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Linear independence of digamma function and a variant of a conjecture of Rohrlich

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ABSTRACT

Let $\psi(x)$ denote the digamma function. We study the linear independence of $\psi(x)$ at rational arguments over algebraic number fields. We also formulate a variant of a conjecture of Rohrlich concerning linear independence of the log gamma function at rational arguments and report on some progress. We relate these conjectures to non-vanishing of certain L-series.

1. Introduction

The classical digamma function is the logarithmic derivative of the gamma function and is given by

$$-\psi(z) = \gamma + \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n}\right) \quad (z \neq 0, -1, -2, \ldots).$$

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Here γ is the Euler's constant. It is known that $\psi(z)$ is a meromorphic function with simple poles at $z = 0, -1, -2, \ldots$ and $\mathrm{Res}(\psi; -n) = -1$. Also

$$\psi(1) = -\gamma.$$

We refer to [1] for various expressions for the digamma function. We only mention the following interesting expression:

$$\psi(1+z) = -\gamma + \sum_{n=2}^{\infty} (-1)^n \zeta(n) z^{n-1}$$

where $\zeta(s)$ is the Riemann zeta function.

Murty and Saradha, in [15], proved the following:

Theorem *(Murty–Saradha). Let $q > 1$ be an integer and $\varphi(q)$ be the Euler's totient function. Let K be an algebraic number field over which the qth cyclotomic polynomial is irreducible. Then the numbers*

$$\psi(a/q) + \gamma,$$

where $1 \leqslant a \leqslant q$ and $(a, q) = 1$ are linearly independent over K. Further, the K-vector space spanned by γ and the $\varphi(q)$ numbers

$$\psi(a/q): \quad 1 \leqslant a \leqslant q, \ (a, q) = 1$$

has dimension at least $\varphi(q)$.

Motivated by the above theorem, the authors, in the same paper, conjectured the following:

Conjecture. *Let K be any number field over which the qth cyclotomic polynomial is irreducible. Then the $\varphi(q)$ numbers $\psi(a/q)$ with $1 \leqslant a \leqslant q$ and $(a, q) = 1$ are linearly independent over K.*

In this context, they also proved [15]:

Theorem *(Murty–Saradha). Either the Euler's constant γ is a Baker period or the above conjecture is true.*

As introduced in the same paper, a *Baker period* is an element of the $\overline{\mathbb{Q}}$ vector space spanned by the logarithms of non-zero algebraic numbers. The notion of *periods* has been introduced by Kontsevich and Zagier [11] and these Baker periods are examples of transcendental periods.

As mentioned by the authors, the co-primality condition cannot be dispensed with in their conjecture as illustrated by the following example:

$$\psi(1/2) = -\gamma - 2\log 2, \qquad \psi(1/4) = -\gamma - 3\log 2 - \pi/2,$$
$$\psi(3/4) = -\gamma - 3\log 2 + \pi/2,$$

so that

$$\psi(1) + \psi(1/4) - 3\psi(1/2) + \psi(3/4) = 0.$$

In relation to the above conjecture, we have the following theorem:

Theorem 1. *Let $q, r > 1$ be two co-prime integers. Let K be a number field over which both the qth and rth cyclotomic polynomials are irreducible. Then at least one of the following sets of real numbers*

$$\left\{\psi(a/q)\colon\ 1 \leqslant a \leqslant q,\ (a,q)=1\right\},$$
$$\left\{\psi(b/r)\colon\ 1 \leqslant b \leqslant r,\ (b,r)=1\right\}$$

is linearly independent over K. Thus in particular, there exists an integer $q_0 > 1$ such that for any integer q co-prime to q_0, the $\varphi(q)$ numbers

$$\psi(a/q)\colon\quad 1 \leqslant a \leqslant q,\ (a,q)=1$$

are linearly independent over $\mathbb{Q}$.

