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The Fibonacci Zeta Function

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Abstract

We consider the lacunary Dirichlet series obtained by taking the reciprocals of the s-th powers of the Fibonacci numbers. This series admits an analytic continuation to the entire complex plane. Its special values at integral arguments are then studied. If the argument is a negative integer, the value is algebraic. If the argument is a positive even integer, the value is transcendental by Nesterenko's work. This is a result of Duverney, Ke. Nishioka, Ku. Nishioka and Shiokawa. We present a simplified proof of their result. If the argument is 1, the value has been shown to be irrational by André-Jeannin. We present a slight modification of Duverney's proof of this fact. At the same time, we highlight the "modular connection" of these questions as well as signal some new results in the theory of special values of q-analogues of classical Dirichlet L-functions.

1 Introduction

The sequence of Fibonacci numbers is defined by the recurrence relation

$$f_{n+1} = f_n + f_{n-1}, \quad n \geq 1$$

with initial values $f_0 = 0$ and $f_1 = 1$. The Fibonacci zeta function is the series

$$F(s) := \sum_{n=1}^{\infty} f_n^{-s}.$$

Since the n-th Fibonacci number has exponential growth, it is easy to see that the series converges for $\Re(s) > 0$. Its analytic continuation is easily derived using a technique the author and Sinha [12] used to derive the analytic continuation for the Riemann, Hurwitz and multiple Hurwitz zeta functions. Indeed, if $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = -1/\alpha$, then,

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

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Thus, using the binomial theorem,

$$\begin{aligned}(\alpha-\beta)^{-s}\sum_{n=1}^{\infty}1/f_n^s &= \sum_{n=1}^{\infty}\alpha^{-ns}\left(1-(\beta/\alpha)^n\right)^{-s}\\ &= \sum_{n=1}^{\infty}\alpha^{-ns}\sum_{j=0}^{\infty}\binom{-s}{j}(-1)^j(\beta/\alpha)^{nj}\\ &= \sum_{j=0}^{\infty}\binom{-s}{j}(-1)^j\frac{\beta^j/\alpha^{s+j}}{1-(\beta^j/\alpha^{s+j})}.\end{aligned}$$

The right hand side is easily seen to converge for all complex values of s. From this derivation, we immediately see that the special values of $F(s)$ are all algebraic (and in fact lying in the quadratic field $\mathbb{Q}(\sqrt{5})$) when s is either zero or a negative integer, since in these cases the sum is a finite sum.

In this paper, we will be interested in the special values $F(k)$ when k is a natural number. What kind of numbers are these? Are they transcendental? Are they algebraic? It turns out that much like the case with the Riemann zeta function, the special values $F(2k)$ are all transcendental and the nature of $F(2k+1)$ is still unknown. Analogous questions can be asked for any second order recurrence sequence. The discussion below extends to some of these sequences but not all. There are some subtle issues that enter into the derivations that seem at present to be insurmountable. We discuss these issues in the final section of the paper. For the sake of elegance of exposition, we focus on the Fibonacci sequence first.

The problem of evaluating $F(1)$ seems to go back to Laisant in 1899 (see [11]). Later that year, Edmund Landau [11] addressed the question and was unable to give a definite answer. However, he noted that the sum

$$\sum_{n=1}^{\infty}\frac{1}{f_{2n+1}}$$

can be expressed as special values of classical theta functions, giving us the first hint of a "modular connection." More precisely, Landau proved that

$$\sum_{n=1}^{\infty}\frac{1}{f_{2n+1}}=\frac{\sqrt{5}}{4}\theta_2^2\left(\frac{3-\sqrt{5}}{2}\right),$$

where

$$\theta_2(q)=\sum_{n=-\infty}^{\infty}q^{(n+1/2)^2}.$$

Beyond this, Landau was unable to say if this number is transcendental but this is now known due to recent advances in transcendental number theory.

Our current state of knowledge on this problem is recorded in the following two theorems:

Theorem 1.1 (André-Jeannin, 1989) *$F(1)$ is irrational.*

Theorem 1.2 (Duverney, Ke. Nishioka, Ku. Nishioka, Shiokawa, 1998, [6]) *$F(2k)$ is transcendental for all $k \geq 1$.*

As will be explained below, the essential ingredient in the proof of Theorem 1.2 is a deep theorem of Nesterenko regarding the transcendence of special values of Eisenstein series, proved in 1996. We will attempt to explain why the case of even exponents can be solved and where the difficulty lies for the odd exponents.

