RAMANUJAN GRAPHS AND ZETA FUNCTIONS

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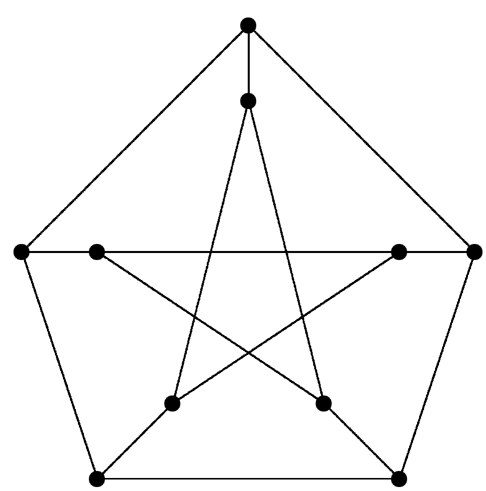
The theory of Ramanujan graphs is a fertile meeting ground for graph theory, number theory, representation theory and arithmetic algebraic geometry. Ramanujan graphs are also expander graphs and these have applications to the "real world" in the optimal construction of telephone networks. (See [3] for a detailed exposition.) In this talk[1], we will focus on the "pure" mathematical aspect of the theory and refer the reader to [17] for an expanded survey as well as the excellent monograph [5].

For the most part, we will be considering only simple graphs, that is, graphs with no loops or multiple edges. Our graphs will also be undirected and finite. Any finite graph $X = (V, E)$ with vertex set V and edge set E is completely determined by its adjacency matrix $A = A_X$ whose rows and columns are parametrized by vertices. We put a 1 in the (i, j)-th position if (i, j) is an edge in X and 0 otherwise. As our graphs are undirected, the matrix A is symmetric. Observe also that given a connected graph, we may define a metric on it as follows. The *distance* between any two vertices is the minimal number of edges needed to traverse from one vertex to the other. The *diameter* of the graph X, denoted $\mathrm{diam}(X)$, is the maximal value of the distance function. The figure below is the celebrated Petersen graph. Here $|V| = 10$ and $|E| = 15$ and the degree of every vertex is 3. It has diameter 3.



The Petersen Graph

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A graph is called *k-regular* if every vertex has degree k. Thus, the Petersen graph is a 3-regular graph. The complete graph K_n is the graph on n vertices in which any two distinct vertices are adjacent. It is an $(n-1)$-regular graph.

The study of the eigenvalues of A and relating them back to properties of the graph X is called spectral graph theory. The process is analogous to the study of prime numbers via the study of the zeros of the Riemann zeta function or the study of Riemannian manifolds by examining the nature of the eigenvalues of the Laplace operator.

As A is a real symmetric matrix, all of its eigenvalues are real and the matrix can be diagonalized by an orthogonal transformation. In particular, if $|V| = n$, we can order the eigenvalues as:

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_n.$$

For a k-regular graph, it is easy to see that $\lambda = k$ is an eigenvalue. Indeed,

$$A \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

In fact $\lambda_1 = k$ for a k-regular graph and all eigenvalues lie in the interval $[-k, k]$. Moreover, one can prove easily that for k-regular graphs X, $-k$ is an eigenvalue if and only if X is bipartite, which means that V can be partitioned into two independent sets.

It is also not hard to show that the multiplicity of $\lambda_1 = k$ is equal to the number of connected components of X. For a k-regular graph, we call any eigenvalue $\lambda \neq \pm k$, a non-trivial eigenvalue.

A *Ramanujan graph* is a connected k-regular graph such that all the non-trivial eigenvalues λ satisfy

$$|\lambda| \leq 2\sqrt{k-1}.$$

In other words,

$$\lambda(X) := \max_{\lambda_i \neq \pm k} |\lambda_i| \leq 2\sqrt{k-1}.$$

This definition seems rather dry without any motivation of why these graphs should be interesting. It also raises the question of what Ramanujan had to do with them. Both of these questions will be answered by the end of the talk.

