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Fundamental units and consecutive squarefull numbers

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We connect several seemingly unrelated conjectures of Ankeny, Artin, Chowla and Mordell to a conjecture of Erdös on consecutive squarefull numbers. We then study the Erdös conjecture and relate it to the *abc* conjecture. We also derive by elementary methods several unconditional results pertaining to the Erdös conjecture.

Keywords: Fundamental units; consecutive squarefull numbers; Diophantine equations; Erdös conjecture; *abc* conjecture.

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1. Introduction

If p is a prime $\equiv 1 \pmod 4$, the fundamental unit of $\mathbb{Q}(\sqrt{p})$ can be written as

$$\varepsilon = \frac{t + u\sqrt{p}}{2} > 1,$$

with $t^2 - pu^2 = -4$. In 1952, Ankeny, Artin and Chowla [1] showed that if h is the class number of the ring of integers of $\mathbb{Q}(\sqrt{p})$, then

$$\frac{uh}{t} \equiv B_{(p-1)/2} \pmod p,$$

where B_k denotes the kth Bernoulli number (see [15, p. 56]) defined by the generating function

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{k=2}^{\infty} \frac{B_k t^k}{k!}.$$

Now $B_k = 0$ for all odd k (see [16, Problem 2.1.8]). For $k \geq 2$ even, we have Euler's celebrated formula

$$\sum_{k=1}^{\infty} \frac{1}{n^k} = \frac{B_k(2\pi i)^k}{2k!}.$$

Ankeny, Artin and Chowla further conjectured that $u \not\equiv 0 \pmod{p}$. From their result, and the fact that $h(p) = O(p^{1/2} \log p)$, we see that for p sufficiently large, $u \not\equiv 0 \pmod{p}$ if and only if $B_{(p-1)/2} \not\equiv 0 \pmod{p}$.

In a similar vein, if $p \equiv 3 \pmod{4}$ is prime, Mordell [15, pp. 56–57] conjectured that the fundamental unit $t + u\sqrt{p}$ of the ring of integers of $\mathbb{Q}(\sqrt{p})$ satisfies $u \not\equiv 0 \pmod{p}$. He also showed that this happens if and only if

$$E_{(p-3)/2} \not\equiv 0 \pmod{p},$$

where E_k denotes the Euler number given by the power series for the secant function:

$$\sec x = \sum_{k=0}^{\infty} \frac{E_k x^k}{k!}.$$

Both of these conjectures are still open (see also [14]). It is not even known if the conjectures are true for infinitely many primes. They have been checked numerically for all primes $p < 10^7$ and the reader can find further details and exposition in [5].

In this paper, we relate these conjectures to a conjecture of Erdös (see [19, p. 234]) on consecutive squarefull numbers (or nearly consecutive squarefull numbers) and then apply the *abc* conjecture to move toward these conjectures. We also formulate what we call a strong Erdös conjecture which then implies that both the Ankeny–Artin–Chowla conjecture and the Mordell conjecture are true for almost all primes p (in the sense of density). More precisely, we show that the set of primes $p \leq x$ for which the conjecture is false is $O(x^\theta)$ for some $\theta < 1$.

Our methods are elementary and do not use anything deeper than the theory of the Brahmagupta–Pell (BP) equation. We also obtain some unconditional results in this context.

The study of consecutive squarefull numbers has also been undertaken (independently) by Chan [6] as well as Blomer and Schöbel [3]. Our work has some overlap with these works. In this context, we highlight our new contributions.

An integer r is called squarefull (or powerful) if, for all primes p dividing r, we have p^2 divides r. In 1970, Golomb [10] proved an estimate for the number of squarefull numbers. More precisely, if $k(x)$ denotes the number of squarefull numbers $\leq x$, then $k(x) \sim c\sqrt{x}$ with the constant $c = \zeta(3/2)/\zeta(3)$ as $x \to \infty$ where $\zeta(s)$ is the Riemann zeta function. Already in 1935, Erdös and Szekeres [8] gave the first result about $k(x)$:

$$k(x) = \frac{\zeta\left(\dfrac{3}{2}\right)}{\zeta(3)} x^{1/2} + O(x^{1/3})$$

as $x \to \infty$; see also [2]. Golomb seemed to be unaware of this when he wrote his paper. (Here and elsewhere, unless otherwise specified, all constants implied by the big "O" symbol are absolute.)