The proof of Theorem 1.2 is somewhat "easier" than Theorem 1.1. But both proofs require our entry into the world of q-series. Theorem 1.1 requires the q-exponential and q-logarithm functions and the identities seem to suggest a connection to Ramanujan's "mock-theta" world.

For the most part, this paper is a survey of known results. However, the presentation and arrangement of ideas is new. In particular, the emphasis on the use of the q-exponential and q-logarithm in the proof of Theorem 1.1 following [5] is with a view to give conceptual unity to these isolated results. We also introduce q-analogues of Dirichlet L-series $L(s, \chi)$ and report on the transcendence of some of their special values. These results will appear in a forthcoming paper [3].

2 Nesterenko's Theorem

To facilitate the proof of Theorem 1.2, we review Nesterenko's theorem regarding transcendental values of Eisenstein series. Recall that the Eisenstein series of weight 2, 4, and 6 for the full modular group are given by

$$E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n,$$

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

where $\sigma_j(n) = \sum_{d|n} d^j$. We will make fundamental use of:

Theorem 2.1 (Nesterenko, 1996) *For any q with $|q| < 1$, the transcendence degree of the field*

$$\mathbb{Q}(q, E_2(q), E_4(q), E_6(q))$$

is at least 3. Thus, for q algebraic, $E_2(q), E_4(q)$, and $E_6(q)$ are algebraically independent.

Let us also recall that the general Eisenstein series

$$E_{2k}(q) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

where B_{2k} is the $2k$-th Bernoulli number, is a polynomial in E_4 and E_6 (see for example, [15]).

We also record here the following theorems.

Theorem 2.2 *For $k \geq 1$ and $m \geq 1$, we have that $E_{2k}(q^m)$ is algebraic over the field generated by E_2, E_4, E_6.*

Theorem 2.3 *Let K be the field generated by E_2, E_4, E_6 over the rational numbers. Suppose that f is a non-constant function which is algebraic over the function field K. If α is an algebraic number with $0 < |\alpha| < 1$ and $f(\alpha)$ is defined, then $f(\alpha)$ is transcendental.*

For proofs, we refer the reader to Lemmas 2 and 3 of [6].

3 Proof of Theorem 1.2

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = -1/\alpha$. Then,

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

as is easily verified by induction. Thus, the study of the series $F(k)$ reduces to the study of the series

$$\sum_{n=1}^{\infty} \frac{1}{(\alpha^n - \beta^n)^k},$$

which is

$$(-1)^k \sum_{n=1}^{\infty} \frac{1}{(\beta^n - (-1)^n \beta^{-n})^k}. \tag{3.1}$$

Breaking up the sum into n even and n odd leads us to consider the two series

$$A_k(q) = \sum_{n=1}^{\infty} \frac{1}{(q^n - q^{-n})^k},$$

and

$$B_k(q) = \sum_{n=1}^{\infty} \frac{1}{(q^n + q^{-n})^k}.$$

For then, we can re-write (3.1) as

$$(-1)^k \left(A_k(\beta^2) + B_k(\beta) - B_k(\beta^2)\right),$$

We now look at $A_k(q)$ and $B_k(q)$ separately. Clearly,

$$(-1)^k A_k(q) = \sum_{n=1}^{\infty} \frac{q^{nk}}{(1 - q^{2n})^k}. \tag{3.2}$$

Now for $|q| < 1$, we have

$$\frac{1}{1-q} = \sum_{m=0}^{\infty} q^m.$$

Differentiating this $(k-1)$ times with respect to q leads to

$$\frac{(k-1)!}{(1-q)^k} = \sum_{m=k-1}^{\infty} m(m-1)\cdots(m-(k-2))q^{m-k+1}.$$