The complete graph K_n is a Ramanujan graph. Its adjacency matrix has characteristic polynomial

$$(\lambda - (n-1))(\lambda + 1)^{n-1}.$$

This involves the computation of the determinant of a circulant matrix which we leave as an exercise to the reader.

The Petersen graph is also Ramanujan. Its characteristic polynomial is

$$(\lambda - 3)(\lambda + 2)^4(\lambda - 1)^5$$

and to ensure that it is Ramanujan, we must check that

$$2 \leq 2\sqrt{2}$$

which is true.

Why are Ramanujan graphs interesting? Of the many reasons, we give one. If we think of a graph as modeling a communication network, then the diameter measures, in some sense, the efficiency of communication of the network. The smaller the diameter, the better is the communication. The following theorem [4] makes the relationship between the diameter and $\lambda(X)$ precise.

Theorem 1. (Chung, 1989) *Let X be a k-regular graph with n vertices. If X is not bipartite,*

$$\mathrm{diam}(X) \leq \frac{\log(n-1)}{\log(k/\lambda(X))} + 1.$$

If X is bipartite,

$$\mathrm{diam}(X) \leq \frac{\log(n-2)/2}{\log(k/\lambda(X))} + 2.$$

Thus, to minimize the diameter of X, we must minimize $\lambda(X)$. So how small can we make $\lambda(X)$? There are several results one can obtain concerning the size of $\lambda(X)$. We begin with the most elementary. For simplicity, we assume X is not bipartite. Consider the matrix A^2. It is not hard to see that $\mathrm{tr}(A^2) = kn$. But $\mathrm{tr}(A^2)$ is equal to

$$\sum_{i=1}^{n} \lambda_i^2.$$

This is

$$\leq k^2 + \lambda(X)^2(n-1).$$

Thus,

$$\lambda(X) \geq \left(\frac{n-k}{n-1}\right)^{1/2} \sqrt{k}.$$

Hence,

$$\lim_{n\to\infty} \lambda(X) \geq \sqrt{k}.$$

The first non-trivial lower bound is given by the Alon-Boppana theorem [21] which states that:

$$\liminf_{n\to\infty} \lambda(X) \geq 2\sqrt{k-1}.$$

Serre [21] gave the following refinement of this theorem. Fix $\epsilon > 0$. Then, there is a positive constant $c = c(k, \epsilon)$ depending only on k and ϵ such that the every adjacency matrix of a k-regular graph on n vertices has at least cn eigenvalues larger than

$$(2-\epsilon)\sqrt{k-1}.$$

From these results, we see that in trying to construct k-regular graphs with increasing number of vertices, we cannot hope to do better than $\lambda(X) \leq 2\sqrt{k-1}$. These results now give partial motivation for our definition of a Ramanujan graph.

In this context, we mention two more related results. Nilli (also known as Alon) proved the following inequality for λ_2. Let X be a k-regular graph with diameter d. Then

$$\frac{\lambda_2}{2\sqrt{k-1}} \geq 1 + O(1/d).$$

The Alon-Boppana theorem follows from this because as n grows, so does d (exercise) so that λ_2 is positive and we have

$$\lambda(X) \geq \lambda_2.$$

λ_2 plays an important role in many problems of graph theory. A striking theorem of Colin de Verdière [20] appeared more than ten years ago. Given a graph X, consider the family $\mathcal{A}$ of weighted adjacency matrices $A = (a_{ij})$ where $a_{ij} > 0$ if (i, j) is an edge and zero otherwise. For each such matrix, let k_2 be the multiplicity of λ_2 and define

$$\mu(X) = \max_{\mathcal{A}} k_2.$$

In [20] it is proved that X is planar if and only if $\mu(X) \leq 3$. If $\chi(X)$ denotes the chromatic number of X (that is, the minimum number of colors needed to color the vertices of X so that no two adjacent vertices receive the same color), then it is conjectured that

$$\mu(X) + 1 \geq \chi(X).$$

If true, the conjecture would give us a spectral proof of the four color theorem.

