MATH 401/801: Assignment 1

Math 401: Do any eight. Math 801: Do all ten.

Due: September 26, 2018

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1. (a) Is there a simple graph of 9 vertices with degree sequence

$$3, 3, 3, 3, 5, 6, 6, 6, 6?$$

   (b) Is there a bipartite graph of 8 vertices with degrees

$$3, 3, 3, 5, 6, 6, 6, 6?$$

2. The Fibonacci sequence is defined recursively as follows. $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Prove that

$$\sqrt{5}F_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1},$$

   for $n \geq 0$.

3. In a simple graph with at least two vertices, show that there are at least two vertices with the same degree. [Hint: what are the possible degrees?]

4. A **directed graph** (or **digraph**) is a graph X together with a function assigning to each edge, an **ordered** pair of vertices. The first vertex is called the **tail** of the edge and the second is called the **head**. To each vertex v, we let $d^+(v)$ be the number of edges for which v is the tail and $d^-(v)$ the number for which it is the head. We call $d^+(v)$ the **outdegree** and $d^-(v)$ the **indegree** of v. Prove that

$$\sum_v d^+(v) = \sum_v d^-(v) = \#E(X)$$

   where the sum is over the vertex set of X.

5. In any digraph, we define a **walk** as a sequence

$$v_0, e_1, v_1, e_2, ..., e_k, v_k$$

   with v_{i-1} the tail of e_i and v_i its head. The analogous notions of trail, path, circuit and cycle are easily extended to digraphs in the obvious way. If X is a digraph such that the outdegree of every vertex is at least one, show that X contains a cycle.

6. An **Eulerian trail** in a digraph is a trail containing all the edges. An **Eulerian circuit** is a closed trail containing all the edges. Show that a digraph X contains an Eulerian circuit if and only if $d^+(v) = d^-(v)$ for every vertex v and the underlying graph has at most one component.

7. Show that the cycle graph C_n with n even is a bipartite graph.

8. (a) Show that

$$n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}.$$

   (b) Calculate

$$\sum_{A \subseteq [n]} |A|^2,$$

   where the summation is over all subsets A of $[n]$.

9. Is there a simple graph with degrees $1, 1, 3, 3, 3, 3, 5, 6, 8, 9$? Prove your assertion.

10. (a) Let X be a simple graph with n vertices and e edges. Prove that

$$\sum_{v \in V(X)} d^2(v) = \sum_{uv \in E(X)} (d(u) + d(v))$$

   (b) Show that X has at least one edge uv such that

$$d(u) + d(v) \geq \frac{4e}{n}.$$