Thus,

$$(-1)^k A_k(q)(k-1)! = \sum_{n=1}^{\infty} q^{nk} \sum_{m=k-1}^{\infty} m(m-1)\cdots(m-(k-2))q^{2n(m-k+1)}$$

which is equal to

$$\sum_{n\geq 1; m\geq k-1} q^{n(2m-k+2)} m(m-1)\cdots(m-(k-2)).$$

Now suppose $k = 2j$ is even. Then,

$$A_{2j}(q)(2j-1)! = \sum_{n\geq 1; m\geq 2j-1} q^{2n(m-j+1)} m(m-1)\cdots(m-2j+2).$$

For $j = 1$, we have

$$A_2(q) = \sum_{n,m\geq 1} q^{2nm}m = \sum_{r=1}^{\infty} \sigma_1(r)q^{2r}.$$

We immediately recognize that this is, apart from the constant term, the weight 2 Eisenstein series, $E_2(q^2)$ (upto a constant). For $j = 2$, we have

$$A_4(q)3! = \sum_{n\geq 1; m\geq 3} q^{2n(m-1)}m(m-1)(m-2).$$

Putting $m-1$ as d, we see that $m(m-1)(m-2)$ is $(d+1)d(d-1)$ so that setting $r = nd$, the above can be re-written as

$$\sum_{r=1}^{\infty} q^{2r}\left(\sum_{d|r} d(d^2-1)\right)$$

and again we see that this is a linear combination of $E_2(q^2)$ and $E_4(q^2)$ (minus the constant terms). A similar calculation shows that $A_6(q)$ is a linear combination of $E_2(q^2)$, $E_4(q^2)$ and $E_6(q^2)$ (minus the constant terms). Since every Eisenstein series $E_{2j}(q)$ with $j \geq 2$ is a polynomial is $E_4(q)$ and $E_6(q)$, we see that $A_{2j}(q)$ is a non-zero polynomial in $E_2(q^2), E_4(q^2)$ and $E_6(q^2)$ with rational coefficients. We now invoke Theorem 2.1 to deduce that $E_2(\beta^2), E_4(\beta^2)$ and $E_6(\beta^2)$ are algebraically independent. Hence, $A_{2j}(\beta^2)$ is transcendental. (From (3.2), it is clear that $A_{2j}(\beta^2) \neq 0$.)

Now, what about $B_k(q)$? Here again, we need only observe that

$$\frac{(k-1)!}{(1+q)^k} = \sum_{m=k-1}^{\infty} m(m-1)\cdots(m-(k-2))q^{m-k+1}(-1)^{m-k+1},$$

so that

$$B_k(q) = \sum_{n=1}^{\infty} \frac{q^{nk}}{(1+q^{2n})^k},$$

and

$$B_k(q)(k-1)! = \sum_{n\geq 1; m\geq k-1} q^{n(2m-k+2)}(-1)^{m-k+1}m(m-1)\cdots(m-(k-2)).$$

For $k = 2j$, we have

$$B_{2j}(q)(2j-1)! = \sum_{n\geq 1; m\geq 2j-1} q^{2n(m-j+1)}(-1)^{m-1}m(m-1)\cdots(m-(2j-2)).$$

For $j = 1$, we have

$$-B_2(q) = \sum_{n,m \geq 1} q^{2nm}(-1)^m m = \sum_{r=1}^{\infty} q^{2r} \left(\sum_{d|r} (-1)^d d \right).$$

Now let us note that

$$-B_2(q) + A_2(q) = 2 \sum_{r=1}^{\infty} q^{2r} \left(\sum_{d|r, d \text{ even}} d \right).$$

Writing $r = 2r_1$ and $d = 2d_1$, we may rewrite this as

$$-B_2(q) + A_2(q) = 2 \sum_{r_1=1}^{\infty} q^{4r_1} \left(\sum_{d_1|r_1} 2d_1 \right).$$