We note that the linear independence of the digamma function at rational arguments is linked to the non-vanishing of L-functions associated to periodic functions. Let f be a periodic arithmetic function with period q. We only consider functions which take algebraic values. The following associated L-series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

is a holomorphic function for $\mathrm{Re}(s) > 1$. The Hurwitz zeta function $\zeta(s,z)$ is of central importance in studying such L-series associated to arbitrary periodic functions. For a pair of complex numbers (s,z) with $\mathrm{Re}(s) > 1$ and $z \neq 0$, the Hurwitz zeta function $\zeta(s,z)$ is initially defined by the series

$$\zeta(s,z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^s}.$$

For $0 < x < 1$, Hurwitz proved that $\zeta(s,x)$ extends analytically to the entire complex plane, apart from $s = 1$, where it has a simple pole with residue 1. Clearly, $\zeta(s,1)$ is the classical Riemann zeta function. Then running over arithmetic progressions modulo q, we have

$$L(s, f) = q^{-s} \sum_{a=1}^{q} f(a)\zeta(s, a/q). \tag{1}$$

Since $\zeta(s,z)$ admits an analytic continuation to the entire complex plane with a simple pole at $s = 1$ with residue 1, we immediately deduce that $L(s, f)$ also extends to the complex plane with a possible simple pole at $s = 1$ with residue $\sum_{a=1}^{q} f(a)$. Further, if the sum of all the values taken by f vanishes, it turns out that the series

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}$$

converges and is equal to $L(1, f)$. In [15], Murty and Saradha show that the value of the series in this case is

$$L(1, f) = -\frac{1}{q} \sum_{a=1}^{q} f(a)\psi(a/q).$$

Now let f be a periodic function supported only at co-prime residue classes modulo q. Following [16], we refer to such functions as *Dirichlet-type* functions. Further, suppose that f is rational-valued. Then the conjecture of Murty and Saradha will imply the non-vanishing of $L(1, f)$ for any rational-valued Dirichlet-type function f. We note that the non-vanishing of $L(1, f)$ for such a function has been established by Baker, Birch and Wirsing [5].

On the other hand, for any even periodic function with period q, we have (see [12, p. 245]),

$$L(s, f) = \left(\frac{2\pi}{q}\right)^s \frac{1}{\pi}\Gamma(1-s)L(1-s, \hat{f})\sin(\pi s/2),$$

where

$$\hat{f}(n) = \sum_{a=1}^{q} f(a)e^{-2\pi i a n/q}.$$

Suppose that f is an even rational-valued Dirichlet-type function such that $L(1, f)$ exists. Differentiating the above relation and evaluating at $s = 0$, we have

$$L'(0, f) = \frac{L(1, \hat{f})}{2}.$$

In the other direction, using the following identities due to Lerch [14]

$$\zeta(0, a/q) = \frac{1}{2} - a/q, \qquad \zeta'(0, a/q) = \log\Gamma(a/q) - \frac{1}{2}\log 2\pi$$

and (1), we have

$$L'(0, f) = -\log q \sum_{a=1}^{q} f(a)(1/2 - a/q) + \sum_{a=1}^{q} f(a)\log\Gamma(a/q) - \frac{\log(2\pi)}{2}\sum_{a=1}^{q} f(a).$$

Since $L(1, f)$ exists, we have $\sum_{a=1}^{q} f(a) = 0$ and thus

$$L'(0, f) = \frac{L(1, \hat{f})}{2} = \frac{\log q}{q}\sum_{a=1}^{q} f(a)a - \sum_{a=1}^{q-1} f(a)\log\Gamma(a/q).$$

Further, since f is even, we have $\sum_{a=1}^{q} af(a) = 0$ and thus

$$L(1, \hat{f}) = -2\sum_{a=1}^{q-1} f(a)\log\Gamma(a/q).$$

Thus the $\mathbb{Q}$-linear independence of the following set of real numbers

$$\log\Gamma(a/q), \quad (a, q) = 1,$$

will imply the non-vanishing of $L(1, \hat{f})$.

Interestingly, the question of linear independence of the log gamma function at rational arguments is more delicate. However, we have a conjecture of Rohrlich about the multiplicative independence of such gamma values. We note that this conjecture is quite important in the theme of special values of L-functions. We refer to [9] for further elaboration. Following is the conjecture of Rohrlich [20]:

Conjecture *(Rohrlich). Any multiplicative dependence relation of the form*

$$\pi^{n/2} \prod_{a\in\mathbb{Q}} \Gamma(a)^{m_a} \in \overline{\mathbb{Q}}, \quad n, m_a \in \mathbb{Z}$$

is a consequence of the following relations

$$\Gamma(z+1) = z\Gamma(z) \quad (\text{Translation}),$$
$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (\text{Reflection}),$$
$$\prod_{k=0}^{n-1} \Gamma\left(a + \frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+1/2} \Gamma(na) \quad (\text{Multiplication}).$$

This is a major unsolved conjecture in transcendental number theory. In Section 3, we mention a more transparent formulation of the above. Motivated by this, we suggest the following conjecture which can be regarded as a variant of Rohrlich's conjecture.

