# A SIMPLE PROOF OF THE WIENER-IKEHARA TAUBERIAN THEOREM

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ABSTRACT. We present a simple and elegant proof of the Wiener-Ikehara Tauberian theorem, relying only upon the technique of contour integration. We also discuss some of its applications in number theory.

## 1. INTRODUCTION

To motivate the notion of Tauberian theorems, let us begin with a brief discussion of Abel's theorem. Let $\sum_{n=0}^{\infty} a_n x^n$, $x \in \mathbb{R}$ be a power series centered at $0$ having radius of convergence $1$. At the boundary of the region of convergence, i.e. at $|x| = 1$, the series may converge or diverge. Abel's theorem states that if the series converges at a boundary point, then it is reasonably well behaved in the sense that it is continuous at that point. More precisely, if

$$\sum_{n=0}^{\infty} a_n = A, \tag{1}$$

then

$$\lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = A. \tag{2}$$

Broadly speaking, Tauberian theorems are conditional converses of Abel's theorem. They derive their name from a theorem of A. Tauber [8] published in $1897$, which states that if $(2)$ is satisfied and we have the growth condition $a_n = o(1/n)$ on the coefficients of the power series, then $(1)$ holds. These growth conditions were subsequently relaxed, most notably by Hardy and Littlewood.

Some of the most interesting applications of Tauberian theorems pertain to analytic number theory. In this context, Tauberian results can be thought of as estimates for the partial sums of coefficients of certain Dirichlet series. An important result of this type is the Wiener-Ikehara theorem. Introduced by Ikehara [1] in $1931$, it generalizes a theorem of Landau [3], by applying a Tauberian result obtained by Wiener. Proofs of this and other Tauberian theorems in the literature are usually found to be quite involved.

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A well known application of the Wiener-Ikehara theorem is to the derivation of the prime number theorem. In 1980, Newman [6] gave an ingenious short proof of the prime number theorem. We modify Newman's proof to derive the Wiener-Ikehara Tauberian theorem. That this can be done was also recognized by Korevaar [2]. However, our presentation is simplified and our theorem more general. We derive as a consequence an assortment of prime number theorems following the arrangement of Serre [7].

## 2. The analytic theorem

The following analytic theorem of Newman [6], is the key result that will be used to prove the Tauberian theorem. The proof is an application of Cauchy's residue theorem. Newman's novel idea was the insertion of a new kernel into the relevant integral, playing a role similar to that of the Fejér kernel in standard proofs of the Tauberian theorem.

**Theorem 1.** *For $t \geq 0$, let $f(t)$ be a bounded and locally integrable function and let $g(s) := \int_0^\infty f(t)e^{-st}dt$ for $\mathrm{Re}(s) > 0$. If $g(s)$ has an analytic continuation to $\mathrm{Re}(s) \geq 0$, then $\int_0^\infty f(t)dt$ exists and equals $g(0)$.*

*Proof.* For $T > 0$, let $g_T(s) = \int_0^T f(t)e^{-st}dt$. This integral converges for all values of $s$ and it is easy to see that $g_T(s)$ is an entire function. We need to show that

$$\lim_{T\to\infty} g_T(0) = g(0).$$

We will denote $\mathrm{Re}(s)$ by $\sigma$. Fix $R > 0$ and consider the positively oriented contour $\mathscr{C}$ shown in Figure 1 below. Here $\delta > 0$ (depending on $R$) is chosen small enough so that $g(s)$ is analytic on $\mathscr{C}$. Indeed, as $g(s)$ is analytic on the line $\sigma = 0$, one can cover the vertical strip from $(0, R)$ to $(0, -R)$ with open balls, on each of which $g(s)$ is analytic. Compactness of this strip allows one to obtain a finite subcover, which then gives the desired $\delta$.

