Canad. Math. Bull. Vol. **20** (1), 1977

IRRATIONAL NUMBERS ARISING FROM CERTAIN DIFFERENTIAL EQUATIONS

BY
M. RAM MURTY AND V. KUMAR MURTY

Niven [3] gave a simple proof that π is irrational. Koksma [2] modified Niven's proof to show that e^r is irrational for every non-zero rational r. Dixon [1] made a similar modification to show that π is not algebraic of degree 2. In this note, we prove a general theorem which gives Niven's and Koksma's results as easy corollaries. A suitable modification in our proof also gives Dixon's result.

THEOREM 1. *Let G be non-trivial solution of the equation*

$$L(u)=p_0u^{(n)}+p_1u^{(n-1)}+\cdots+p_nu=0$$

where p_i are rational numbers and $p_n\neq 0$. If $b>0$ is such that $G(x)\geq 0$ on $[0, b]$ and $G^{(i)}(0)$, $G^{(i)}(b)$ are rational for $0\leq i\leq n-1$, then b is irrational.

Proof. Without any loss of generality, we may suppose that the p_i are integers. Suppose b is rational and set $b=p/q$, $(p, q)=1$, $p, q\in\mathbb{Z}$. Set $f_m(x)=1/m!(qx)^m(p-qx)^m$, where m is a natural number. It is easy to see that $f_m^{(k)}(0)$ are integers for $k\geq 0$ and since $f_m(x)=f_m(b-x)$, the same is true of $f_m^{(k)}(b)$. Now define the sequence $\{t_k\}$ recursively as follows:

$$t_0=1,$$
$$p_nt_1-p_{n-1}t_0=0,$$
$$p_nt_2-p_{n-1}t_1+p_{n-2}t_0=0,$$
$$p_nt_{n-1}-p_{n-1}t_{n-2}+\cdots+(-1)^{n-1}p_1t_0=0,$$
$$p_nt_{n+r}-p_{n-1}t_{n+r-1}+\cdots+(-1)^np_0t_r=0 \quad \text{for} \quad r\geq 0.$$

Clearly, $p_n^kt_k$ is an integer for $k\geq 0$. Let

$$F_m(x)=\sum_{r=0}^{2m} t_rf_m^{(r)}(x).$$

Received by the editors Sept. 15, 1976.

If L^* is the adjoint of L, we have

$$\begin{aligned} L^*(F_m(x)) &= \sum_{k=0}^{n} (-1)^k p_{n-k} F_m^{(k)}(x) \\ &= \sum_{k=0}^{n} (-1)^k p_{n-k} \sum_{r=0}^{2m} t_r f_m^{(r+k)}(x) \\ &= \sum_{s=0}^{2m} f_m^{(s)}(x) \sum_{r+k=s} (-1)^k p_{n-k} t_r = p_n f_m(x). \end{aligned}$$

Letting

$$P(u, v) = u\left[p_{n-1}v - \frac{d}{dx}(p_{n-2}v) + \cdots + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(p_0 v)\right] +$$
$$\frac{du}{dx}\left[p_{n-2}v - \frac{d}{dx}(p_{n-3}v) + \cdots + (-1)^{n-2}\frac{d^{n-2}}{dx^{n-2}}(p_0 v)\right] +$$
$$+ \cdots + \frac{d^{n-1}u}{dx^{n-1}}(p_0 v),$$

we have by Lagrange's identity,

$$F_m(x)L(G) - G(x)L^*(F_m(x)) = \frac{d}{dx}P(G, F_m),$$

so that

$$-\int_0^b p_n f_m(x)G(x)\,dx = [P(G, F_m)]_0^b$$

since $L(G) = 0$. As $p_n^k t_k$ is an integer, it follows that $p_n^{2m} F_m^{(w)}(x)$ is an integer for $x = 0$ and b, $w \geq 0$. Thus, if A denotes the products of the denominators of $G^{(i)}(0)$ and $G^{(i)}(b)$, $0 \leq i \leq n-1$ (when expressed in lowest terms), $Ap_n^{2m}[P(G, F_m)]_0^b$ is an integer for every m. Now

$$Ap_n^{2m}[P(G, F_m)]_0^b = -Ap_n^{2m+1}\int_0^b f_m(x)G(x)\,dx.$$

If B and C are such that $|G(x)| \leq B$, $|qx(p-qx)| \leq C$ on $[0, b]$ we have

$$0 < Ap_n^{2m+1}\left|\int_0^b f_m(x)G(x)\,dx\right| < \frac{bBAp_n^{2m+1}C^{2m}}{m!}.$$