This is equal to

$$4 \sum_{r_1=1}^{\infty} q^{4r_1} \sigma_1(r_1) = 4A_2(q^2).$$

It follows that

$$B_2(q) = A_2(q) - 4A_2(q^2).$$

Since the sum in question is (3.1) which is equal to

$$(-1)^k \left(A_k(\beta^2) + B_k(\beta) - B_k(\beta^2) \right),$$

we deduce from the above calculations that for $k = 2$, the series $F(2)$ can be expressed as a polynomial in $E_2(\beta^2), E_2(\beta^4)$ and $E_2(\beta^8)$. Now, the function

$$f(q) := A_2(q^2) + B_2(q) - B_2(q^2)$$

is a non-constant polynomial in $E_2(q^2), E_2(q^4)$ and $E_2(q^8)$. By Theorem 2.2, $E_2(q^m)$ is algebraic over K. Thus, $f(q)$ is algebraic over K and by Theorem 2.3, the specialization $f(\beta)$ is transcendental. Proceeding inductively, we see that in the general case, the non-constant function

$$\sum_{n=1}^{\infty} \frac{1}{(q^n - (-1)^n q^{-n})^k}$$

is algebraic over the function field K. By Theorem 2.3, the specialization (3.1) is transcendental. This completes the proof of Theorem 1.2.

The method can be used to treat the case of the series

$$\sum_{n=1}^{\infty} \frac{1}{f_{2n+1}^k}.$$

We need only make a few elementary observations and reduce the calculation to the above. First, we observe that the sum in question is essentially (apart from an algebraic factor) $B_k(\beta) - B_k(\beta^2)$. If k is even, we are done by the above argument. If k is odd, the argument is more delicate. We make some remarks on this point at the end of the paper.

4 The Case of Odd Arguments

The situation of the special values of the Fibonacci zeta function at odd arguments is somewhat similar to the case of the Riemann zeta function and there are some striking analogies between the two. In later sections, we will outline the proof of André Jeannin that $F(1)$ is irrational. This can be viewed as the "Fibonacci analogue" of Apéry's theorem that $\zeta(3)$ is irrational. As for $\zeta(2k+1)$, we know from the work of Ball and Rivoal [2] that there is a positive constant $c > 0$ such that

$$\dim_{\mathbb{Q}} \mathbb{Q}(\zeta(3), \zeta(5), ..., \zeta(2a+1)) \geq c \log a.$$

In particular, there are infinitely many $\zeta(2k+1)$ that are irrational. There is a "Fibonacci analogue" of this result too. To state it, we need to introduce the following q-analogues of the Riemann zeta function, namely,

$$\zeta_q(s) = \sum_{n=1}^{\infty} q^n \left(\sum_{d|n} d^{s-1} \right). \tag{4.1}$$

We will not go into a discussion of why this can be considered as the q-analogue of the Riemann zeta function here except to say that some hint will be given in the next section when we review q-series. The problem of transcendence of $F(s)$ at $s = k$ leads to the study of $\zeta_q(s)$ at $s = k$ for $q = (\sqrt{5} - 1)/2$ and other algebraic numbers. When $s = 2k$, then $\zeta_q(2k)$ is the classical Eisenstein series (without the constant term) $E_{2k}(q)$. Since these are polynomials in $E_4(q)$ and $E_6(q)$ (or equal to $E_2(q)$ when $k = 1$) with algebraic coefficients, the theory of Nesterenko enters into the discussion in a fundamental way in the even case. In the odd case, nothing is known about $\zeta_q(2k+1)$ and these are not modular forms. Their nature is a mystery. Motivated by the work on $\zeta(2k+1)$, Krattenthaler, Rivoal and Zudilin [10] proved:

Theorem 4.1 (2006) *Fix q such that $1/q$ is an integer different from ± 1. Then, infinitely many $\zeta_q(2k+1)$ are irrational. More precisely,*

$$\dim_{\mathbb{Q}} \mathbb{Q}(\zeta_q(3), ..., \zeta_q(2a+1)) \geq c\sqrt{a}.$$

5 Introduction to q-series

It is becoming increasingly clear that the theory of q-series, often relegated to the theory of combinatorics, actually will come to play a dominant conceptual role unifying large tracts of the mathematical world. Often called "quantum analogues", the q-analogues provide not only an appealing domain of investigation but also give rise to greater conceptual understanding of the world of "natural" numbers.

