The Central Limit Theorem in Algebra and Number Theory

Arpita Kar and M. Ram Murty

Abstract As its name suggests, the central limit theorem occupies a central position in all of mathematics and perhaps all of science. From its humble origins in combinatorics, it has evolved into a powerful tool through which we can understand the mysteries of nature. In this paper, we survey how it has led to the development of new insights in algebra and number theory.

Keywords Central limit theorem · Erdős-Kac theorem, Goncharov's theorem, Riemann Hypothesis

2010 Mathematics Subject Classification 11M06 · 20C15

1 Introduction

The central limit theorem is one of the remarkable theorems of twentieth-century mathematics, and there are even good reasons to say it is the most remarkable discovery of our time. With its humble origins emanating from a simple combinatorial problem, it has evolved over decades into a profound principle of probability theory. It gave birth to statistical methods and now influences disciplines outside of mathematics such as biology, medical science, economics and artificial intelligence. This paper will not expound these connections and ramifications outside the mathematical field, but rather, our goal is to elucidate its impact on number theory and algebra, which surprisingly, is not so well-known even among pure mathematicians.

Research of the M. R. Murty partially supported by an NSERC Discovery grant.

A. Kar · M. R. Murty (✉)
Department of Mathematics and Statistics, Queen's University,
Kingston, ON K7L 3N6, Canada
e-mail: murty@mast.queensu.ca

A. Kar
e-mail: arpita@mast.queensu.ca

B. Ramakrishnan et al. (eds.), *Modular Forms and Related Topics in Number Theory*, Springer Proceedings in Mathematics & Statistics 340,
https://doi.org/10.1007/978-981-15-8719-1_8

Apart from its intrinsic beauty, the central limit theorem offers us a metaphor for approaching and understanding a notorious unsolved problem in number theory, namely the Riemann hypothesis. This celebrated problem and its manifold incarnations as the Generalized Riemann hypothesis, both in the context of algebraic number fields and the wider context of automorphic L-functions, occupies a pivotal place in the landscape of pure mathematics. Like the Himalayan peaks, these hypotheses brood ominously over all of mathematics and invite us to scale their celestial summits. The central limit theorem, and probability theory in general, offers us a method to approach these summits.

To motivate our discussion, let us begin with a study of the Liouville function $\lambda(n)$ defined as follows. If $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$ is the unique factorization of a natural number n into distinct prime powers, then

$$\lambda(n) = (-1)^{\Omega(n)} \qquad \text{where} \qquad \Omega(n) = a_1 + a_2 + \cdots a_k.$$

In other words, $\lambda(n)$ is $+1$ if n has an even number of prime factors and -1 if n has an odd number of prime factors (counted with multiplicity). The Riemann hypothesis is then equivalent to the assertion that for every $\epsilon > 0$,

$$\sum_{n \leqslant N} \lambda(n) = O(N^{\frac{1}{2}+\epsilon}). \tag{1}$$

This was first noted by Pólya [28] in 1919.

This suggests the following "thought experiment" where we would treat the $\lambda(n)$ as "random variables" taking the values ± 1. Indeed, consider the set

$$\S_N := \{\sigma = (a_1, a_2, \dots, a_N) : \quad a_i = \pm 1\}.$$

Evidently, $|\S_N| = 2^N$. For each $\sigma \in \S_N$, we define

$$s(\sigma) = a_1 + a_2 + \cdots + a_N.$$

Since each of ± 1 is taken with probability $1/2$, we would expect $s(\sigma)$ to be zero "on average." In fact,

$$\sum_{\sigma \in \S_N} s(\sigma) = \frac{1}{2} \sum_{\sigma \in \S_N} (s(\sigma) + s(-\sigma)) = 0.$$

Writing $\sigma = (a_1(\sigma), \dots, a_N(\sigma))$, we also see that

$$\begin{aligned}\sum_{\sigma\in\S_N} s(\sigma)^2 &= \sum_{\sigma\in\S_N}\sum_{1\leqslant i,j\leqslant N} a_i(\sigma)a_j(\sigma)\\ &= \sum_{\sigma\in\S_N}\left(N+\sum_{i\neq j} a_i(\sigma)a_j(\sigma)\right)\\ &= N2^N + \sum_{\sigma\in\S_N}\sum_{i\neq j} a_i(\sigma)a_j(\sigma).\end{aligned}$$

For each $\sigma \in \S_N$ and $i \neq j$, define $\check{\sigma}$ to be the same as σ except $a_i(\check{\sigma}) = -a_i(\sigma)$, $a_j(\check{\sigma}) = a_j(\sigma)$. It is then transparent that interchanging sums in the last summation and pairing σ with $\check{\sigma}$, the sum vanishes and we obtain

$$\sum_{\sigma\in\S_N} s(\sigma)^2 = N2^N.$$

In other words, $s(\sigma)$ has mean zero and variance N. Inspired by the Chebycheff inequality, we deduce

$$P(\sigma : |s(\sigma)| > N^{1/2+\epsilon}) \leqslant \frac{1}{N^{2\epsilon}},$$

which goes to zero as N tends to infinity. In other words, we can expect that for any random sequence $a_1, a_2, \ldots$ of ±1's that

$$a_1 + a_2 + \cdots + a_N = O(N^{\frac{1}{2}+\epsilon})$$

with probability 1. This is essentially Chebycheff's inequality (1867). In fact, more can be shown:

Theorem 1.1 (de Moivre (1738), Laplace (1812))

$$P\left(\sigma : \alpha \leqslant \frac{s(\sigma)}{\sqrt{N}} \leqslant \beta\right) \to \frac{1}{\sqrt{2\pi}}\int_\alpha^\beta e^{-t^2/2}dt$$

as $N \to \infty$.

