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SUMMATION METHODS AND DISTRIBUTION OF EIGENVALUES OF HECKE OPERATORS

Sanoli Gun[1], M. Ram Murty[2], Purusottam Rath[3]

Dedicated to Professor Władysław Narkiewicz

Abstract: Let p be a fixed prime number. Let $S_k(N)$ be the space of cusp forms of weight k and level N. We prove a weighted equidistribution theorem for the eigenvalues of the p-th Hecke operator T_p acting on $S_k(N)$. This is a variant of a celebrated theorem of Serre.

1. Introduction

Let X be a compact Hausdorff space with a regular normalized Borel measure μ. We denote by $\mathcal{R}(X)$ the space of all real-valued continuous functions on X. A sequence $\{x_n\}$ of elements in X is called μ-equidistributed or μ-uniformly distributed (μ-u.d in short) if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(x_n) = \int_X f d\mu$$

for all $f \in \mathcal{R}(X)$. We note that if there is a class $\mathcal{C}$ of real-valued functions on X which are dense in $\mathcal{R}(X)$ (with respect to the sup norm), then it suffices to check the above condition for all the elements in $\mathcal{C}$. It is interesting to note that if X has a countable basis, then in a certain sense, almost all sequences in X are μ-u.d. To be more precise, if the product space $X^{\mathbb{N}}$ is given the product measure μ_∞ induced from μ, then the set of all μ-u.d sequences in X, viewed as a subset of $X^{\mathbb{N}}$, has full μ_∞ measure in $X^{\mathbb{N}}$ (see [6], for instance).

Let us now restrict ourselves to compact subsets X of real numbers $\mathbb{R}$. To start with, suppose it is endowed with the familiar Lebesgue measure. Since trigonometric polynomials are dense in $\mathcal{R}(X)$, linearity of integrals and limits gives us the following criterion due to Weyl, when X is the unit interval:

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Theorem 1.1 (Weyl's criterion). *A sequence of real numbers* $\{x_n\}$ *in* $[0, 1]$ *is uniformly distributed with respect to Lebesgue measure* dx *if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(hx_n) = 0$$

for all non-zero integers h *where* $e(x) := e^{2\pi i x}$.

While most of the sequences in X will be equidistributed with respect to the Lebesgue measure, we do come up with important sequences which are not equidistributed with respect to the Lebesgue measure. We are concerned here with one such sequence, namely the eigen values of the normalized Hecke operators (we will elaborate a bit later). One way to redress such a scenario is to alter the measure so that the given sequence is equidistributed with respect to the new measure. A natural way to proceed is as follows. Suppose $\{x_n\}$ is a sequence of real numbers in $X \subset \mathbb{R}$ which is not equidistributed with respect to Lebesgue measure. Thus for some non-zero integer h, the sequence $\frac{1}{N}\sum_{n=1}^{N} e(hx_n)$ does not converge to zero as $N \to \infty$. However, suppose for each $h \in \mathbb{Z}$, the limits

$$a_h = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(hx_n)$$

exist. Then we can construct a measure on X when the sequence $\{a_h\}$ has finite l^2-norm. To be more precise, we have the following theorem which follows from the Wiener-Schoenberg theorem (see for instance [6], also [8]).

Theorem 1.2. *Let* $\{x_n\}$ *is a sequence in* X *such that for all integers* h*, the limits* a_h *defined as above exist. Suppose further that*

$$\sum_{h=-\infty}^{\infty} |a_h|^2 < \infty.$$

Then the sequence $\{x_n\}$ *is equidistributed with respect to the measure* $g(-x)dx$ *where* $g(x)$ *is given by*

$$g(x) = \sum_{h=-\infty}^{\infty} a_h e(hx).$$

This approach has been exploited in [11] in the context of distribution of eigen values of normalized Hecke operators. Let N, k be natural numbers with k even and let $S(N, k)$ be the space of cusp forms of weight k and level N with dimension $d(N, k)$. Let p be a prime not dividing the level and T_p be the p-th Hecke operator acting on $S(N, k)$. If $a_{p,i}, 1 \le i \le d(N, k)$ are the eigenvalues of T_p, then a result of Deligne [2] states that these eigenvalues lie in the interval