Returning to the study of $\lambda(X)$, the Alon-Boppana theorem tells us that to minimize $\lambda(X)$, the best we can hope for is $\lambda(X) \leq 2\sqrt{k-1}$. The question now arises if for each k, there exists a sequence of graphs X_i with an increasing number of vertices, satisfying this bound. That is, can we give an explicit construction of Ramanujan graphs? The only case known for which such sequences have been constructed is when $k - 1$ equals a prime power. In all these cases, the proof that the eigenvalues satisfy the required bound is by means of the Ramanujan conjecture in the theory of modular forms, proved by Deligne [6] in 1974 in the case when $k-1$ is prime, and by the work of Drinfeld [8] in the case when $k - 1$ is a prime power [16]. This explains how Ramanujan's name has entered into the definition of these graphs.

When $k-1$ is prime, the first explicit construction seems to be due to Ihara [11] in 1965. He used the theory of modular curves. Later Margulis [15] and independently Lubotzky, Phillips and Sarnak [14] gave explicit constructions using the theory of automorphic forms.

F. Chung [4] and Winnie Li [21], have constructed more examples of Ramanujan graphs using an idea to be described below. However, their constructions do not give infinite families of k-regular Ramanujan graphs. In a sense to be made precise below, these examples are "abelian" and in some recent joint work with J. Friedman and J.-P. Tillich [9], it is shown that such constructions always lead to only a finite number of such examples. More precisely, we show that for abelian Cayley graphs which are k-regular, the second largest eigenvalue is greater than $k - O(kn^{-4/k})$. Thus, as n goes to infinity, the second largest eigenvalue has a limit value of k. Such a limit result was first proved by Alon and Roichman [1] in 1984. They however, did not obtain any error term for it. The error term in [9] seems to be the best possible at the moment. Contrary to popular misconception, the work in [12] does not establish these results or even the limit result. Concerning the "non-abelian" case, we note that the argument in the paper [9] can be refined to show that one must consider "highly non-abelian" Cayley graphs (defined below) to produce infinitely many examples of k-regular graphs which are Ramanujan.

We will indicate below a simple argument that shows that for abelian Cayley graphs, the second eigenvalue is greater than $k - o(1)$ which is of course weaker than the result proved in [9] but still of interest in the present discussion. As the proof of this result proceeds differently from the arguments of [9], we will give it below after reviewing some basic definitions.

When $k-1$ is a prime power, Morgenstern (1994) constructed Ramanujan graphs of degree k using Drinfeld's theory of automorphic representations of function fields over finite fields, where the Ramanujan conjecture is known.

The first open case is $k = 7$. Are there infinitely many 7-regular graphs which are Ramanujan? A partial answer to this question is provided by:

Theorem 2. (Friedman, 1991) *A random k-regular graph has*

$$\lambda_2 < 2\sqrt{k-1} + 2\log k + O(1).$$

Cayley graphs give us a natural family of regular graphs. They are defined as follows. Let G be a finite group and S a symmetric subset. That is $s \in S$ implies $s^{-1} \in S$. We can construct a k-regular graph with $k = |S|$ as follows. The vertex set is the set of elements of G. We join x and y if and only if $xy^{-1} \in S$. We will denote this graph by $X(G, S)$.

If G is abelian and S symmetric, then, it is not hard to show that the eigenvalues of $X(G, S)$ can be given explicitly by

$$\lambda_\chi = \sum_{s \in S} \chi(s),$$

as χ ranges over all the irreducible characters of G.

This is the theorem used by Chung[4] and Winnie Li [21] in their construction of Ramanujan graphs.