In its simplest form, the analogy is best understood by replacing the natural number n by $q^n - 1$ to obtain the "q-analogue." Until recently, mathematicians preferred

$$\frac{q^n - 1}{q - 1}$$

instead of $q^n - 1$, simply because

$$\lim_{q \to 1} \frac{q^n - 1}{q - 1} = n.$$

But the "Birch-Swinnerton-Dyer philosophy" has taken over in which we divide by the "obvious" factor eliminating the zero at $q = 1$ and study the constant term. Thus, the exponential function

$$e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

has the q-analogue:

$$E_q(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)}.$$

The logarithm function

$$-\log(1 - x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

would have the q-analogue

$$L_q(x) = \sum_{n=1}^{\infty} \frac{x^n}{q^n - 1}.$$

Let us quickly return to our original motivating question and study the number

$$\theta = \sum_{n=1}^{\infty} 1/f_n.$$

Since

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2} = -\frac{1}{\alpha},$$

we see that

$$\theta = (\alpha - \beta)\sum_{n=1}^{\infty} \frac{1}{\alpha^n - \beta^n} = (\alpha-\beta)\sum_{n=1}^{\infty}\frac{1}{\alpha^n - (-1/\alpha)^n} =$$

$$(\alpha-\beta)\sum_{n=1}^{\infty}\frac{(-\alpha)^n}{(-\alpha^2)^n - 1} = (\alpha-\beta)L_q(-\alpha),$$

with $q = -\alpha^2$. Thus, the original question of the algebraic nature of $F(1)$ reduces to the study of a special value of the q-logarithm.

Returning to the q-analogue of the Riemann zeta function, it would seem logical, in view of the discussion above, to define it as

$$Z_q(s) := \sum_{n=1}^{\infty} \frac{1}{(q^n-1)^s}.$$

That this is related to $\zeta_q(s)$ becomes evident when we write $\zeta_q(s)$ given by (4.1) as a Lambert series:

$$\zeta_q(s) = \sum_{d=1}^{\infty}\sum_{e=1}^{\infty} d^{s-1}q^{de} = \sum_{d=1}^{\infty}\frac{d^{s-1}q^d}{1-q^d}.$$

Indeed,

$$Z_q(s) = (-1)^s \sum_{n=1}^{\infty}\frac{1}{(1-q^n)^s}$$

and when $s = k$,

$$\frac{1}{(1-q^n)^k} = \frac{1}{(k-1)!}\sum_{m=k-1}^{\infty} m(m-1)\cdots(m-(k-2))q^{n(m-k+1)}.$$

Putting $d = m - k + 2$ and writing $m(m-1)\cdots(m-(k-2))$ as $d(d+1)\cdots(d+k-2)$ we see that

$$\sum_{n=1}^{\infty}\frac{1}{(1-q^n)^k}$$

is a linear combination of series of the form

$$\sum_{d,n=1}^{\infty} d^j q^{dn} = \sum_{d=1}^{\infty} \frac{d^j q^d}{1-q^d}$$

and these are the values $\zeta_q(j+1)$. So this gives some clue as to why the q-Riemann zeta function is defined as (4.1). For other perspectives on this definition, we refer the reader to [9].

6 The q-exponential and q-logarithm

Now

$$E_q(x) - E_q(x/q) = \sum_{n=1}^{\infty} \frac{x^n - (x/q)^n}{(q^n-1)\cdots(q-1)} = \sum_{n=1}^{\infty} \frac{(x/q)^n}{(q^{n-1}-1)\cdots(q-1)} = \frac{x}{q} E_q(x/q).$$

Thus, we get the functional equation

$$E_q(x) = (1 + x/q) E_q(x/q).$$

Iterating this, we get

$$E_q(x) = \left(1 + \frac{x}{q}\right)\left(1 + \frac{x}{q^2}\right)\cdots$$

so that we deduce:

Theorem 6.1 *For $|q| > 1$, we have*

$$E_q(x) = \prod_{n=1}^{\infty} \left(1 + \frac{x}{q^n}\right).$$

We recognize immediately that the q-exponential function has some resemblance to the Dedekind η-function and thus we see the emergence of a "modular" link.