Conjecture. *For any positive integer $q > 1$, let $V_\Gamma(q)$ denote the vector space over $\mathbb{Q}$ spanned by the real numbers*

$$\log \Gamma(a/q), \quad 1 \leqslant a \leqslant q, \ (a, q) = 1.$$

Then the dimension of $V_\Gamma(q)$ is $\varphi(q)$.

Almost nothing is known about the above conjecture. We illustrate the following few cases where we have some knowledge of the dimension of $V_\Gamma(q)$.

Theorem 2. *We have*:

1. *The dimension of $V_\Gamma(3)$ is $2 = \varphi(3)$.*
2. *The dimension of $V_\Gamma(4)$ is $2 = \varphi(4)$.*
3. *The dimension of $V_\Gamma(6)$ is $2 = \varphi(6)$.*
4. *The dimension of $V_\Gamma(5)$ is at least $3 = \varphi(5) - 1$.*

When q is a prime power, we have the following theorem:

Theorem 3. *Let q be a prime power. Then*

$$\dim_{\mathbb{Q}} V_\Gamma(q) \geqslant \varphi(q)/2.$$

Finally, for any positive integer $q > 1$, let $\overline{V_\Gamma(q)}$ denote the vector space over the field of algebraic numbers $\overline{\mathbb{Q}}$ spanned by the real numbers

$$\log \Gamma(a/q), \quad 1 \leqslant a \leqslant q, \ (a, q) = 1.$$

Then, we have the following theorem about the dimension of $\overline{V_\Gamma(q)}$:

Theorem 4. *Let q, r be distinct prime powers. Then as $\overline{\mathbb{Q}}$-linear spaces,*

$$\dim_{\overline{\mathbb{Q}}} \overline{V_\Gamma(q)} \geqslant \varphi(q)/2 \quad \text{or} \quad \dim_{\overline{\mathbb{Q}}} \overline{V_\Gamma(r)} \geqslant \varphi(r)/2.$$

Thus in particular, there exists a prime power q_0 such that for any other prime power q co-prime to q_0,

$$\dim_{\overline{\mathbb{Q}}} \overline{V_\Gamma(q)} \geqslant \varphi(q)/2.$$

2. Proof of Theorem 1

An important ingredient in the present work is the following theorem due to Baker [4]:

Theorem *(Baker). Suppose that $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers such that the numbers*

$$\log \alpha_1, \ldots, \log \alpha_n$$

are linearly independent over $\mathbb{Q}$. Then

$$1, \log \alpha_1, \ldots, \log \alpha_n$$

are linearly independent over the field of algebraic numbers $\overline{\mathbb{Q}}$. In particular, a non-zero Baker period is transcendental.

We shall also need the following non-vanishing theorem proved by Baker, Birch and Wirsing [5]:

Theorem *(Baker–Birch–Wirsing). Let $f : \mathbb{Z}/q\mathbb{Z} \to \overline{\mathbb{Q}}$ be a non-vanishing algebraic-valued periodic function with period q. Also let $f(n) = 0$ whenever $1 < (n, q) < q$ and the qth cyclotomic polynomial be irreducible over $\mathbb{Q}(f(1), \ldots, f(q))$, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

Proof of Theorem 1. Assume that

$$\sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} f(a)\psi(a/q) = 0 \quad \text{and} \quad \sum_{\substack{b=1 \\ (b,r)=1}}^{r-1} g(b)\psi(b/r) = 0$$

where f and g are K-valued. We extend f and g to functions mod q and mod r respectively as

$$f(a) := 0, \quad \text{for } 1 < (a, q) < q; \qquad f(q) := -\sum_{a=1}^{q-1} f(a),$$

$$g(b) := 0, \quad \text{for } 1 < (b, r) < r; \qquad g(r) := -\sum_{b=1}^{r-1} g(b).$$