If m is sufficiently large, the right hand side is < 1, giving a contradiction. Hence b is irrational.

COROLLARY. (1) π^2 *is irrational,* (*hence so also is* π). (2) $\log r$ *is irrational for every rational* $r > 0$, $r \neq 1$. (3) e^r, $\sin r$, $\cos r$, $\cosh r$ $\sinh r$ *are irrational for every non-zero rational* r.

Proof. If π^2 is rational, consider $y'' + \pi^2 y = 0$ which has as a solution $(1/\pi)\sin \pi x$. For $b = 1$, we get a contradiction. This proves (1). (2) and (3) are proved similarly, using the equation $y' - y = 0$ or $y'' \pm y = 0$.

The following theorem is more arithmetical in nature.

THEOREM 2. *Let G be a non-trivial solution of $y^{(n)} + ty = 0$ where $t = (u/v,)(u, v) = 1$, is a non-zero rational. Suppose $G^{(i)}(0)$ is rational for $0 \le i \le n-1$ and for some $r \ne 0$ with $(r, n) = 1$, we have $G^{(r)}(0) \ne 0$. If β is a non-zero rational, then $G^{(n-1)}(\beta)$ is irrational.*

Proof. Let $\beta = (a/b,)(a, b) = 1$. Define

$$f_p(x) = \frac{(\beta - x)^{np}[\beta^n - (\beta - x)^n]^{p-1} b^{np+(n-1)(p-1)}}{(p-1)!},$$

where p is a prime soon to be specified. If we compute the t_k in Theorem 1 for the equation $y^{(n)} + ty = 0$, we find $t_k = 0$ if $k \not\equiv 0 \pmod n$ and in case $k = sn$, $t_{sn} = (-1)^{sn-s} t^s$. If we set

$$F_p(x) = \sum_{k=0}^{M} t_k f_p^{(k)}(x)$$

where $M = n(2p-1)$, we have as in Theorem 1, $L^*(F_p(x)) = tf_p(x)$, where L^* is the adjoint of $L(y) = y^{(n)} + ty$. Since f_p is a polynomial of degree M, $f_p^{(k)}(x) \equiv 0$ for $k > M$. If we set $\beta - x = y$ and $g_p(y) = y^{np}(\beta^n - y^n)^{p-1}$, then

$$g_p(y) = \sum_{i=0}^{p-1} (-1)^i \beta^{n(p-1-i)} \binom{p-1}{i} y^{n(p+i)}$$

from which it follows at once that $f_p^{(k)}(\beta) = 0$ for all $k \ne n(p+i)$, $0 \le i \le p-1$ and

$$f_p^{(k)}(\beta) = (-1)^{n(p+i)+i} \frac{b^{M-p+1}}{(p-1)!} \beta^{n(p-1-i)} \binom{p-1}{i} [n(p+i)]!$$

