilon)\} &\leqslant \sum_{u=1}^{n-1}\operatorname{Prob}\left\{\lambda^u \geqslant d(1-\varepsilon)\right\}\\ &\leqslant (n-1)\,(1+O(\varepsilon))\left(\frac{n^2\varepsilon e}{(n-1)^2\pi}\right)^{d/4}.\end{aligned}$$

This proves the following theorem.

**Theorem 7.** *Fix an even integer $d \geqslant 2$. For any $\theta \in [0,1]$, we have*

$$\operatorname{Prob}\left\{\lambda_2(X) \geqslant d - d\frac{\theta\pi}{e}n^{-4/d}\right\} \leqslant \left(1+O(n^{-\frac{4}{d}})\right)\theta^{d/4}.$$

The bound on the deviations of $\lambda_u$ obtained by the approach in this section is sharp. However the union bound is not sharp and is the reason that we do not recover what happens for random Cayley graphs of degree 2 defined on $\mathbf{Z}/n\mathbf{Z}$.

## 5. Concluding remarks

If $G$ is an arbitrary finite group, not necessarily Abelian, we may still consider the $d$-regular Cayley graph $X = \mathrm{Cay}(G; \{g_i\}; \{\widetilde{g}_i\})$. If $G'$ denotes the commutator subgroup of $G$, then it is well-known that $G$ has $n_1 = |G/G'|$ Abelian characters. Thus, by the argument of Section 2, we deduce in a similar way that

$$\lambda_2 \geqslant d - O(d/n_1^{4/d}).$$

As a consequence, we see that if $X_j = \mathrm{Cay}(G_j; \{g_i^j\}; \{\widetilde{g}_i^j\})$ is a sequence of $d$-regular graphs with $\lambda_2(X_j)$ bounded away from $d$, we must necessarily have that indices $[G_j : G'_j]$ must be bounded as $j$ tends to infinity. In particular, if we want to construct an infinite sequence of $d$-regular Ramanujan graphs using Cayley graphs, we must look in the category $\mathscr{C}_t$ of finite groups whose commutator subgroup has a fixed bounded index $t$ in the group. The category $\mathscr{C}_1$ is the set of so-called perfect groups which is a subset of non-solvable groups. The results of this paper justify the intuitive idea that the search for $d$-regular Ramanujan graphs, $X_j$, with $X_j$ Cayley graphs whose number of vertices tends to infinity, cannot be made in the category of groups where the commutator subgroup has bounded size.

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