It is easy to see that there are infinitely many consecutive squarefull numbers. Indeed 8 and 9 are both squarefull, and if $n, n+1$ are both squarefull, then so are the consecutive numbers $4n(n+1)$ and $4n(n+1)+1 = (2n+1)^2$. But it is expected that there are not too many of them. More precisely, Erdös [7] conjectured (see also [19, p. 234]) that there is a constant $A > 0$ such that

$$S(N) = \#\{n \leqslant N \,|\, n \text{ and } n+1 \text{ are squarefull}\} < (\log N)^A.$$

More generally, for any fixed d, we expect the following *Strong Erdös Conjecture*:

$$S_d(N) = \#\{n \leqslant N \,|\, n \text{ and } n+d \text{ are squarefull}\} < (\log N)^A.$$

There is a heuristic argument to support these conjectures. By the Erdös–Szekeres theorem, the probability that a random number n is squarefull is approximately $1/\sqrt{n}$, so the number of $n \leq N$ such that n and $n+d$ are both squarefull is approximately

$$\sum_{n \leq N} \left(\frac{1}{\sqrt{n}}\right)^2 = \sum_{n \leq N} \frac{1}{n} \approx \log N.$$

Motivated by this reasoning, we formulate a strong Erdös conjecture: for some $A < 2$,

$$S_d(N) := \#\{n \leqslant N \,|\, n \text{ and } n+d \text{ are squarefull}\} < (\log N)^A.$$

It is not difficult to see that

$$S_1(N) \gg \log N.$$

Indeed, from our above example, we may consider the real quadratic field $\mathbb{Q}(\sqrt{2})$ in which $N(3 + 2\sqrt{2}) = 1$. If we write

$$U_n + V_n\sqrt{2} := (3 + 2\sqrt{2})^n, \quad n = 1, \ldots$$

with U_n, V_n integers, then it is clear that $U_n^2 - 2V_n^2 = 1$. Moreover, it is also clear that $2|V_n$ for otherwise, both U_n and V_n would be odd and hence $\equiv 1 \pmod 4$ which means $1 \equiv U_n^2 - 2V_n^2 \equiv -1 \pmod 4$, a contradiction. Therefore both U_n^2 and $2V_n^2$ are powerful and consecutive. The number of such numbers $\leq N$ is clearly $\gg \log N$.

Our first theorem is the following.

Theorem 1.1. *The strong Erdös conjecture implies that the Ankeny–Artin–Chowla conjecture and the Mordell conjecture are true for all but $O(x^\theta)$ primes $p \leq x$, for some $\theta < 1$.*

It therefore seems reasonable to ask if one can prove the Erdös conjecture. In this direction, there is an assortment of results. First, we show the following.

Theorem 1.2. *The abc conjecture implies $S_d(N) = O(N^\epsilon)$ for any $\epsilon > 0$.*

This theorem was also proved by Chan [6] who relied on a non-trivial result of Estermann [9]. Our proof of the above theorem is slightly simpler and will be of independent interest. Unconditionally, we also prove the following independent from d.

Theorem 1.3.

$$S(N) = O(N^{2/5}).$$

Chan [6] obtained a slightly weaker version, namely an estimate of

$$O(\tau_3(d)N^{2/5}\log^2 N),$$

where $\tau_3(d)$ is the number of ways of writing d as a product of three positive integers. If the explicit dependence on d is dropped, he obtains an improved estimate of

$$O(N^{7/19}\log N).$$

All of Chan's results rely on non-trivial results of Bombieri and Schmidt [4] as well as Evertse on estimates for the number of solutions of the Thue equation. These results comprise a deep chapter in transcendental number theory. By contrast, our approach is elementary. We will prove the above theorems using only rudimentary facts of the BP equation. Recently, Reuss [18] established a better error term in the case $d = 1$. He showed $S_1(N) = O(N^{29/100+\epsilon})$, for any $\epsilon > 0$. This result uses more sophisticated methods and relies on earlier deep work of Heath-Brown [11]. Earlier, Blomer and Schöbel [3] obtained the exponent of $61/180$.

It has also been asked by Erdös whether consecutive squarefull numbers may be obtained other than as solutions of appropriate equations of the form $mX^2 - nY^2 = 1$. We will show this is not the case (see Lemma 2.4).

Generalizations of these problems to consecutive k-full numbers have been studied by Blomer and Schöbel [3].

2. Preliminaries

Golomb [10] discussed consecutive integer pairs of squarefull numbers which fall into one or the other of two types. Type I consists of pairs of consecutive squarefull numbers one of which is a perfect square; and Type II consists of pairs of consecutive squarefull numbers neither of which is a perfect square. He showed that there are infinite families of examples where two consecutive integers are squarefull which correspond to the solutions through the theory of Pell's equation which give the solutions for Type I.