Thus,

$$\sum_{a=1}^{q} f(a) = 0 \quad \text{and} \quad \sum_{b=1}^{r} g(b) = 0.$$

Then (see [15, Theorem 16]) the functions

$$L(s, f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

and

$$L(s, g) := \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

converge at $s = 1$ and

$$L(1, f) = \frac{-1}{q} \sum_{a=1}^{q} f(a)\psi(a/q) = \frac{f(q)}{q}\gamma,$$

$$L(1, g) = \frac{-1}{r} \sum_{b=1}^{q} g(b)\psi(b/r) = \frac{g(r)}{r}\gamma.$$

By the theorem of Baker, Birch and Wirsing, we see that

$$L(1, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$$

and

$$L(1, g) = \sum_{n=1}^{\infty} \frac{g(n)}{n} \neq 0.$$

Also, we have (see [15, Theorem 19])

$$L(1, f) = -\sum_{a=1}^{q-1} \widehat{f}(a) \log\bigl(1 - \zeta_q^a\bigr)$$

and

$$L(1, g) = -\sum_{b=1}^{r-1} \widehat{g}(b) \log\bigl(1 - \zeta_r^b\bigr).$$

Thus we have

$$\frac{g(r)}{r} \sum_{a=1}^{q-1} \widehat{f}(a) \log\bigl(1 - \zeta_q^a\bigr) - \frac{f(q)}{q} \sum_{b=1}^{r-1} \widehat{g}(b) \log\bigl(1 - \zeta_r^b\bigr) = 0.$$

Let S and T be maximal linearly independent subsets of

$$\bigl\{\log\bigl(1 - \zeta_q^a\bigr):\ 1 \leqslant a \leqslant q - 1\bigr\} \quad \text{and} \quad \bigl\{\log\bigl(1 - \zeta_r^b\bigr):\ 1 \leqslant b \leqslant r - 1\bigr\}$$

respectively and let I and J be the corresponding indexing sets. That is,

$$I = \{a \mid \log(1-\zeta_q^a) \in S\}, \qquad J = \{b \mid \log(1-\zeta_r^b) \in T\}.$$

Thus,

$$\frac{g(r)}{r}\sum_{a=1}^{q-1}\widehat{f}(a)\log(1-\zeta_q^a) = \sum_{a\in S}\alpha_a\log(1-\zeta_q^a)$$

and

$$\frac{f(q)}{q}\sum_{b=1}^{r-1}\widehat{g}(b)\log(1-\zeta_r^b) = \sum_{b\in T}\beta_b\log(1-\zeta_r^b)$$

where $\alpha_a, \beta_b \in \overline{\mathbb{Q}}$. Further,

$$\sum_{a\in S}\alpha_a\log(1-\zeta_q^a) = \sum_{b\in T}\beta_b\log(1-\zeta_r^b) \neq 0.$$

By Baker's theorem on linear forms in logarithms, the numbers

$$\log(1-\zeta_q^a), \quad \log(1-\zeta_r^b): \quad a\in S,\ b\in T$$

are linearly dependent over $\mathbb{Q}$. The linear dependence of the numbers

$$\log(1-\zeta_q^a), \quad \log(1-\zeta_r^b): \quad 1\leqslant a\leqslant q-1,\ 1\leqslant b\leqslant r-1$$

results in a non-trivial linear expression of the form

$$\sum_{a\in S}n_a\log(1-\zeta_q^a) + \sum_{b\in T}m_b\log(1-\zeta_r^b) = 0$$

where $n_a, m_b \in \mathbb{Z}$ with at least one $n_a \neq 0$ and at least one $m_b \neq 0$. Thus we have

$$\prod_{a\in I}(1-\zeta_q^a)^{n_a} = \prod_{b\in J}(1-\zeta_r^b)^{-m_b} = \alpha. \tag{1}$$

Since

$$\mathbb{Q}(\zeta_q)\cap\mathbb{Q}(\zeta_r) = \mathbb{Q},$$

we see that α is a rational number.

However, the first product in (1) is supported only at prime divisors of q while the second product is supported only at prime divisors of r. Since q and r are co-prime, comparing the norms of the two products, we have

$$\alpha = \pm 1.$$

Hence, we have

$$\prod_{a\in S}\left(1-\zeta_q^a\right)^{2n_a}=\prod_{b\in T}\left(1-\zeta_r^b\right)^{2n_b}=1.$$

This contradicts the linear independence of the elements of S and T. Thus the theorem follows. □