$$[-2p^{\frac{k-1}{2}}, 2p^{\frac{k-1}{2}}]\ .$$

Thus, the eigenvalues of the normalized Hecke operator,

$$T_p^{'}(N,k) = \frac{T_p(N,k)}{p^{(k-1)/2}}$$

lie in the interval $[-2,2]$. If we fix the level N and weight k and vary the prime p, we have the Sato-Tate conjecture which predicts that the eigenvalues are equidistributed with respect to the measure

$$\mu_\infty = \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx$$

Recently, Taylor [14] has announced a proof of the Sato-Tate conjecture for weight 2 eigenforms with square-free level and rational integer coefficients, hence for modular forms corresponding to elliptic curves over $\mathbb{Q}$.

Serre, in his paper [12] considered the effect of fixing the prime p and varying the parameters N and k. In this set up, he proved the following interesting theorem:

Theorem 1.3. *Let N and k be varying over infinite families of positive integers with k even and $N+k\to\infty$. Let $S(N,k)$ be the space of cusp forms of weight k and level N and p be a prime not dividing any of the $N's$. Then the family of eigenvalues of the normalized Hecke operator*

$$T_p^{'}(N,k) = \frac{T_p(N,k)}{p^{(k-1)/2}}$$

is equidistributed in $[-2,2]$ with respect to the measure

$$\mu_p = \frac{p+1}{\pi}\frac{(1-x^2/4)^{1/2}}{(p^{1/2}+p^{-1/2})^2-x^2}dx\ .$$

In a recent work, the second author and K. Sinha [11] re-established the above theorem using the recipe suggested by Theorem 1.2. Furthermore, they derived an effective version of the above theorem. More precisely, they proved the following:

Theorem 1.4. *Let N be a positive integer and p be a prime coprime to N. Then for any $[\alpha,\beta]\subset[-2,2]$,*

$$\frac{1}{d(N,k)}\#\left\{1\le i\le d(N,k) : \frac{a_{p,i}}{p^{\frac{k-1}{2}}}\right\} = \int_\alpha^\beta \mu_p + O\left(\frac{\log p}{\log kN}\right)\ ,$$

where the constant is effective

As mentioned in [11], the result follows from a general Erdös-Tuŕan type theorem and using the idea of constructing a measure out of the a_h's referred to as the Weyl limits of the given sequence.

As we see, Theorem 1.2 does suggest a way out when the Weyl limits a_h's of the sequence $\{x_n\}$ exist and have finite l^2 norm. But the recipe fails when the

Weyl limits of a sequence do not exist or do not have finite l^2 norm. There do exist lot of sequences of this type. A rather simple example is given by the sequence of the fractional parts of $\log n$ in $[0,1]$. The Weyl limits of this sequence do not exist. We refer to a paper of Helson and Kahane [4] where a large class of sequences are discussed which cannot be made equidistributed with respect to any measure.

The matrix summation methods which we discuss in the next section offers alternatives to remedy this scenario. The basic premise is that a suitable altering or scaling of a given sequence may make its distribution more uniform with respect to the same measure. As we shall see in the last section, employing a matrix method which is natural in relation to the set up, the sequence of eigenvalues considered in Theorem 1.3 becomes uniformly distributed with respect to a simpler measure, quite similar to the measure expected in the Sato-Tate conjecture.

2. Matrix Summation Methods

The notion of equidistribution is based on the hypothesis that the sequence

$$\frac{1}{N}\sum_{n=1}^{N} f(x_n)$$

must converge to $\int_X f d\mu$. But this sequence is nothing but the Cesaro sum of the sequence $\{f(x_n)\}$. This is a very special case of a matrix limitation method.