We use the following notation :

$$\mathscr{C}_+ = \mathscr{C} \cap \{s : \sigma > 0\}, \quad \mathscr{C}_- = \mathscr{C} \cap \{s : \sigma < 0\}.$$

We also denote the semicircle of radius $R$ to the left of the line $\sigma = 0$ by $C_-$. We will use the big $O$ notation, treating everything other than the variables $T, R$ and $\sigma$ as constants.

Cauchy's theorem gives us

$$I_{\mathscr{C}} := \frac{1}{2\pi i}\int_{\mathscr{C}} (g(s) - g_T(s))e^{sT}\left(1 + \frac{s^2}{R^2}\right)\frac{ds}{s} = g(0) - g_T(0), \tag{3}$$

as the integrand is analytic inside $\mathscr{C}$ except for a simple pole at $s = 0$. We denote the corresponding integrals over $\mathscr{C}_+$ and $\mathscr{C}_-$ as $I_{\mathscr{C}_+}$ and $I_{\mathscr{C}_-}$ respectively. Let

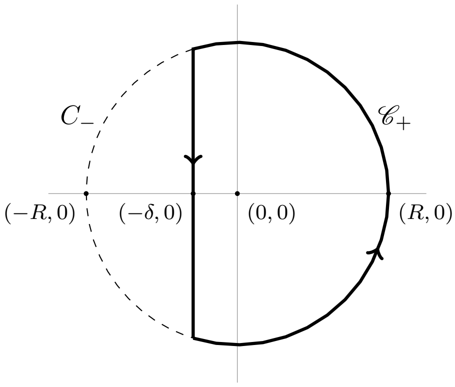


FIGURE 1. The contour $\mathscr{C}$

$M = \sup_{t\geq 0}|f(t)|$. On $\mathscr{C}_+$, as $\sigma > 0$, we have

$$|g(s) - g_T(s)| = \left|\int_T^\infty f(t)e^{-st}dt\right| \leq M\int_T^\infty e^{-\sigma t}dt \ll \frac{e^{-\sigma T}}{\sigma}.$$

Using $s = Re^{i\theta}$ and $R\cos\theta = \sigma$ on $\mathscr{C}_+$, we obtain the following estimate for the kernel

$$\left|e^{sT}\frac{1}{s}\left(1+\frac{s^2}{R^2}\right)\right| = e^{\sigma T}\left|\frac{1}{Re^{i\theta}} + \frac{e^{i\theta}}{R}\right| = e^{\sigma T}\left|\frac{2\cos\theta}{R}\right| \ll e^{\sigma T}\frac{|\sigma|}{R^2}. \tag{4}$$

Thus, the contribution to (3) from the path $\mathscr{C}_+$ of length $\pi R$ is

$$|I_{\mathscr{C}_+}| \ll \frac{1}{R^2}\left|\int_{\mathscr{C}_+} ds\right| \ll \frac{1}{R}.$$

On $\mathscr{C}_-$, we examine $g_T(s)$ and $g(s)$ separately. Consider first the integral

$$I_1 := \frac{1}{2\pi i}\int_{\mathscr{C}_-} g_T(s)e^{sT}\left(1+\frac{s^2}{R^2}\right)\frac{ds}{s}$$

As $g_T(s)$ is entire and the rest of the integrand is analytic to the left of $\sigma = 0$, we have by Cauchy's theorem,

$$I_1 = \frac{1}{2\pi i}\int_{C_-} g_T(s)e^{sT}\left(1+\frac{s^2}{R^2}\right)\frac{ds}{s}.$$

That is, we can integrate over the semicircle $C_-$ instead of $\mathscr{C}_-$, with $C_-$ oriented in the same manner as $\mathscr{C}_-$. Then, noting that $\sigma < 0$ in this case, we have

$$|g_T(s)| = \left|\int_0^T f(t)e^{-st}dt\right| \leq M\int_0^T e^{-\sigma t}dt \ll \frac{e^{-\sigma T}}{|\sigma|},$$

and the estimate $(4)$ holds on $C_-$ exactly as it did on $\mathscr{C}_-$. We obtain $|I_1| \ll 1/R$ in the same way as done for $|I_{\mathscr{C}_+}|$ above. This leaves us with the integral

$$I_2 := \frac{1}{2\pi i}\int_{\mathscr{C}_-} g(s)e^{sT}\left(1+\frac{s^2}{R^2}\right)\frac{ds}{s}.$$