3. Proofs of Theorems 2, 3, and 4

We begin with a brief account of the conjecture of Rohrlich. It asserts that any multiplicative relation of the form

$$\pi^{b/2}\prod_{a\in\mathbb{Q}}\Gamma(a)^{m_a}\in\overline{\mathbb{Q}}$$

with b and m_a in $\mathbb{Z}$ is a consequence of the standard relations mentioned in the introduction. This has another formulation as follows. Let $\mathbb{A}$ be the free abelian group generated by the symbols of the form $[a]$ where $a\in\mathbb{Q}/\mathbb{Z}$. For $\mathbf{a}=\sum m_i[a_i]\in\mathbb{A}$, let us define

$$\Gamma(\mathbf{a}):=\prod_{i:\ a_i\neq 0}\left(\frac{\Gamma(\langle a_i\rangle)}{\sqrt{2\pi}}\right)^{m_i}$$

where $\langle a_i\rangle$ is the smallest positive rational representing the class of a_i in $\mathbb{Q}/\mathbb{Z}$. Then the conjecture of Rohrlich can be stated as:

$\Gamma(\mathbf{a})$ *is algebraic if and only if* $\sum m_i\langle a_i\rangle=\sum m_i\langle ta_i\rangle$ *for all* $t\in(\mathbb{Z}/d\mathbb{Z})^\times$ *where* d *is the lcm of the denominators of the* $\langle a_i\rangle$.

We note that Koblitz and Ogus [7] have shown that $\Gamma(\mathbf{a})$ is algebraic if $\sum m_i\langle a_i\rangle=\sum m_i\langle ta_i\rangle$ for all $t\in(\mathbb{Z}/d\mathbb{Z})^\times$. However the converse is the essence of Rohrlich's conjecture which is yet to be settled.

For the proof of Theorem 2, we will need the following result proved by Chudnovsky [6]:

Theorem *(Chudnovsky). The numbers* $\Gamma(1/4)$ *and* π *are algebraically independent and so are the numbers* $\Gamma(1/3)$ *and* π.

Proof of Theorem 2. For the proof of the first assertion, suppose that

$$a\log\Gamma(1/3)+b\log\Gamma(2/3)=0$$

for rational numbers a and b. Then

$$\Gamma(1/3)^a\Gamma(2/3)^b=1.$$

Since

$$\Gamma(1/3)\Gamma(2/3)=\frac{2\pi}{\sqrt{3}},$$

we have

$$\Gamma(1/3)^{a-b}\left[\frac{2\pi}{\sqrt{3}}\right]^b=1.$$

But by Chudnovsky's theorem, $\Gamma(1/3)$ and π are algebraically independent. Thus

$$a-b=0=b$$

and hence the dimension of $V_\Gamma(3)$ is 2. The proof of the second assertion follows along similar lines.

For the proof of the third assertion, we note that (see [19], for instance):

$$\frac{\Gamma(1/3)^2}{\sqrt{\pi}}=2^{1/3}3^{-1/2}\Gamma(1/6).$$

By Chudnovsky's theorem, $\Gamma(1/3)$ and π are algebraically independent. Thus by the above, we see that $\Gamma(1/6)$ and π are algebraically independent. We also have

$$\log\Gamma(5/6)+\log\Gamma(1/6)=\log(2\pi).$$

Suppose that the dimension of $V_\Gamma(6)$ is 1. Then $\log\Gamma(5/6)$ is a rational multiple of $\log\Gamma(1/6)$ and hence by the above relation,

$$\Gamma(1/6)^r=2\pi$$

for some rational r. This contradicts the algebraic independence of $\Gamma(1/6)$ and π.

The proof of the final assertion is more delicate. We shall need a result of Grinspan [8] (see also Vasilév [18]) who showed that at least two of the three numbers $\Gamma(1/5)$, $\Gamma(2/5)$ and π are algebraically independent. We note that the $V_\Gamma(5)$ is generated by the numbers

$$\log\Gamma(1/5),\quad \log\Gamma(2/5),\quad \log\Gamma(3/5),\quad \log\Gamma(4/5)$$

which is same as the space generated by

$$\log\Gamma(1/5),\quad \log\Gamma(2/5),\quad \log\pi-\log\sin(\pi/5),\quad \log\pi-\log\sin(2\pi/5).$$