Let $A = (a_{N,n})_{N,n}$ be an infinite real matrix. Then a real sequence $\{x_n\}$ is said to be A-limitable to x if

$$\lim_N \sum_{n=1}^{\infty} a_{N,n} x_n = x$$

which we write as

$$A - \lim x_n = x$$

So any infinite real matrix A gives a matrix-limitation method. A matrix-limitation method given by A is called regular if it preserves the limits of convergent sequences, i.e.

$$\lim x_n = x \implies A - \lim x_n = x\ .$$

The following theorem due to Toeplitz [3] gives a characterization of regular summation methods:

Theorem 2.1. *An infinite matrix method $A = (a_{N,n})$ is regular if and only if it satisfies the following conditions:*

$$\lim_N \sum_{n=1}^{\infty} a_{N,n} = 1$$

$$\lim_N \sum_{n=1}^{\infty} |a_{N,n}| < \infty \quad \text{and}$$

$$\lim_N a_{N,n} = 0, \quad \forall n.$$

One of the most common examples of a regular matrix method is the simple Riesz means method given as follows; let (p_n) be a sequence of non-negative real numbers with $p_1 > 0$. For $n \geq 1$, let $P_n = p_1 + \cdots + p_n$ and suppose $\lim P_N = \infty$. Then the matrix $A = (a_{N,n})$ is defined by

$$a_{N,n} = \begin{cases} p_n/P_n & \text{for } n \leq N; \\ 0 & \text{for } n > N. \end{cases}$$

We generalize the notion of uniform distribution by using matrix-limitation methods. Given a matrix method A as before, a sequence $\{x_n\}$ of real numbers in a compact set X is said to be A-equidistributed with respect to a measure μ if

$$A - \lim f(x_n) = \lim_{N\to\infty} \sum_{n=1}^{\infty} a_{N,n} f(x_n) = \int_X f(x) d\mu$$

When A is the matrix method corresponding to Riesz means method with $p_n = 1$ for all n, we get the standard equidistribution.

In this set up also, we have Weyl's criterion which reads as:

Theorem 2.2 (Weyl's Criterion). *Given a regular matrix method A as above, a sequence $\{x_n\}$ of real numbers is A-equidistributed with respect to a measure μ on a compact subset X of real numbers if*

$$\lim_{N\to\infty} \sum_{n=1}^{\infty} a_{N,n} e^{2\pi i m x_n} = \int_X e^{2\pi i m x} d\mu$$

for every natural number m.

The proof is very much in the spirit of the proof of the classical case, the relevant part is that the infinite matrix $A = (a_{N,n})$ has finite norm, i.e.

$$\lim_N \sum_{n=1}^{\infty} |a_{N,n}| < \infty$$

As we have mentioned before, the sequence given by the fractional parts of $\log n$ is not equidistributed with respect to any regular Borel measure on $[0, 1]$. The matrix summation method does suggest a way out. We have the following neat theorem due to Baker and Harman [1]:

Theorem 2.3. *Let A be the matrix summation method given by the simple Riesz means with $p_n = \frac{1}{n}$. Then for any real numbers α and β with $\beta \neq 0$, the fractional parts of the sequence $\alpha n + \beta \log n$ is A-equidistributed with respect to the standard Lebesgue measure.*

In our main theorem in the last section, we consider the sequence of eigenvalues of normalized Hecke operators and consider a matrix limitation method which is natural in this set up. We show that with respect to this matrix method, this sequence is equidistributed with respect to a measure which is simpler than the measure given by Theorem 1.3 and much in the spirit of the Sato-Tate measure.

3. Modular requisites

In this section, we enlist some of the results from the theory of modular forms which will be required in the proof of our theorem. As before, let $S(N,k)$ be the space of cusp forms of weight k and level N. For our purposes, we fix a weight k and a prime p. We are interested in varying N over positive integers coprime to p. Since we are fixing the weight k, we denote $S(N,k)$ by $S_k(N)$ and its dimension $d(N,k)$ by d_N. Denote by

$$\mathcal{F}_N = \{f_1^N, \cdots, f_{d_N}^N\} \ ,$$

the set of normalized Hecke eigenforms for $S_k(N)$ and by

$$T_p' = \frac{T_p}{p^{\frac{k-1}{2}}} \ ,$$

the normalized p-th Hecke operator acting on $S_k(N)$.
Let $a_p(f_i^N)$, $1 \leq i \leq d_N$ be the eigenvalues of the normalized p-th Hecke operator T_p'. We write $a_p(f_i^N) = 2\cos\theta_{i,p}^N$, where $\theta \in [0,\pi]$ for $1 \leq i \leq d_N$. We are interested in the A-equidistribution of these $2\cos\theta_{i,p}^N$'s as $N \to \infty$ with respect to a suitable matrix method A.