for $k = n(p+i)$. Hence $v^{2p-1} F_p(\beta)$ is an integer divisible by p. Since f_p has a zero of order $p-1$ at $x = 0$, we have $f_p^{(k)}(0) = 0$ for $k < p-1$. Writing $f_p(x) = (x^{p-1}/(p-1)!) h_p(x)$ we see from $(p-1)! f_p^{(k)}(x) = \sum_{s=0}^{k} \binom{k}{s} [x^{p-1}]^{(s)} [h_p(x)]^{(k-s)}$ that $f_p^{(k)}(0) = \binom{k}{p-1} h_p^{(k-p+1)}(0)$ for $k \ge p-1$. Clearly $f_p^{(k)}(0)$ is an integer as $h_p^{(k-p+1)}(0)$ is an integer. Also $\binom{k}{p-1}$ is divisible by p if $k \ge p$, and $k \not\equiv -1 \pmod p$. If $k \ge p$ and $k - p + 1 \equiv 0 \pmod p$, then $h_p^{(k-p+1)}(0)$ is divisible by p. If $k = p-1$, $f_p^{(p-1)}(0) = n^{p-1} a^{np+(n-1)(p-1)}$. Hence, $f_p^{(k)}(0)$ is divisible by p unless $k = p-1$. As $(r, n) = 1$, let p be a prime $> na$, congruent to $-r \pmod n$. As $G^{(r)}(0) \ne 0$, and $p - 1 \equiv n - r - 1 \pmod n$, the term $f_p^{(p-1)}(0)$ occurs once and only once in $\sum_{k=0}^{n-1}(-1)^k G^{(k)}(0)F_p^{(n-k-1)}$ and that is in the expression for $F_p^{(n-r-1)}(0)$. Let N be the product of all the denominators of the rationals $G(0)$, $G'(0), \ldots, G^{(n-1)}(0)$, $G^{(n-1)}(\beta)$. (Here, we are supposing $G^{(n-1)}(\beta)$ is rational and will arrive at a contradiction). Thus, if $p > \max(na, NG^{(r)}(0), uv)$, all terms in

$$Nv^{2p-1}\{G^{(n-1)}(\beta)F_p(\beta) - \sum_{k=0}^{n-1}(-1)^k G^{(k)}(0)F_p^{(n-k-1)}(0)\}$$

are divisible by p except one term (the one involving $G^{(r)}(0) \neq 0$). Now, as in the proof of Theorem 1,

$$-uNv^{2p}\int_0^\beta G(x)f_p(x)\,dx = Nv^{2p-1}\{G^{(n-1)}(\beta) \times F_p(\beta) - \sum_{k=0}^{n-1}(-1)^k G^{(k)}(0)F_p^{(n-k-1)}(0)\}.$$

Thus, it follows $uNv^{2p}\int_0^\beta G(x)f_p(x)\,dx \neq o$ for an infinity of primes p, using Dirichlet's theorem. On the other hand, we know $uNv^{2p}\int_0^\beta G(x)f_p(x)\,dx$ is an integer. This is a contradiction since

$$\lim_{p\to\infty}\left|uNv^{2p}\int_0^\beta G(x)f_p(x)\,dx\right| = 0.$$

This proves the theorem.

COROLLARY. *Let p be an odd prime and G a non-trivial solution of $y^{(p)} + ty = 0$, t a non-zero rational. If $G(0), \ldots, G^{(p-1)}(0)$ are rational and at least two of them are non-zero, then $G(\beta), G'(\beta), \ldots, G^{(p-1)}(\beta)$ are irrational for any non-zero rational β.*

REMARK. The case $p = 2$ has been covered by a corollary of Theorem 1.

Proof. As at least two of $G(0)$, $G'(0), \ldots, G^{(p-1)}(0)$ are non-zero, there is an r such that $G^{(r)}(0) \neq 0$ and $(r, p) = 1$. The conditions of the theorem are satisfied and so $G^{(p-1)}(\beta)$ is irrational. As $G^{(i)}(x)$ also satisfies the conditions of the theorem for $0 < i \leq p-1$ the result follows.

ACKNOWLEDGEMENT. We would like to thank Professor P. R. Beesack for his helpful suggestions.

REFERENCES

1. J. D. Dixon, *π is not algebraic of degree one or two*, *Amer. Math. Monthly*, **69** (1962), 632.
2. J. F. Koksma, *On Niven's proof that π is irrational*, *Nieuw Archief voor Wiskunde*, **(2) 23** (1949), 39.
3. I. Niven, *A simple proof that π is irrational*, *Bull. Amer. Math. Soc.*, **53** (1947), 509.

DEPARTMENT OF MATHEMATICS
CARLETON UNIVERSITY
OTTAWA, ONTARIO