As $\mathscr{C}_-$ is contained in a compact set on which $g(s)$ is analytic, $|g(s)|$ can be bounded in terms of $R$ on $\mathscr{C}_-$. As the estimate $(4)$ holds on the arcs of $\mathscr{C}_-$, the integrand in this region is of the order of

$$|\sigma|e^{\sigma T} \text{ as } T\to\infty,$$

with the implicit constant depending on $R$. Recalling that $\sigma < 0$ in this region, the above quantity can be compared to the real valued function $xe^{-x}$, which attains a global maximum of $e^{-1}$ (as can be checked by standard derivative tests). Thus,

$$|\sigma|e^{\sigma T} \le e^{-1}/T,$$

giving a bound of $O_R(1/T)$ for the integrand over the arcs of $\mathscr{C}_-$. As the length of the arcs is again a function of $R$ which gets absorbed into the implied constant, we see that the contribution to $|I_2|$ from the arcs of $\mathscr{C}_-$ is $O_R(1/T)$ as $T\to\infty$. On the vertical strip of $\mathscr{C}_-$, as $\sigma = -\delta$, we have

$$|e^{sT}| = e^{-\delta T}.$$

The rest of the integrand of $I_2$ is analytic in this region and hence absolutely bounded in terms of $R$. The contribution to $|I_2|$ from this strip is thus $O_R(e^{-\delta T})$. Putting everything together, we have obtained, as $T\to\infty$,

$$\begin{aligned}|g(0)-g_T(0)| = |I_{\mathscr{C}}| &\le |I_{\mathscr{C}_+}| + |I_1| + |I_2| \\ &\ll O\left(\frac{1}{R}\right) + O_R\left(\frac{1}{T}\right) + O_R(e^{-\delta T}).\end{aligned}$$

As $R$ is arbitrary, the right hand side can be made as small as needed. This completes the proof. □

## 3. The proof of the Tauberian theorem

We establish the following version of the Tauberian theorem, applicable in many settings.

**Theorem 2.** *Let*

$$G(s) = \sum_{n=1}^{\infty} b_n/n^s$$

*be a Dirichlet series with non-negative coefficients, satisfying*

(a) *$G(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$.*
(b) *The function $G(s)$ extends meromorphically to the region $\mathrm{Re}(s) \ge 1$, having no poles except possibly a simple pole at $s = 1$ with residue $R$.*

(c) $B(x) := \sum_{n \le x} b_n = O(x)$.

*Then, as* $x \to \infty$,

$$B(x) = Rx + o(x).$$

We begin by making some elementary observations. For any $\epsilon > 0$,

$$\sum_{n \le x} b_n \le \sum_{n=1}^{\infty} b_n \left(\frac{x}{n}\right)^{1+\epsilon}$$

The right hand side is $x^{1+\epsilon}G(1+\epsilon)$ which is of the order of $x^{1+\epsilon}/\epsilon$ since $G(1+\epsilon) \ll 1/\epsilon$. Choosing $\epsilon = (\log x)^{-1}$ gives $B(x) \ll x \log x$. Note that this estimate does not use any information about the behaviour of $G(s)$ on $\mathrm{Re}(s) = 1$, except at $s = 1$. Normally (c) is not needed in the general Wiener-Ikehara Tauberian theorem. One can deduce it from the other assumptions, as indicated in the concluding remarks. However, in practically all applications, this condition is found to be readily available and we retain it for the sake of a shorter proof.