Suppose that $\Gamma(1/5)$ and $\Gamma(2/5)$ are algebraically independent. Then $\log\Gamma(1/5)$ and $\log\Gamma(2/5)$ are linearly independent over $\mathbb{Q}$. Suppose that dimension of $V_\Gamma(5)$ is 2. Then the numbers

$$\log\pi-\log\sin(\pi/5),\quad \log\pi-\log\sin(2\pi/5)$$

are expressible as rational linear combinations $\log\Gamma(1/5)$ and $\log\Gamma(2/5)$. Thus,

$$\log\frac{\sin(\pi/5)}{\sin(2\pi/5)}=a\log\Gamma(1/5)+b\log\Gamma(2/5)$$

where $a,b\in\mathbb{Q}$ and hence

$$\frac{\sin(\pi/5)}{\sin(2\pi/5)}=\Gamma(1/5)^a\log\Gamma(2/5)^b.$$

This contradicts that $\Gamma(1/5)$ and $\Gamma(2/5)$ are algebraically independent. A similar argument will work when $\Gamma(1/5)$ (or $\Gamma(2/5)$) and π are algebraically independent. $\square$

Before we proceed to prove Theorem 3, we note that $V_\Gamma(q)$ is also generated by the following sets of numbers:

$$\log\Gamma(a/q)+\log\Gamma(1-a/q),\quad 1\leqslant a<q/2,\ (a,q)=1,$$
$$\log\Gamma(a/q)-\log\Gamma(1-a/q),\quad 1\leqslant a<q/2,\ (a,q)=1.$$

For $q>1$, let $V_\Gamma(q)^+$ be the $\mathbb{Q}$-linear space generated by the numbers

$$\log\Gamma(a/q)+\log\Gamma(1-a/q),\quad 1\leqslant a<q/2,\ (a,q)=1,$$

while $V_\Gamma(q)^-$ be the $\mathbb{Q}$-linear space generated by the numbers

$$\log\Gamma(a/q)-\log\Gamma(1-a/q),\quad 1\leqslant a<q/2,\ (a,q)=1.$$

Clearly these are subspaces of $V_\Gamma(q)$ and our conjecture is equivalent to the assertion that

$$\text{Dimension of } V_\Gamma(q)^+=\text{Dimension of } V_\Gamma(q)^-=\varphi(q)/2\quad\text{and}$$
$$V_\Gamma(q)=V_\Gamma(q)^+\oplus V_\Gamma(q)^-.$$

Proof of Theorem 3. Let $q=p^m$ be a prime power. We claim that dimension of the space $V_\Gamma(q)^+$ is exactly equal to $\varphi(q)/2$. This will prove the theorem.

We have, for $1\leqslant a\leqslant q/2$, $(a,p)=1$,

$$\log\Gamma(a/q)+\log\Gamma(1-a/q)=\log\pi-\log\sin\frac{\pi a}{q}.$$

Suppose that

$$\sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a\left[\log\pi-\log\sin\frac{\pi a}{q}\right]=0$$

where c_a's are integers. This implies that

$$\left[\sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a\right]\log\pi=\sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a\log\sin\frac{\pi a}{q}=\log\prod_{\substack{a=1\\(a,p)=1}}^{q/2}\left[\sin\frac{\pi a}{q}\right]^{c_a}.$$

But c_a's are integers and hence the numbers $(\sin\frac{\pi a}{q})^{c_a}$ are all algebraic numbers. This forces that

$$\sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a=0.$$

Hence we have

$$\sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a\log\sin\frac{\pi a}{q}=0.$$

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Then we have

$$\sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a \log\left[\frac{\sin\frac{\pi a}{q}}{\sin\frac{\pi}{q}}\right] = 0 \quad \text{since} \quad \sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a = 0.$$

But

$$\frac{\sin\frac{\pi a}{q}}{\sin\frac{\pi}{q}} = \zeta_q^{(1-a)/2}\frac{1-\zeta_q^a}{1-\zeta_q}$$

where $\zeta_q := e^{\frac{2\pi i}{q}}$. Thus we have

$$\sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a \log\left[\zeta_q^{(1-a)/2}\frac{1-\zeta_q^a}{1-\zeta_q}\right] = 0$$

and hence

$$\prod_{\substack{a=1\\(a,p)=1}}^{q/2} \left[\zeta_q^{(1-a)/2}\frac{1-\zeta_q^a}{1-\zeta_q}\right]^{c_a} = 1.$$

But the numbers

$$\zeta_{p^m}^{(1-a)/2}\,\frac{1-\zeta_{p^m}^a}{1-\zeta_{p^m}}, \quad 1 \leqslant a < p^m/2,\ (a,p)=1$$

are multiplicatively independent. We refer to the book of Washington [21] (Lemma 8.1) for a proof. Hence all the c_a's are equal to zero. This completes the proof. □