It may be enlightening to make some historical remarks here. The equation $x^2 - dy^2 = 1$ whose solutions give the pairs of squarefull numbers of Type I was labeled as Pell's equation by Euler. However, this is one instance (among many in mathematics) where a name has been attached in error and the same was so

commonly accepted in the literature that it stuck. Actually, Pell had little to do with Pell's equation. The first mention of the equation seems to date back to ancient India, starting with Brahmagupta, who developed the *chakravala* method to solve the equation in his Brahma Sphuta Siddhanta in 628 C.E., about a thousand years before Pell's time. The word "chakra" in Sanskrit means "wheel" and it is quite clear that Brahmagupta and his successors like Jayadeva and Bhaskaracharya (see [22]) were aware of the "cyclic" nature of all solutions. In modern language, we know these solutions really correspond to the cyclic group of units of a real quadratic field. The reader can find further information and the fascinating history about the method in Weil's book [22]. That is the reason why due to the long journey of this type of equation, we will call the equation

$$mX^2 - nY^2 = \pm 1 \tag{1}$$

as the generalized BP equation.

This Eq. (1) was studied by Walker [20] in 1967. He showed how the elementary theory of the BP equation can be used to derive the following. We record below his main result. To this end, we say $x\sqrt{m} + y\sqrt{n}$ is solution of (1), if $X = x$, $Y = y$ is a solution of (1). It is called a positive solution if both x and y are positive. Walker [20] has shown that among all solutions (x, y) with $x, y > 0$, there is (x_0, y_0) (say) in which both x and y have their least values. For such a solution, we call $x_0\sqrt{m} + y_0\sqrt{n}$ the smallest solution of (1).

Theorem 2.1. *If* (1) *is solvable for both of* $m, n > 1$ *and has* $x_0\sqrt{m} + y_0\sqrt{n}$ *as its smallest solution; and if* i *is a nonnegative integer, then the formula*

$$x_i\sqrt{m} + y_i\sqrt{n} = (x_0\sqrt{m} + y_0\sqrt{n})^{2i+1}$$

gives all positive solutions of (1).

Let us note that

$$(x_0\sqrt{m} + y_0\sqrt{n})^{2i+1} = \sum_{i=j}^{2i+1} \binom{2i+1}{j} (x_0\sqrt{m})^j (y_0\sqrt{n})^{2i+1-j}.$$

Thus, in each summand, either j is even or $2i + 1 - j$ is even so that $(x_0\sqrt{m} + y_0\sqrt{n})^{2i+1}$ has the form $x_i\sqrt{m} + y_i\sqrt{n}$.

Conjecture 2.2. *Let us recall the abc conjecture of Oesterlé and Masser* (1985). *Fix* $\epsilon > 0$. *Then, there exists a constant* $\kappa(\epsilon)$ *such that whenever* $a + b = c$, *where* a, b *and* c *are positive integers and* $(a, b) = (b, c) = (a, c) = 1$, *then*

$$c \leqslant \kappa(\epsilon) \left(\prod_{p|abc} p \right)^{1+\epsilon}.$$

We apply the *abc* conjecture to study certain special solutions of the generalized BP equation (1) where m and n are positive integers and neither is a perfect square. We refer the reader to [20] for a general exposition of this equation.

Lemma 2.3. *Any squarefull number can be written as $a^2 b^3$ with b squarefree.*

Proof. Every squarefull number n can be written in the form $p_1{}^{\alpha_1} p_2{}^{\alpha_2} \cdots p_k{}^{\alpha_k}$ where $p_1 < p_2 < \cdots < p_k$ are distinct primes and each exponent $\alpha_i \geq 2$ where $i = \{1, 2, \ldots, k\}$. So let p^α be one of these prime powers dividing n, where $\alpha \geq 2$. Writing $\alpha = 2k + \delta$ with $\delta = 0$ or 1, we see that in the latter case, we may write $\alpha = 2(k-1) + 3$ as $k \geq 1$. In either case, this shows that each prime power can be written in the desired form. Hence the result. □

Lemma 2.4. *If n and $n+1$ are consecutive squarefull numbers, then there are b and d both squarefree such that we have a solution of the generalized BP equation*

$$dx^2 - by^2 = 1, \tag{2}$$

where $b \mid y$ and $d \mid x$.