The proof that graphs constructed in this way are Ramanujan reduces to the estimation of character sums in number theory. Here is a concrete example. Let p be a prime congruent to 1 (mod 4) and $G = \mathbb{Z}/p\mathbb{Z}$. Join x to y if and only if $x - y$ is a square. This gives a $(p+1)/2$-regular graph. The non-trivial eigenvalues turn out to be (for $a \neq 0$),

$$\frac{1}{2} + \frac{1}{2} \sum_{x \bmod p} \left(\frac{x}{p}\right) e^{2\pi i a x/p}$$

The last sum is a classical Gauss sum with absolute value $\sqrt{p}$.

Here is the promised argument concerning the second largest eigenvalue. The argument is a nice application of the pigeonhole principle and is based on a well-known result of Dirichlet.

Lemma 1. (Dirichlet) *If $s_1, ..., s_k$ are arbitrary positive real numbers, and N is a given number > 1, then there is an $0 < m < N^k$ and integers $m_1, ..., m_k$ so that*

$$|ms_j - m_j| < 2/N \quad \text{for} \quad 1 \leq j \leq k.$$

Theorem 3. *If $X(G, S)$ is an abelian Cayley graph with $|S| = k$, then the second largest eigenvalue is $k - o(1)$ as $|G|$ tends to infinity. In particular, there are only finitely many k-regular abelian Cayley graphs which are Ramanujan.*

Proof. We begin with some preliminary observations. Suppose that $G = \mathbb{Z}/f\mathbb{Z}$ and S consists of $a_1, ..., a_k$ (say). We may apply the lemma with $s_j = a_j/f$ and $N = [f^{1/k}]$. As

$$g \mapsto e^{2\pi\sqrt{-1}mg/f}$$

is a non-trivial character of G, we see that

$$\sum_j e^{2\pi\sqrt{-1}ma_j/f}$$

is a non-trivial eigenvalue of the adjacency matrix of $X(G, S)$. But by the lemma, we have $|ma_j/f - m_j| < 2/N$ so that the (real) eigenvalue is easily seen to be

$$k - O(k/N^2) = k - O(k/f^{2/k}),$$

by using a simple approximation for the cosine function. This allows us to deduce the result stated in the theorem if we had a sequence "cyclic" Cayley graphs. We modify this idea for the general case. Let $X(G_i, S_i)$ be a family of k-regular connected abelian Cayley graphs with $|G_i|$ tending to infinity. Then, the exponent of G_i (which is the smallest f_i so that $x^{f_i} = 1$ for all x in G_i) must also go to infinity, for otherwise, each element of S_i has bounded order and as G_i is generated by the S_i (via connectedness) we have that G_i has bounded order, a contradiction. By the structure theory of finite abelian groups, we may view G_i as a direct sum of cyclic groups and one of these cyclic components must be isomorphic to $\mathbb{Z}/f_i\mathbb{Z}$. Let $\phi : G_i \to \mathbb{Z}/f_i\mathbb{Z}$ be the projection map. Let us fix i and write $a_1, ..., a_k$ for the images under ϕ of the elements of S_i in the cyclic component $\mathbb{Z}/f_i\mathbb{Z}$. Let f_i be the exponent of G_i and choose N so that $N = [f_i^{1/k}]$ in the lemma with $s_j = a_j/f_i$. Then there is a non-zero $m < f_i$ so that

$$|ms_j - m_j| < 2/N \quad \text{for} \quad 1 \leq j \leq k.$$

Since

$$g \mapsto e^{2\pi\sqrt{-1}m\phi(g)/f_i}$$

is a non-trivial character of G_i, we deduce that for the non-trivial eigenvalue

$$\sum_j e^{2\pi\sqrt{-1}ma_j/f_i},$$

we have that this is $k - O(k/N^2)$ by an easy application of Taylor's theorem to the cosine function. This gives a final estimate for λ_2 as greater than $k - O(k/f_i^{2/k})$, which completes the proof, since we already noted that f_i tends to infinity as i tends to infinity. □

If G is non-abelian, the description of the eigenvalues of $X(G, S)$ is more difficult. However, if S is assumed to be invariant under conjugation, one can show the eigenvalues are parametrized by

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{s \in S} \chi(s).$$

This was discovered independently by L. Babai[2] and Diaconis-Shahshahani[7].