Proof of Theorem 4. For $q > 1$, let $\overline{V_\Gamma(q)^+}$ be the $\overline{\mathbb{Q}}$-linear subspace of $\overline{V_\Gamma(q)}$ generated by the numbers

$$\log\Gamma(a/q) + \log\Gamma(1-a/q), \quad 1 \leqslant a < q/2,\ (a,q)=1.$$

Let $q = p^m$ and $r = P^n$ be distinct prime powers. We claim that either the dimension of the space $\overline{V_\Gamma(q)^+}$ as a $\overline{\mathbb{Q}}$-vector space is exactly equal to $\varphi(q)/2$ or that of the space $\overline{V_\Gamma(r)^+}$ is exactly equal to $\varphi(r)/2$. This will prove the theorem.

Suppose our claim is false. Then we have

$$\sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a\left[\log\pi - \log\sin\frac{\pi a}{q}\right] = 0,$$

$$\sum_{\substack{b=1\\(b,P)=1}}^{r/2} d_b\left[\log\pi - \log\sin\frac{\pi b}{r}\right] = 0$$

where c_a and d_b are all algebraic numbers.

Case 1. Either the c_a's or the d_b's sum up to zero.

Without loss of generality, suppose

$$\sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a = 0.$$

Then we have

$$\sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a \log\sin\frac{\pi a}{q} = 0.$$

By Baker's theorem, this will imply that the numbers

$$\log\sin\frac{\pi a}{q}, \quad 1 \leqslant a < q/2,\ (a,p) = 1$$

are linearly dependent over the rationals. But as shown in the previous theorem, they are linearly independent over $\mathbb{Q}$. So in this case,

$$\dim_{\overline{\mathbb{Q}}} \overline{V_\Gamma(q)} \geqslant \varphi(q)/2.$$

Case 2. Neither the c_a's nor the d_b's sum to zero.

In this case, we have

$$\Bigg[\sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a\Bigg](\log\pi + \log 2i) = \sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a \log\big[\zeta_{2q}^a - \zeta_{2q}^{-a}\big] \neq 0,$$

$$\Bigg[\sum_{\substack{b=1\\(b,P)=1}}^{r/2} d_b\Bigg](\log\pi + \log 2i) = \sum_{\substack{b=1\\(b,P)=1}}^{r/2} d_b \log\big[\zeta_{2r}^b - \zeta_{2r}^{-b}\big] \neq 0.$$

Thus a suitable linear combination of the left-hand sides of the above two equations can be equated to zero and we will have

$$\sum_{\substack{a=1\\(a,p)=1}}^{q/2} C_a \log\big[\zeta_{2q}^a - \zeta_{2q}^{-a}\big] = \sum_{\substack{b=1\\(b,P)=1}}^{r/2} D_b \log\big[\zeta_{2r}^b - \zeta_{2r}^{-b}\big] \neq 0$$

where C_a and D_b are algebraic numbers. The proof then follows mutatis mutandis the proof of Theorem 1. □

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4. Concluding remarks

1. It is worth mentioning that the co-primality condition is required in our conjecture. More precisely, the dimension of the vector space over $\mathbb{Q}$ spanned by the real numbers

$$\log\Gamma(a/q), \quad 1 \leqslant a \leqslant q-1$$

is not always equal to $q-1$. For instance, for the case $q=6$, we have the following relations: (see [19], for instance)

$$\Gamma(2/3) = \frac{2\pi}{\sqrt{3}\Gamma(1/3)}, \qquad \Gamma(5/6) = \frac{2\pi}{\Gamma(1/6)},$$
$$\frac{\Gamma(1/3)^2}{\sqrt{\pi}} = 2^{1/3}3^{-1/2}\Gamma(1/6).$$

Thus,

$$\left[\frac{\Gamma(1/3)}{\Gamma(2/3)}\right]^3 \left[\frac{\Gamma(5/6)}{\Gamma(1/6)}\right]\left[\frac{\Gamma(5/6)}{\Gamma(3/6)}\right] = 1$$

and hence

$$\log\Gamma(1/6) - 3\log\Gamma(2/6) + \log\Gamma(3/6) + 3\log\Gamma(4/6) - 2\log\Gamma(5/6) = 0.$$

2. We have the conjecture of Schanuel which asserts that the transcendence degree of the field $\mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n})$ is at least n when $\alpha_1, \ldots, \alpha_n$ are complex numbers linearly independent over $\mathbb{Q}$. This is a very strong conjecture. For instance, it implies that e and π are algebraically independent.