Proof. If n and $n+1$ are consecutive squarefull numbers, by Lemma 2.3, we can write $n = a^2 b^3$ and $n+1 = c^2 d^3$ where b and d are squarefree. So, we have

$$c^2 d^3 - a^2 b^3 = 1.$$

Thus we get a solution of the generalized BP equation

$$dx^2 - by^2 = 1,$$

with $x = cd$, $y = ab$. Clearly, $d \mid x$ and $b \mid y$. □

This lemma answers the question of Erds regarding consecutive squarefull numbers. They all arise from solutions of equations of Type II (see [19, p. 234]).

Proposition 2.5. *Suppose n, $n+1$ are consecutive squarefull numbers with $n = a^2 b^3$, $n+1 = c^2 d^3$ with b, d squarefree and $n \leq N$. The abc conjecture implies $|d|, |b| \leqslant \kappa(\epsilon) N^\epsilon$, for any $\epsilon > 0$.*

Proof. From Lemma 2.4, we have

$$dx^2 - by^2 = 1$$

with $x = cd, y = ab$. Noting that $|x|, |y| \leq N$, we apply the *abc* conjecture to the above equation. We also see that $d \mid x$ and $b \mid y$ and b, d are both squarefree. Thus, we have

$$dx^2 \leq \kappa(\epsilon) \left(d \cdot \frac{x}{d} \cdot b \cdot \frac{y}{b} \right)^{1+\epsilon}.$$

Thus

$$dx^2 \leq \kappa(\epsilon)(xy)^{1+\epsilon}.$$

Hence

$$d \le \kappa(\epsilon)\left(\frac{y}{x}\right)N^{2\epsilon}.$$

Since Eq. (2) implies

$$\left(\frac{y}{x}\right) \asymp \sqrt{\frac{d}{b}}$$

we get

$$\sqrt{bd} \le \kappa(\epsilon)N^{2\epsilon} \Rightarrow bd \le \kappa_1(\epsilon)N^{4\epsilon},$$

$$bd \ll N^{4\epsilon}$$

from which our results follows. □

3. Proof of Theorem 1.1

The strong Erdös conjecture predicts that $S_d(N) = O(\log^A N)$ with $A < 2$. We now show that the Ankeny–Artin–Chowla conjecture holds for almost all primes p. The latter conjecture [1] is the assertion that the fundamental unit $\varepsilon = \frac{T+U\sqrt{p}}{2}$ of $\mathbf{Q}(\sqrt{p})$ satisfies $p \nmid U$. If $p \mid U$, then from $T^2 - pU^2 = \pm 4$, we have two squarefull numbers n and $n \pm 4$ with $n = pU^2$, $n \pm 4 = T^2$. By the analytic class number formula (see, for example, [17, Exercise 10.5.12]), we have

$$\frac{h(p)\log\varepsilon}{\sqrt{p}} = O(\log p),$$

where $h(p)$ is the class number of $\mathbb{Q}(\sqrt{p})$. Using $h(p) \ge 1$, we have

$$\varepsilon \le \exp(B\sqrt{p}\log p)$$

for some constant B. Then we see that $\frac{T+U\sqrt{p}}{2} \le \exp(B\sqrt{p}\log p)$ and so in particular, $T, U \le 2\exp(B\sqrt{p}\log p)$. To count the primes $p \le N$ for which the Ankeny–Artin–Chowla conjecture is false, we are led to count

$$\#\{n \le 2\exp(c\sqrt{N}\log N) : n, n \pm 4 \text{ are squarefull}\}.$$

By the strong Erdös conjecture with $A < 2$, we see the number $\ll (\sqrt{N}\log N)^A \ll N^\theta$, for some $\theta < 1$. This means that if there are counterexamples to the Ankeny–Artin–Chowla conjecture, they are extremely rare. A similar estimate and proof holds for the Mordell conjecture also.

4. Proof of Theorem 1.2

By Proposition 2.5 and assuming abc conjecture, any squarefull numbers n, $n + 1$ can be written as $n = a^2b^3$, $n + 1 = c^2d^3$, $n, n + 1 \le N$ with b, d squarefree and $|b|, |d| \le N^\epsilon$. We fix b, d. We count the number of solutions of Eq. (2) using

Lemma 2.4. By the theory of the generalized BP equation, all solutions of Eq. (2) are generated by fundamental solutions [20]: We have

$$(\sqrt{d}x_0 + \sqrt{b}y_0)^{2i+1} = \sqrt{d}x_i + \sqrt{b}y_i.$$

In our case, $|x_i| \le N$, $|y_i| \le N$ imply

$$(2i + 1)\log(\sqrt{b}x_0 + \sqrt{d}y_0) \le 2\log N \Rightarrow (2i + 1) \ll (\log N),$$

$$S(N) = \#\{n \le N : n, n + 1 \text{ squarefull numbers}\} \le \sum_{b,d \le N^\epsilon} (\log N) \ll N^\epsilon.$$