In 1988, Lubotzky, Phillips, and Sarnak [14] gave the following construction. Let p and q be unequal primes $p \equiv q \equiv 1 (\bmod 4)$. Let u be an integer with $u^2 \equiv -1 (\bmod q)$.. By a classical theorem of Jacobi, there are exactly $8(p+1)$ ways of writing p as a sum of four squares:

$$p = a^2 + b^2 + c^2 + d^2.$$

If we specify $a > 0$, b, c, d even, there are exactly $p+1$ solutions $v = (a, b, c, d)$. To each such v, we associate

$$\tilde{v} = \begin{pmatrix} a + ub & c + ud \\ -c + ud & a - ub \end{pmatrix}$$

a matrix in $PGL_2(\mathbb{Z}/q\mathbb{Z})$. One can verify the set S of such matrices $\tilde{v}$ is a symmetric subset and the group generated by them is $PSL_2(\mathbb{Z}/q\mathbb{Z})$ (see page 97 of [13]). To prove that $X(G, S)$ is Ramanujan, one needs the Jacquet-Langlands correspondence and the full strength of the Ramanujan conjecture for weight 2 forms (proved by Eichler).

There is an alternate formulation of the whole problem due to Ihara [11] which is highly suggestive. Let us first observe that the (i, j)-entry of A^r counts the number of walks of length r from i to j. Let A_r be the matrix whose (i, j)-th entry is the number of "proper" walks of length r from i to j without backtracking. Then, it is easy to show that for $r \geq 2$,

$$AA_r = A_{r+1} + (k-1)A_{r-1}.$$

For $r = 1$, we have

$$A^2 = A_2 + kI.$$

Indeed, using the notation M_{ij} to denote the (i, j)-th entry of the matrix M, we have

$$(A_{r+1})_{ij} = \sum_{t=1}^{n} (A_r)_{it} A_{tj} - (k-1)(A_{r-1})_{ij}.$$

To see this, note that the left hand side represents the number of walks of length $r+1$ from i to j without backtracking. This number can also be obtained by counting first the number of ways of extending a proper walk (without backtracking) of length r from i to t to the vertex j and this can be done in A_{tj} ways. We must remove from this count those which now have backtracking. This backtracking could have been introduced only in the last step, which means that we retraced our step at the last stage. Thus, the number to remove is $(A_{r-1})_{ij}(k-1)$ since there is only one vertex excluded from the possible choices. This represents the right hand side of the above equation.

We can use this recurrence to give, following Pizer [18], a simple proof of the Alon-Bopanna theorem. To this end, it is convenient to define the matrices B_r as follows:

$$B_0 = I, \quad B_1 = A, \quad \text{and} \quad B_1 B_r = B_{r+1} + (k-1)B_{r-1}, \quad \text{for } r \geq 1.$$

An easy induction argument shows that

$$B_r = A_r + A_{r-2} + \cdots + A_1$$

if r is odd and

$$B_r = A_r + A_{r-2} + \cdots + A_2 + I$$

if r is even. In either case, we see that the trace of B_r is non-negative, a fact we will utilise below. At this point, we make the following variation of the argument in [18]. We may re-normalize and set $C_r = B_r(k-1)^{-r/2}$ so that the the above recursion becomes

$$C_1 C_r = C_{r+1} + C_{r-1},$$

for $r \geq 1$. This is reminiscent of the recursion for the cosine function:

$$(2\cos\theta)(2\cos r\theta) = 2\cos(r+1)\theta + 2\cos(r-1)\theta.$$

Accordingly, we define $C_{-r} = C_r$ for $r \geq 0$ and verify that the recursion holds for the C_i's with negative subscripts also. Consequently, we are led to conjecture that for $r \geq 0$,