The first result which we require is the Petersson trace formula (see for instance [9]). The space $S_k(N)$ has an inner product, the Petersson inner product:

$$\langle f, g\rangle = \int_{\Gamma_0(N)\backslash\mathcal{H}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}$$

where $\mathcal{H}$ is the upper half plane. Let $\{g_1, \cdots g_{d_N}\}$ be any orthonormal basis for $S_k(N)$. The space $S_k(N)$ is also spanned by the Poincaré series: for $m \geq 1$

$$P_m(z,k,N) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} j(\gamma,z)\Gamma_0(N)^{-k} e(m\gamma z)$$

where Γ_∞ is the stabilizer of $i\infty$ in $\Gamma_0(N)$, (c,d) gives the second row of γ and

$$e(z) = e^{2\pi i z}, \ j(\gamma,z) = (cz+d)(\det\gamma)^{-1/2}$$

Let $f(z) \in S_k(N)$ whose fourier expansion at $i\infty$ is given by

$$f(z) = \sum_{n=1}^{\infty} \widehat{f}(n) e(nz)$$

Petersson proved that

$$\widehat{f}(n) = \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \langle f, P_n(\cdot, k, N) \rangle$$

Suppose that

$$P_n(\cdot, k, N) = \sum_i c_i g_i \ .$$

Then we have,

$$c_i = \langle P_n(\cdot, k, N), g_i \rangle = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \overline{\widehat{g}_i(n)}$$

which gives

$$\frac{(4\pi n)^{k-1}}{\Gamma(k-1)} P_n(\cdot, k, N) = \sum_i \overline{\widehat{g}_i(n)} g_i \ .$$

Comparing the m-th coefficients on either side;

$$\frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \widehat{P}_n(m, k, N) = \sum_i \overline{\widehat{g}_i(n)} \widehat{g}_i(m) \tag{3.1}$$

Petersson obtained the m-th Fourier coefficient of the n-th Poincaré series explicitly as:

$$\widehat{P}_n(m, k, N) = (\frac{m}{n})^{\frac{k-1}{2}} \left\{ \delta_{mn} + (2\pi i)^{-k} \sum_{c \equiv 0 \,(\mathrm{mod}\ N)} c^{-1} J_{k-1}(\frac{4\pi\sqrt{mn}}{c}) S(m, n, c) \right\}, \tag{3.2}$$

where $\delta_{m,n} = 0$ unless $m = n$ in which case it is 1, $J_{k-1}(x)$ is the Bessel function of order $k-1$, $S(m, n, c)$ is the Kloosterman sum:

$$S(m, n, c) = \sum_{d \,(\mathrm{mod}\ c)} e\left(\frac{md + n\bar{d}}{c}\right)$$

where $d\bar{d} \equiv 1 (\mathrm{mod}\ c)$. Using (3.1) and (3.2), we have

$$\frac{\Gamma(k-1)}{4\pi(\sqrt{mn})^{k-1}} \sum_{f \in \mathcal{F}} \overline{\widehat{g}_i(n)} \widehat{g}_i(m) \tag{3.3}$$

$$= \delta(m, n) + \frac{1}{(2\pi i)^k} \sum_{\substack{c > 0 \\ c \equiv 0 \ (\mathrm{mod}\ N)}} c^{-1} S(m, n, c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

We have the following estimate for the Bessel function

$$J_{k-1}(x) \ll \min\left(1, \frac{x^{k-1}}{(k-1)!}\right)$$

We also have the following estimate for the Kloosterman sum due to Weil:

$$\mid S(m,n,c) \mid \leq (m,n,c)^{\frac{1}{2}}\ d(c)\ c^{\frac{1}{2}}\ ,$$

$d(c)$ is the number of positive divisors of c. Using these estimates in (3.3) and the estimate $d(N) \ll N^{\epsilon}$, we get the following version of Petersson trace formula:

Lemma 3.1. *Let $\mathcal{F}$ be any orthonormal basis of $S_k(N)$. For $f \in \mathcal{F}$, let $a_f(n)$ be its n-th Fourier coefficient. Then*

$$\sum_{f\in\mathcal{F}} a_f(n) = \delta(n,1)\frac{4\pi n^{(k-1)/2}}{\Gamma(k-1)} + O\left(N^{-k+\frac{1}{2}+\epsilon}\right),$$

where the implied constant depends on k and n.