Finally, we vary over $|b|, |d| \le N^\epsilon$ to get a total contribution of $N^{4\epsilon}$. □

5. Proof of Theorem 1.3

Theorem 5.1. *Let $S(N)$ be the number of integers $n \le N$ such that both n and $n + 1$ are squarefull. Then we have for all $\epsilon > 0$ and sufficiently large N that $S(N) = O(N^{2/5})$.*

Proof. We have

$$d(cd)^2 - b(ab)^2 = 1$$

such that $n = a^2b^3$ and $n + 1 = c^2d^3$ from Lemma 2.4 where both b and d are squarefree. We already know from Golomb's paper that the number of squarefull integers $\ll N$ is $k(N) \sim c\sqrt{N}$. Let Z be a parameter to be chosen later. First we count our squarefull numbers n with $n + 1$ squarefull and $n = a^2b^3$ with $b > Z$. Thus we have

$$\#\{n \le N : \exists\, a, \exists\, b > Z, \ n = a^2b^3\} \le \sum_{b>Z} \frac{\sqrt{N}}{b^{3/2}}.$$

Finally we get the number of squarefull integers $\ll \frac{\sqrt{N}}{\sqrt{Z}}$. Similarly, we count our $n + 1 = c^2d^3$ squarefull with $d > Z$:

$$\#\{n \le N : \exists\, c, \exists\, d > Z, \ n + 1 = c^2d^3\} \le \sum_{d>Z} \frac{\sqrt{N}}{d^{3/2}} \ll \frac{\sqrt{N}}{\sqrt{Z}}.$$

We may now suppose that both $b, d \le Z$. Let us fix b, d. Thus, we have

$$dx^2 - by^2 = 1,$$

which is a BP equation by Theorem 2.1. This gives that the number of consecutive squarefull integer pairs is

$$\ll \sum_{b,d \le Z} (\log N) \ll Z^2 (\log N).$$

Thus we have a total estimate of $Z^2(\log N) + O\left(\frac{\sqrt{N}}{\sqrt{Z}}\right)$ for any Z. We choose $Z = \frac{N^{1/5}}{(\log N)^{2/5}}$ to get a final estimate for $S(N) = N^{2/5}(\log N)^{1/5}$. This can be refined to deduce the result stated as follows. We note that

$$(2i + 1)\log(x_0\sqrt{d} + y_0\sqrt{b}) \leq 2\log N$$

so that, in fact,

$$(2i + 1)\frac{1}{2}(\log b) \leq 2\log N.$$

Our sum above becomes

$$\sum_{b,d\leq Z} \frac{\log N}{\log b} = O\left(\frac{Z^2 \log N}{\log Z}\right)$$

which gives

$$S(N) = O(N^{2/5}). \qquad \blacksquare$$

This theorem gives a non-trivial result using more elementary results than Reuss [18] used in his paper to estimate the error term.

As a result of the question of finding consecutive squarefull pairs, one example has been found where neither is a perfect square as $d = 23^3 = 12{,}167$ and $b = 2^3 3^2 13^2 = 12{,}168$ for the first time by Golomb [10]. This was the first example of Type II pairs. Another example was given by Walker [21].

6. Concluding Remarks

Our heuristic also predicts that the number of three consecutive squarefull numbers is bounded. It is well known that the *abc* conjecture implies this (see, for example, [17, Exercise 1.3.6]). It would be interesting to explore if this fact can be deduced without the *abc* conjecture. On the other hand, with recent announcements from Mochizuki [12] that he has a proof of the *abc* conjecture, there is hope that we are moving toward the conjectures of Ankeny, Artin, Chowla and Mordell.

Mollin, Walsh and Erdös [13] have conjectured that for a positive squarefree composite integer $d \equiv 7 \pmod 8$, the fundamental unit $t + u\sqrt{d}$ of the ring of integers of $\mathbb{Q}(\sqrt{d})$ satisfies $u \equiv 0 \pmod d$ with t powerful. As an example of the fundamental unit, $\epsilon = 24{,}335 + 3{,}588\sqrt{46}$ of the ring of integers of $\mathbb{Q}\sqrt{46}$ satisfies $3{,}588 \equiv 0 \pmod{46}$ for a composite $d = 46$. Our arguments above on primes extend *mutatis mutandis* to this conjecture too. We leave the details to the reader.

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