Let us look at the q-logarithm. We have

$$L_q(x) - L_q(x/q) = \sum_{n=1}^{\infty} \frac{x^n}{q^n-1} - \sum_{n=1}^{\infty} \frac{(x/q)^n}{q^n-1} = \sum_{n=1}^{\infty} \frac{q^n x^n - x^n}{q^n(q^n-1)} = \sum_{n=1}^{\infty} (x/q)^n = \frac{x/q}{1-x/q} = \frac{x}{q-x},$$

provided $|x| < |q|$. Iterating this, we obtain

$$L_q(x) = \sum_{n=1}^{\infty} \frac{x}{q^n - x}.$$

This leads to

Theorem 6.2 *For* $|q| > \max(1, |x|)$*, we have*

$$L_q(x) = \sum_{n=1}^{\infty} \frac{x}{q^n - x}.$$

Taking the logarithmic derivative of $E_q(x)$, we deduce immediately that

Theorem 6.3 *For* $|q| > \max(1, |x|)$*, we have*

$$L_q(x) = x\frac{E_q'(-x)}{E_q(-x)},$$

where the derivative is with respect to x*.*

7 Proof of Theorem 1.1.

Now suppose that θ is rational and equal to $-A/B$ (say), with A, B coprime integers. Recall that we showed

$$\theta = \sqrt{5}L_q(-\alpha), \quad q = -\alpha^2, \quad \alpha = \frac{1+\sqrt{5}}{2}.$$

Then,

$$B\sqrt{5}L_q(-\alpha) + A = 0.$$

But

$$L_q(x) = x\frac{E_q'(-x)}{E_q(-x)}.$$

Thus, we may re-write this as

$$-B\sqrt{5}\alpha E_q'(\alpha) + AE_q(\alpha) = 0.$$

Using the q-series for $E_q(x)$, we obtain

$$A + \sum_{n=1}^{\infty} \frac{A - Bn\sqrt{5}}{(1+\alpha^2)(1-\alpha^4)\cdots(1-(-\alpha^2)^n)}(-\alpha)^n = 0.$$

We split this as

$$A + \sum_{n=1}^{N} \frac{A - Bn\sqrt{5}}{(1+\alpha^2)(1-\alpha^4)\cdots(1-(-\alpha^2)^n)}(-\alpha)^n = \\ - \sum_{n=N+1}^{\infty} \frac{A - Bn\sqrt{5}}{(1+\alpha^2)(1-\alpha^4)\cdots(1-(-\alpha^2)^n)}(-\alpha)^n.$$

We clear the denominators and estimate:

$$|A(1+\alpha^2)\cdots(1-(-\alpha^2)^N) + \sum_{n=1}^{N}(A - Bn\sqrt{5})(-\alpha)^n(\cdots)| \\ \leq \sum_{n=N+1}^{\infty} |A + Bn\sqrt{5}| \frac{\alpha^n}{(\alpha^{2N+2}-1)\cdots(\alpha^{2n}-1)}.$$

To estimate this tail, we proceed as follows. The summand in the tail is easily seen to be

$$\ll \frac{n}{\alpha^n}$$

so that the tail is estimated by

$$\ll \sum_{n=N+1}^{\infty} \frac{n}{\alpha^n} \ll \frac{N}{\alpha^N}.$$

The number X_N (say) given by

$$X_N := A(1+\alpha^2)\cdots(1-(-\alpha^2)^N) + \sum_{n=1}^{N}(A - Bn\sqrt{5})(-\alpha)^n(\cdots)$$

lies in $\mathbb{Q}(\sqrt{5})$ and is an algebraic integer. We take the algebraic conjugate $\widetilde{X_N}$. Since the conjugate β of α satisfies $|\beta| < 1$, one finds

$$|\widetilde{X_N}| \ll N^2.$$

Thus,

$$|X_N \widetilde{X_N}| \ll N^3 \alpha^{-N},$$

which tends to zero as N tends to infinity. Now, $X_N \widetilde{X_N}$ is an integer, and so we conclude that either X_N or $\widetilde{X_N}$ is zero for N large. But since X_N and $\widetilde{X_N}$ are conjugates of each other, we deduce that both X_N and $\widetilde{X_N}$ are zero for N sufficiently large. But then, this implies that $\sqrt{5}$ is rational, a contradiction. This completes the proof.