For our result, we need the following upper bound for the Petersson norm $||f|| = \sqrt{\langle f, f\rangle}$.

Lemma 3.2. *Let $f \in S_k(N)$ be a normalized Hecke eigenform. Then*

$$||f||^2 \ \ \ll \frac{\Gamma(k)}{(4\pi)^k}N^{1+\epsilon}$$

For the sake of completeness, we sketch a proof of this. For this, we will need some generalities on Dirichlet series and L-functions (see also [10]). Let

$$L(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

be a Dirichlet series absolutely convergent for $\Re(s) = \sigma > 1$ and admitting

1. an Euler product for $\sigma > 1$

$$L(s) = \prod_p L_p(s)$$

   where $L_p(s)$ is the inverse of a polynomial of degree d_p in p^{-s};
2. a positive integer d such that for $d_p \leq d$ for all p and equality holds for almost all p;
3. an analytic continuation to an entire function satisfying a functional equation of the form

$$\Lambda(s) = \omega\Lambda^{\star}(1-s)$$

where $|\omega| = 1$ and

$$\Lambda(s) = A^{s/2} \prod_{i=1}^{m} \Gamma(a_i s + r_i) L(s) \ ,$$

$$\Lambda^\star(s) = A^{s/2} \prod_{i=1}^{m} \Gamma(a_i s + r_i) L^\star(s) \ .$$

Further,

$$L_p(s) = \prod_{i=1}^{d_p} (1 - \alpha_i p^{-s})^{-1}, \qquad |\alpha_i| = 1$$

and

$$L_p^\star(s) = \prod_{i=1}^{d_p} (1 - \overline{\alpha_i} p^{-s})^{-1} \ ,$$

where

$$L^\star(s) := \prod_p L_p^\star(s) \ .$$

Here $A > 0$, α_i and r_i are real numbers and $m \in \mathbb{N}$.

Under the above formalism, we have Rademacher's version of the Phragmén-Lindelöf theorem:

Proposition 3.1. *For $0 \leq \sigma \leq 1$, we have*

$$|L(\sigma + it)| \quad \ll \quad (A(|t| + 2)^d)^{(1-\sigma)/2} (\log(A(|t| + 2)^d))^d$$

where the implied constants depend only on the r_i's.

Let f be a normalized Hecke eigenform with Fourier expansion at infinity $f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$. Write

$$\frac{a(p)}{p^{(k-1)/2}} = \alpha_p + \overline{\alpha_p}$$

By the theory of Rankin and Selberg, any two cusp forms $f, g \in S_k(N)$ admit a Rankin–Selberg convolution $L(f \otimes g, s)$. If f is a normalized Hecke eigenform, the symmetric square L function (same as the adjoint square in this case) of f is an L-function of degree 3 obtained by dividing the Rankin–Selberg convolution $L(f \otimes f, s)$ by the Riemann zeta function and is of the form:

$$L(Sym^2 f, s) = \prod_p L_p(s) \ ,$$

where the local factors corresponding to the unramified primes ($p \nmid N$) are given by

$$L_p(Sym^2 f, s) = \left(1 - \frac{\overline{\alpha}_p^2}{p^s}\right)^{-1} \left(1 - \frac{\overline{\alpha}_p \alpha_p}{p^s}\right)^{-1} \left(1 - \frac{\alpha_p^2}{p^s}\right)^{-1}$$

while for $p \mid N$, it can be defined appropriately (see [13]).

This L-function converges for $\Re(s) > 1$. By the work of Shimura [13], we know that $L(Sym^2 f, s)$

- extends to an entire function
- satisfies a functional equation of the form:

$$\Lambda(s) = A^s \Gamma\left(\frac{s+k-1}{2}\right)\Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s+1}{2}\right) L(Sym^2 f, s)$$
$$= \omega \Lambda(1-s)$$

where $|\omega| = 1$, and $\log A = O(\log N)$.