A natural starting point for this and indeed most proofs of the Tauberian theorem is what is known as Abel's trick: for $\mathrm{Re}(s) > 1$, we have

$$G(s) = s \int_1^{\infty} \frac{B(x)}{x^{s+1}} dx. \tag{5}$$

This can be derived using partial summation, as is done in Exercise 2.1.5 of [4]. We proceed to prove the above theorem.

*Proof of Theorem* 2. Without loss of generality, we may suppose $R > 0$. Indeed, if $R \le 0$, it is enough to prove the result for $G(s) + m\zeta(s)$, where $\zeta$ is the Riemann-zeta function and $m$ is an integer greater than $|R|$. For $R > 0$, replacing $b_n$ by $b_n/R$ if needed, we may assume $R = 1$. From our discussion above, we have for $\mathrm{Re}(s) > 1$,

$$\frac{G(s)}{s} - \frac{1}{s-1} = \int_1^{\infty} \frac{B(x) - x}{x^{s+1}} dx \tag{6}$$

After the change of variable $x$ to $e^u$ and then $s$ to $s+1$, we have for $\mathrm{Re}(s) > 0$,

$$\frac{G(s+1)}{s+1} - \frac{1}{s} = \int_0^{\infty} \frac{B(e^u) - e^u}{e^u} e^{-su} du,$$

which is suitable for application of Theorem 1 because the function

$$f(u) := (B(e^u) - e^u)/e^u$$

is bounded on account of (c) and the left hand side has an analytic continuation to $\mathrm{Re}(s) \ge 0$ by (b). Hence, by Theorem 1, the integral

$$\int_0^{\infty} \frac{B(e^u) - e^u}{e^u} du = \int_1^{\infty} \frac{B(t) - t}{t^2} dt \tag{7}$$

converges. We will show that $B(x) \sim x$ as $x \to \infty$. Suppose not. Then either $\lim_{x\to\infty} B(x)/x$ does not exist or does not equal 1 if it exists. In either case, we see that $\limsup_{x\to\infty} B(x)/x > 1$ or $\liminf_{x\to\infty} B(x)/x < 1$. Suppose the former inequality holds (the latter case can be treated similarly). Then there exists some $\lambda > 1$ such that $B(x) \geq \lambda x$ for infinitely many $x$. As there exists $x$ arbitrarily large with $B(x) \geq \lambda x$ and $B(x)$ is an increasing function, we have

$$\int_x^{\lambda x} \frac{B(t)-t}{t^2} dt \geq \int_x^{\lambda x} \frac{\lambda x - t}{t^2} dt$$
$$= \int_1^{\lambda} \frac{\lambda x - vx}{(vx)^2} x dv = \int_1^{\lambda} \frac{\lambda - v}{v^2} dv,$$

which is a positive quantity $c(\lambda)$ (say) depending only on $\lambda$. This gives

$$\left| \int_x^{\infty} \frac{B(t)-t}{t^2} dt - \int_{\lambda x}^{\infty} \frac{B(t)-t}{t^2} dt \right| = c(\lambda)$$

For fixed $\lambda$, as $x \to \infty$, the above integrals are tails of the convergent integral (7) and can be made arbitrarily small, thereby giving a contradiction. This completes the proof. □

The result can be extended to Dirichlet series with complex coefficients as follows.

**Corollary 3.** *Let*

$$F(s) = \sum_{n=1}^{\infty} a_n/n^s$$

*be a Dirichlet series with complex coefficients. Let $A(x)$ denote the partial sum of the coefficients:*

$$A(x) = \sum_{n\leq x} a_n.$$

*Suppose there exists a Dirichlet series $G(s) = \sum_{n=1}^{\infty} b_n/n^s$ with non-negative coefficients, such that*

(a) $|a_n| \leq b_n$ *for all* $n$.
(b) $G(s)$ *is absolutely convergent for* $\mathrm{Re}(s) > 1$.
(c) *The function $G(s)$ (resp. $F(s)$) extends meromorphically to the region* $\mathrm{Re}(s) \geq 1$, *having no poles except for a simple pole at $s = 1$ with residue $R$ (resp. $r$).*
(d) $B(x) := \sum_{n\leq x} b_n = O(x)$.