8 q-analogues of Dirichlet L-series

In analogy with our definition of the q-Riemann zeta function (4.1), it seems natural to define for each Dirichlet character $\chi \pmod N$, the q-Dirichlet L-function by

$$L_q(s,\chi) := \sum_{n=1}^{\infty} q^n \left(\sum_{d|n} \chi(d) d^{s-1} \right).$$

As before, this can be written as a Lambert series of the form:

$$\sum_{d=1}^{\infty} \frac{\chi(d) d^{s-1} q^d}{1-q^d}.$$

For certain values of k and suitable characters χ, this function coincides with a classical Eisenstein series of level N. More precisely, if $\chi(-1) = (-1)^k$, and $k \geq 3$, then, $L_q(k,\chi)$ coincides (apart from a rational scalar factor) to an Eisenstein series relative to to the congruence subgroup $\Gamma_1(N)$ (see Theorem 4.5.1 of [4]). One has similar results for $k=1$ and $k=2$ also. In each of these cases, with q algebraic, one can deduce the transcendence of $L_q(k,\chi)$. The essential idea is to use the modularity of $L_q(k,\chi)$ and observe that if Δ is Ramanujan's cusp form of weight 12, then $L_q(k,\chi)^{12}/\Delta^k$ is a modular function of level N. Since the field of modular functions of level N is algebraic over the field $\overline{\mathbb{Q}}(j)$, where j denotes the j-function, we derive a contradiction to Nesterenko's therem if $L_q(k,\chi)$ is algebraic. The details of this result along with a discussion of the cases not covered by modularity will appear in [3].

9 Concluding Remarks

A natural question that arises is to what extent these results can be generalized for other second-order recurrence sequences. In [6], the authors show the following: let α, β be algebraic numbers with $\alpha \neq \beta$ and $|\beta| < 1$. Put

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad V_n = \alpha^n + \beta^n.$$

If $\alpha\beta = \pm 1$, then the numbers

$$\sum_{n=1}^{\infty} \frac{1}{U_n^{2s}}, \quad \sum_{n=1}^{\infty} \frac{1}{V_n^{2s}}$$

are transcendental for any positive integer s. If $\alpha\beta = 1$, then the number

$$\sum_{n=1}^{\infty} \frac{1}{V_n^s}$$

is transcendental for any positive integer s and if $\alpha\beta = -1$, then

$$\sum_{n=1}^{\infty} \frac{1}{U_{2n+1}^s}$$

is transcendental for any positive integer s. It would be interesting to investigate the cases not covered by this theorem.

If k is odd, it is possible to show that $B_k(q)$ is algebraic over the field generated by E_2, E_4 and E_6. But this is not a totally trivial deduction (see [6]). However, it is possible to deduce the transcendence result for the sum

$$\sum_{n=1}^{\infty} \frac{1}{f_{2n+1}}.$$

Indeed,

$$B_1(q) = \sum_{n=1}^{\infty} \frac{q^n}{(1+q^{2n})}.$$

We recall the classical Jacobi identity:

$$\theta(q)^2 = 1 + 4\sum_{n=1}^{\infty} \frac{q^n}{1+q^{2n}},$$

where

$$\theta(q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

is one of the basic theta functions. Now, $\theta(q)^{24}/\Delta$ is a modular function of level 4. Thus, it is algebraic over the field $\overline{\mathbb{Q}}(j)$ where j denotes the j-function. Since the j function can be expressed as a rational function involving E_4 and E_6, we immediately deduce the desired transcendence result. The discussion for

$$\sum_{n=1}^{\infty} \frac{1}{f_{2n+1}^k}$$

proceeds along similar lines.

There are still many open problems concerning reciprocal sums of Fibonacci numbers. For instance, is $F(3)$ irrational. Recently, the authors in [8] showed that $F(2s_1)$, $F(2s_2)$ and $F(2s_3)$ are algebraically independent if and only if the three integers s_1, s_2, s_3 are all distinct and at least one of them is even thus settling an old problem of whether $F(2)$, $F(4)$ and $F(6)$ are algebraically independent posed in [7]. No doubt, there are still more fascinating facts yet to be discovered about modularity and non-modulairty of the Fibonacci zeta function $F(s)$.

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