So it belongs to the class of L functions described above and thus we can apply the Radamacher's version of the Phragmén-Lindelöf above to estimate its value at $s = 1$. Applying the estimate in Proposition 3.1, we get

$$L(Sym^2 f, 1) = O\left((\log N)^3\right) .$$

On the other hand, the Rankin–Selberg method also provides a link between the Petersson norm $||f||$ of f and the special value of the symmetric square L-function of f at $s = 1$. More precisely (see [5], for instance), we have

$$L(Sym^2, 1) = Res_{s=1} L(f \otimes f, s) = \frac{w_k ||f||^2}{\mathrm{Vol}(\Gamma_0(N) \setminus \mathcal{H})}$$

where $w_k = \frac{(4\pi)^k}{\Gamma(k)}$. Using this and the previous estimate on $L(Sym^2 f, 1)$, we arrive at a proof of Lemma 3.2.

Finally, we will need the following result (see [12]);

Lemma 3.3. *For $n \geq 0$, let $X_n(t)$ be the nth Chebychev polynomial defined as:*

$$X_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad \text{where} \quad t = 2\cos\theta .$$

Then for any prime p, we have

$$T'_{p^n} = X_n(T'_p) .$$

4. Matrix distribution of eigenvalues of Hecke operators

As before, let k be a fixed even positive integer, N be a positive integer and p be a prime not dividing N. Also let $S_k(N)$ be the space of newforms of weight k and level N. Let $d_N = \dim S_k(N)$.
Denote by

$$\mathcal{F}_N = \left\{ f_1^N, \cdots, f_{d_N}^N \right\} ,$$

the set of normalized Hecke eigen forms for $S_k(N)$ and

$$T_p^{'} = \frac{T_p}{p^{\frac{k-1}{2}}} ,$$

the normalized p-th Hecke operator acting on $S_k(N)$. Let $a_p(f_i^N)$, $1 \le i \le d_N$ be the eigenvalues of the normalized p-th Hecke operator T_p'. We write $a_p(f_i^N) = 2\cos\theta_{i,p}^N$, where $\theta \in [0,\pi]$ for $1 \le i \le d_N$. We are interested in the A-equidistribution of these $2\cos\theta_{i,p}^N$'s as $N \to \infty$ with respect to a suitable matrix method A.

We construct a regular matrix summation method in the following way:
For N, let

$$W_N = 2 \sum_{f \in \mathcal{F}_N} \frac{1}{\|f\|^2} ,$$

where $\|f\|^2$ is the Petersson norm. Then the N-th row of A is as follows:

$$a_{N,n} = \begin{cases} 0 & \text{if} \quad n \le 2\sum_{j<N} d_j \\ \frac{1}{W_N \|f_k^N\|^2} & \text{if} \quad n = k + sd_N + 2\sum_{j<N} d_j, k = 1, \cdots, d_N, s = 0, 1; \\ 0 & \text{otherwise} \quad . \end{cases}$$

This clearly satisfies the regularity criterion of Toeplitz and hence is a regular matrix method. It is very much similar to the Riesz means method, but not exactly the same. With respect to this matrix summation method, we have the following theorem:

Theorem 4.1. *Let N, k be positive integers such that k is even and let p be a prime not dividing N. Then as $N \to \infty$, the family of eigenvalues of the normalized p-th Hecke operator*

$$T_p^{'} = \frac{T_p}{p^{\frac{k-1}{2}}}$$

is A-equidistributed in the interval $[-2, 2]$ with respect to the measure

$$\mu = \frac{1}{2\pi} \frac{dx}{\sqrt{1 - \frac{x^2}{4}}} .$$

Proof. Consider the sequence

$$\{\frac{\theta^1_{1,p}}{2\pi}, \cdots, \frac{\theta^1_{d_1,p}}{2\pi}, -\frac{\theta^1_{1,p}}{2\pi}, \cdots, -\frac{\theta^1_{d_1,p}}{2\pi}, \frac{\theta^2_{1,p}}{2\pi}, \cdots\}\ .$$

Then for any natural number m,

$$\begin{aligned}\sum_{n=1}^{\infty} a_{N,n} e^{2\pi i m(\pm\frac{\theta^N_{n,p}}{2\pi})} &= \sum_{n=1}^{d_N} \frac{2}{W_N} \frac{\cos m\theta^N_{n,p}}{\|f^N_n\|^2} \\ &= \frac{2}{W_N} \sum_{n=1}^{d_N} \frac{\cos m\theta^N_{n,p}}{\|f^N_n\|^2}\end{aligned}$$