*Then, as $x \to \infty$,*

$$A(x) = rx + o(x).$$

*Proof.* If $a_n$'s are real, we consider the series $G(s) - F(s)$, which has non-negative coefficients and satisfies the conditions of Theorem 2, giving

$$\sum_{n \leq x} (b_n - a_n) = (R - r)x + o(x),$$

as $x \to \infty$. As $B(x) = Rx + o(x)$, this proves the result in the case of real coefficients. If the coefficients $a_n$ are not real, we define

$$F^*(s) = \sum_{n=1}^{\infty} \bar{a}_n / n^s$$

so that

$$F = \frac{F + F^*}{2} + i \left( \frac{F - F^*}{2i} \right).$$

and apply the result for real coefficients separately to the real and imaginary part above after checking that the necessary conditions are satisfied. □

## 4. Applications

In this section we demonstrate some applications of the Tauberian theorem, following the treatment of Serre [7] who gives a general set-up for the same in the context of equidistribution.

We make this more precise in an abstract setting as follows. Let $G$ be a compact group and $X$ be the space of conjugacy classes of $G$. Let $x_v$ be a family of elements of $X$, indexed by a countably infinite set $\mathcal{P}$. Let $N : \mathcal{P} \to \mathbb{Z}$ be a function taking values $\geq 2$, $\rho$ an irreducible complex representation of $G$ with character $\chi$. We define

$$\zeta_{\mathcal{P}}(s) = \prod_{v \in \mathcal{P}} \left( 1 - \frac{1}{(Nv)^s} \right)^{-1}, \qquad L(s, \rho) = \prod_{v \in \mathcal{P}} \det \left( 1 - \frac{\rho(x_v)}{(Nv)^s} \right)^{-1}.$$

Thus, for the trivial representation $\rho = 1$, $L(s, 1) = \zeta_{\mathcal{P}}(s)$.

**Theorem 4.** *Suppose $L(s, \rho)$ is absolutely convergent for $\mathrm{Re}(s) > 1$ and extends to a meromorphic function on $\mathrm{Re}(s) \geq 1$ with no zeros or poles except for a pole of order $c_\chi$ at $s = 1$. Then,*

$$\sum_{Nv \leq n} \chi(x_v) = (1 + o(1)) c_\chi \frac{n}{\log n}.$$

The proof of the above theorem follows by applying the Tauberian theorem to $L'/L$. We refer the reader to the appendix of Chapter 1 of [7] for the same. If Theorem 4 holds for all irreducible representations $\rho \neq 1$ with $c_\chi = 0$, then the Peter-Weyl theorem allows us to deduce that the $x_v$'s are equidistributed with respect to the normalized Haar measure of $G$. Special cases of this theorem lead to important results, among them being the prime number theorem, Chebotarev density theorem and the Sato-Tate theorem. An excellent reference for the interested reader wishing to delve deeper into these topics is [5].

## 5. CONCLUDING REMARKS

As remarked earlier, the added condition (c) in Theorem 2 is not restrictive for most practical purposes. However, it is possible to eliminate this condition altogether. We give a brief sketch of the argument. The key idea is to notice that the known bound $B(x) \ll x \log x$ implies that for any $\epsilon > 0$, the function

$$f_\epsilon(t) := \frac{f(t)}{e^{\epsilon t}} = \frac{B(e^t)}{e^{t(1+\epsilon)}} - \frac{1}{e^{\epsilon t}}$$

is bounded and satisfies the conditions of Theorem 1. Applying this theorem to $f_\epsilon(t)$ and following an elementary argument that exploits the increasing behaviour of the function $B(e^t)/e^{t(1+\epsilon)}$, one obtains a uniform bound on $\sup_{t\geq 0} |f_\epsilon(t)|$. Letting $\epsilon \to 0$, we see that $f(t)$ must be bounded. A more detailed proof can be found in [2].


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