$$C_1^r = \sum_{j \geq 0} \binom{r}{j} C_{r-2j},$$

which is easily established by induction. Indeed, for $r = 0$, this is clear. Then,

$$C_1^{r+1} = \sum_{j \geq 0} \binom{r}{j} C_1 C_{r-2j}$$

which by the recursion formula for the C_i's is

$$\sum_{j\geq 0}\binom{r}{j}\left(C_{r-2j+1}+C_{r-2j-1}\right)$$

$$=\sum_{j\geq 0}\left\{\binom{r}{j}+\binom{r}{j-1}\right\}C_{r+1-2j}=\sum_{j\geq 0}\binom{r+1}{j}C_{r+1-2j},$$

as desired. We can immediately deduce the Alon-Bopanna theorem from this as follows. By taking traces of both sides of the identity, and using the fact that the traces are non-negative, we get for $r=2s$

$$2k^{2s}+(n-2)\lambda(X)^{2s}\geq\binom{2s}{s}(k-1)^s n.$$

Now

$$(2s+1)\binom{2s}{s}\geq(1+1)^{2s}\geq\binom{2s}{s}$$

so that

$$\lim_{s\to\infty}\binom{2s}{s}^{1/2s}=2.$$

Thus, letting n tend to infinity, we get

$$\lim_{|X|\to\infty}\lambda(X)^{2s}\geq\binom{2s}{s}(k-1)^s$$

and then taking $2s$-th roots and letting s tend to infinity gives us the result.

These recursions allow us to associate a "zeta function" to a regular graph and establish the rationality of this zeta function, somewhat analogous to the situation of zeta functions attached to algebraic varieties over a finite field. Indeed, Using the recurrence for the A_i's, one can show the formal identity

$$\sum_{r=0}^{\infty}A_r t^r=(1-t^2)\left(1-At+(k-1)t^2 I\right)^{-1}.$$

Following Ihara, a proper walk whose endpoints are equal is called a closed geodesic. If γ is a closed geodesic, then γ^r is just the closed geodesic obtained by repeating γ r times. A closed geodesic which is not the power of another one is called a prime geodesic. Two closed geodesics $(x_0,...,x_a)$ and $(y_0,...,y_b)$ are called equivalent if $a=b$ and there is a d such that $y_i=x_{i+d}$ for all i. An equivalence class of a prime geodesic is called a prime geodesic cycle.

Inspired by the theory of the Selberg zeta function, Ihara defined the zeta function of X as follows: put $q=k-1$.

$$Z_X(s)=\prod_p\left(1-q^{-s\ell(p)}\right)^{-1}$$

where the product is over prime geodesic cycles and $\ell(p)$ is the length of any element in the cycle p.

Theorem 4. (Ihara, 1966) *Let X be a k-regular graph. Put $g=(q-1)|X|/2$ and $u=q^{-s}$. Then*

$$Z_X(s)=(1-u^2)^{-g}\det(1-Au+qu^2 I)^{-1}.$$

Moreover, $Z_X(s)$ satisfies the "Riemann hypothesis" (that is, all the singular points of $Z_X(s)$ in the region $0 < \Re(s) < 1$ lie on $\Re(s) = 1/2$) if and only if X is a Ramanujan graph.

Hashimoto [10] as well as Stark and Terras [19] have extended the notion of a zeta function for an arbitrary graph. Let N_r be the number of closed walks γ of length r such that neither γ or γ^2 have backtracking. Let

$$Z_X(t) = \exp\left(\sum_{r=1}^{\infty} \frac{N_r t^r}{r}\right).$$

By a theorem of Hyman Bass, this is a rational function of t. What is the meaning of a "Riemann hypothesis" for this zeta function? This zeta function is not well-understood. However, some theory is slowly emerging. For example, Hashimoto has proved that the residue at $t = 1$ is related to the number of spanning trees of the graph X. This is analogous to the class number formula for the Dedekind zeta function of an algebraic number field. It is hoped that this analogy would lead to new insights in graph theory.

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