For any natural number m, the m'th Chebychev polynomial $X_m(x)$ is defined as follows:

$$X_m(x) = \frac{\sin(m+1)\theta}{\sin\theta}, \quad \text{where} \quad x = 2\cos\theta$$

So we have,

$$\begin{aligned}&X_m(2\cos\theta^N_{n,p}) - X_{m-2}(2\cos\theta^N_{n,p}) \\ &\quad = \frac{\sin(m+1)\theta^N_{n,p}}{\sin\theta^N_{n,p}} - \frac{\sin(m-1)\theta^N_{n,p}}{\sin\theta^N_{n,p}} \\ &\quad = 2\cos m\theta^N_{n,p}\end{aligned}$$

Now by Lemma 3.1, we have

$$X_m(T^{'}_p) = T^{'}_{p^m}.$$

Hence for any eigenvalue α of $T^{'}_p$, $X_m(\alpha)$ is an eigenvalue of $T^{'}_{p^m}$. This gives

$$2\cos m\theta^N_{n,p} = a_{p^m}(f^N_n) - a_{p^{m-2}}(f^N_n)\ .$$

So

$$\frac{2}{W_N}\sum_{n=1}^{d_N} \frac{\cos m\theta^N_{n,p}}{\|f^N_n\|^2} = \frac{1}{W_N}\left\{\sum_{f\in\mathcal{F}_N} \frac{a_{p^m}(f)}{\|f\|^2} - \sum_{f\in\mathcal{F}_N} \frac{a_{p^{m-2}}(f)}{\|f\|^2}\right\}$$

Invoking the Petersson inner product formula in Lemma 3.1 to both the sums inside the parenthesis, we have

$$\left\{\sum_{f\in\mathcal{F}_N} \frac{a_{p^m}(f)}{\|f\|^2} - \sum_{f\in\mathcal{F}_N} \frac{a_{p^{m-2}}(f)}{\|f\|^2}\right\} \ll N^{-k+1/2+\epsilon}\ .$$

Further, by Lemma 3.2, we have

$$W_N \gg \frac{d_N}{N^{1+\epsilon}}$$

Hence we have:

$$\lim_{N\to\infty} \sum_{n=1}^{\infty} a_{N,n} e^{2\pi i m(\pm \frac{\theta_{n,p}^N}{2\pi})} = 0 \ .$$

By Weyl's criterion for A-equidistribution, the sequence $\left\{\pm\frac{\theta_{i,p}^N}{2\pi}\right\}$ is A-uniformly distributed with respect to the standard Lebesgue measure dx on $[0,1]$. So the A-distribution of the sequence $\{\cos\theta_{i,p}^N\}$ is given by the measure $d(\frac{\cos^{-1}x}{2\pi})$. After a suitable change of variable to normalize the measure. We get that the family of eigenvalues of the normalized p-th Hecke operator $T_p^{'}$ is A-equidistributed in the interval $[-2,2]$ with respect to the measure

$$\mu = \frac{1}{2\pi}\frac{dx}{\sqrt{1-\frac{x^2}{4}}} \ .$$

■

Remark. We note that the measure above has a simpler form than the following measure arising in the classical set up

$$\mu_p = \frac{p+1}{\pi}\frac{(1-x^2/4)^{1/2}}{(p^{1/2}+p^{-1/2})^2 - x^2}\ dx \ .$$

Interestingly, it is independent of the prime p and is very much in the spirit of the Sato-Tate measure

$$\mu_\infty = \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}\ dx \ .$$

Remark. It has been brought to our notice by the referee that questions of similar type have been considered by Li [7]. However, the motivation as well as the methodology adopted by him is different from those of ours. The main technique used by us is the theorem of Toeplitz and the Petersson trace formula while the main technique in Li's paper is the Kuznietsov trace formula. The fact that the particular matrix summation method considered by us is a regular method (using Toeplitz's criterion) made it convenient for us to explicitly derive the measure in our theorem.

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Addresses: Department of Mathematics and Statistics, Queen's University, Jeffrey Hall, 99 University Avenue, Kingston, ON, Canada K7L 3N6.

E-mail: sanoli@mast.queensu.ca, murty@mast.queensu.ca, rath@mast.queensu.ca

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