**MATH 401/801: Assignment 2**

Due: October 19, 2018

Students in Math 401: Do any eight. Students in Math 801: Do all ten.

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1. Write down the adjacency matrix of the cycle graph $C_3$ on 3 vertices and compute its characteristic polynomial as well as its roots.
2. The path graph on 3 vertices is the graph $\circ\!\!-\!\!\!-\!\!\circ\!\!-\!\!\!-\!\!\circ$. Calculate the characteristic polynomial and its roots.
3. Show that a simple graph $X$ with $n$ vertices and adjacency matrix $A$ is connected if and only if every entry of $(I+A)^{n-1}$ is non-zero.
4. The **eccentricity** of a vertex $u$ in a graph $X$ is the maximum of $d(u,v)$ as $v$ ranges over the vertices of $X$. The minimum of all the possible eccentricities is called the **radius**, denoted $\mathbf{rad}(X)$ of the graph $X$. Show that if $X$ is connected, then
$$\operatorname{diam}(X) \leq 2\operatorname{rad}(X).$$
5. Let $M$ be the **incidence matrix** of a simple graph $X$. That is, the rows are parametrized by vertices $v_i$ and columns by the edges $e_j$ so that the $i,j$-th entry of $M$ is 1 if $v_i$ is incident with edge $e_j$ and zero otherwise. Prove that
$$\operatorname{tr}(M^t M) = 2e$$
where $e$ is the number of edges.
6. In a simple graph $X$, we choose an **orientation** by assigning a direction to each edge. The modified incidence matrix $N$ is defined as follows. Its rows are parametrized by the vertices $v_i$ and the columns by the edges $e_j$, as before. The $i,j$-th entry of $N$ is $+1$ if $v_i$ is the tail of $e_j$, $-1$ if it is the head and zero otherwise. If $D$ is the diagonal matrix consisting of degrees of $v_1, ..., v_n$, prove that
$$NN^t = D - A_X,$$
where $A_X$ is the adjacency matrix of $X$.
7. Prove that $\lambda^4+\lambda^3+2\lambda^2+\lambda+1$ cannot be the characteristic polynomial of an adjacency matrix of any simple graph.
8. Show that
$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^{n+1} = \binom{n+1}{2} n!$$

9. For any matrix $A$, show that

$$\det(I - A\lambda) = \exp\left(-\sum_{n=1}^{\infty} \frac{\operatorname{tr}(A^n)\lambda^n}{n}\right).$$

10. Let $X$ be a simple graph on $n$ vertices with characteristic polynomial

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + a_3\lambda^{n-3} + \cdots + a_n.$$

Show that $a_1 = 0$, $-a_2$ is the number of edges of $X$, and that $-a_3$ is twice the number of triangles in $X$.

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**Theorem.** If $X$ is a simple graph with finite diameter, then the diameter is strictly less than the number of distinct eigenvalues of the characteristic polynomial of $X$.

**Proof.** Let $k$ be the diameter of $X$. Then, there are two vertices $i, j$ such that there is a path of length $k$ from $i$ to $j$ and no path of shorter length exists between them. Thus, $(A^r)_{i,j} = 0$ for $r < k$ and is non-zero for $r = k$. Therefore, $I, A, A^2, ..., A^k$ are linearly independent. Denoting the minimal polynomial of $A$ by $g(\lambda)$ we note that $g(A) = 0$ using the spectral theorem for real symmetric matrices. If $m$ is the degree of the minimal polynomial then $m$ is the number of distinct eigenvalues of the characteristic polynomial of $A$. In other words, $I, A, ..., A^m$ are linearly *dependent*. Therefore, $k < m$. $\diamondsuit$