It has been established in [10] that if we assume the conjecture of Schanuel, $\log\pi$ is not a Baker period. Thus, we see that in the proof of Theorem 3, the term

$$\left[\sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a\right]\log\pi = \sum_{\substack{a=1\\(a,p)=1}}^{q/2} c_a \log\sin\frac{\pi a}{q}$$

is necessarily equal to zero when the coefficients c_a are algebraic numbers. This is because the right-hand side is a Baker period. Then using Baker's theorem and proceeding exactly along the lines adopted in the proof of Theorem 3, the fugitive exceptional prime power alluded to in Theorem 3 can be dispensed with. More precisely, we have the following theorem:

Theorem 5. *Let q be a prime power. Assume that Schanuel's conjecture is true. Then,*

$$\dim_{\overline{\mathbb{Q}}} \overline{V_\Gamma(q)} \geqslant \varphi(q)/2.$$

3. We note that Lang formulated the following conjecture [13] in relation to the conjecture of Rohrlich which deals not only with monomial relations, but more generally with polynomial relations.

Conjecture *(Lang–Rohrlich). The ideal over $\overline{\mathbb{Q}}$ of all algebraic relations among values of the function $(2\pi)^{-1/2}\Gamma(a)$ with $a \in \mathbb{Q}$ is generated by the distribution relations, the functional equation and the oddness of the gamma function.*

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This can be stated as:

For any integer $q > 1$, the extension of $\mathbb{Q}$ generated by the set

$$\{\pi\} \cup \left\{\Gamma(a/q)\colon\ 1 \leqslant a \leqslant q,\ (a,q) = 1\right\}$$

has transcendence degree $1 + \varphi(q)/2$.

The precise relation between the original conjecture of Rohrlich, the above formulation by Lang and the Grothendieck's conjecture (for certain abelian varieties) is explained in detail in the article of André [3].

4. We end by noting that in the function field set up, in relation to the Lang–Rohrlich conjecture, the story has a happy ending thanks to the seminal work of Anderson, Brownawell and Papanikolas [2]. In this set up, we have the "geometric Γ-function" defined as

$$\frac{1}{\Gamma(z)} = z \prod_{\substack{a \in A \\ a \text{ monic}}} \left(1 + \frac{z}{a}\right)$$

where $A = \mathbb{F}_q[T]$ is the analog of $\mathbb{Z}$ while $k = \mathbb{F}_q(T)$ is the analog of $\mathbb{Q}$ in this set up.

Due to the fundamental work of Thakur [17], we know that these functions satisfy analogs of the reflection and multiplication formula. Also for all $z \in A$, $\Gamma(z)$, when defined, belongs to k while for all $z \in k \setminus A$, $\Gamma(z)$ is transcendental over k.

Let f be a monic polynomial in A of positive degree and ω be the fundamental period of the Carlitz module defined as

$$\omega := T(-T)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} \left(1 - T^{1-q^i}\right)^{-1}.$$

This can be regarded as the $\mathbb{F}_q[T]$ analog of $2\pi i$. We refer to the paper of Anderson, Brownawell and Papanikolas for a thorough exposition of the notion of Carlitz module and other related notions. We also recommend the motivating review of the above paper by D. Goss. Let A_+ be the subset of A of monic polynomials.

In their paper, Anderson, Brownawell and Papanikolas [2] prove the following remarkable result:

The extension of k generated by the set

$$\{\omega\} \cup \left\{\Gamma(x)\colon\ x \in \frac{1}{f} A \setminus \left\{\left(\{0\} \cup -A_+\right)\right\}\right\}$$

is of transcendence degree $1 + \frac{q-2}{q-1} \,.\, \#(A/f)^\times$ over k.

We note that the transcendence degree above is $1 + \frac{q-2}{q-1} \,.\, \#(A/f)^\times$ which in the classical set up is analogous to

$$1 + \left(1 - \frac{1}{\#\mathbb{Z}^\times}\right) .\, \#(\mathbb{Z}/q\mathbb{Z})^\times = 1 + \varphi(q)/2.$$

This is the transcendence degree as predicted by Lang–Rohrlich conjecture in the classical case.

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