This beautiful theorem originates from the combinatorial problem of counting the number of heads or tails one can expect in N random flips of a fair coin. What is intriguing about the theorem is its suggestive connection to the Riemann hypothesis where we can view the values of $\lambda(n)$ as a random collection of ±1's. Viewed in this way, Eq. (1) seems plausible. We will return later to the theme of the Riemann hypothesis and its connection to probability theory.

2 Review of Some Probabilistic Concepts

Some remarks pertaining to "probabilistic thought" are worth repeating here. Let X be a discrete random variable assuming the values $x_1, x_2, \ldots$. The probability of the event $X = x_j$ will be denoted as $P(X = x_j)$. The function

$$f(x_j) := P(X = x_j)$$

is called the *probability distribution* of the random variable X. Clearly,

$$f(x_j) \geqslant 0 \text{ and } \sum_j f(x_j) = 1. \tag{2}$$

It is possible to go in the "reverse" direction. Suppose we are given a set of points $x_1, x_2, \ldots$ and a function f defined on these points satisfying (2), then it is customary to speak of a random variable X assuming values $x_1, x_2, \ldots$ with probabilities $f(x_1), f(x_2), \ldots$. Thus, given f satisfying (2), we say "let X be a random variable with distribution f."

The notion of "independent random variables" is undoubtedly familiar to the reader. However, it may help to elucidate our understanding if we recall this notion from the "probabilistic mind set." Indeed, we say $X_1, X_2, \ldots X_n$ are *independent* if

$$P(X = \alpha_1, X_2 = \alpha_2, \ldots X_n = \alpha_n) = P(X_1 = \alpha_1)P(X_2 = \alpha_2) \cdots P(X_n = \alpha_n).$$

Thus, if X_k depends only on the outcome of the kth trial and not on the previous outcomes, then the variables $X_1, X_2, \ldots X_n$ are mutually independent. We refer the reader to p. 205 of [17] for further clarification of these concepts.

We have discussed these ideas in the discrete random variable case with a view to our applications below. They undoubtedly apply in an analogous fashion to the non-discrete case.

3 The Evolution of the Central Limit Theorem

The de Moivre–Laplace theorem evolved for almost a century into the modern central limit theorem. Beginning with the work of Chebycheff (1887), and then his two pupils Markov (1898) and Lyapunov (1901), Lindeberg (1922), and finally Levy (1935) and Feller (1935), the central limit theorem morphed into its modern form. Feller [16] writes that "For more than one hundred years a great many mathematicians have been working on the problem discovering many special cases to which the theorem applies and gradually establishing, and relaxing step for step, sufficient conditions under which the theorem holds. To the less critical mind, the law appeared as a universal law or, occasionally, as a law of nature."

Theorem 3.1 (The central limit theorem) *Suppose for each n,*

$$X_{n1}, X_{n2}, \ldots, X_{nr_n}$$

are independent random variables on some probability space Υ_n with measure P_n. Put

$$S_n = X_{n1} + X_{n2} + \cdots + X_{nr_n}.$$

Assume the expectation $E(X_{nk}) = 0$ and set

$$\sigma_n^2 = \sum_{k=1}^{r_n} \sigma_{nk}^2$$

where $\sigma_{nk}^2 = E(X_{nk}^2)$. Then

$$P\left(\omega : \alpha \leqslant \frac{S_n(\omega)}{\sigma_n} \leqslant \beta\right) \to \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-t^2/2} dt,$$

and

$$\max_{k \leqslant r_n} \frac{\sigma_{nk}^2}{\sigma_n^2} \to 0$$

as $n \to \infty$ if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{r_n} \frac{1}{\sigma_n^2} \int_{|X_{nk}|>\epsilon\sigma_n} X_{nk}^2 dP_n = 0 \tag{3}$$

for every $\epsilon > 0$.

Remark 1 The sufficiency was shown by Lindeberg (1922) and the necessity by Feller (1935). The condition (3) is often referred to as the Lindeberg condition. According to the historical article [31], Alan Turing also discovered this independently in 1934 while still an undergraduate at the age of 22. The version of the central limit theorem given in our theorem can be found on p. 408 of [4]).

Here is an elegant application of the central limit theorem (which was discovered by Ramanujan independently and without considerations of probability). Ramanujan ([29], p. 323) showed

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

We can deduce this via the central limit theorem as follows. If $X_1, X_2, \ldots$ is a sequence of independent random variables each having the Poisson distribution with parameter 1, then each has mean 1 and variance 1. By the central limit theorem, the sum

$$\frac{X_1 + X_2 + \cdots + X_n - n}{\sqrt{n}}$$

approaches the normal distribution as $n \to \infty$. However, the sum of two Poisson random variables with parameter x and y is again Poisson with parameter $x + y$. Thus,

$$X_1 + X_2 + \cdots + X_n$$

is again Poisson with parameter n with mean n and variance n. We immediately deduce that

$$P\left(\frac{X_1 + X_2 + \cdots + X_n - n}{\sqrt{n}} \leqslant 0\right) = e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-t^2/2} dt = \frac{1}{2}.$$

4 The Evolution of Probabilistic Number Theory

In 1917, Hardy and Ramanujan [20] proved the following theorem.

Theorem 4.1 *Let $\omega(n)$ denote the number of distinct prime factors of n. For any fixed $\epsilon > 0$, the number of $n \leqslant x$ such that*

$$|\omega(n) - \log\log n| > (\log\log n)^{1/2+\epsilon}$$

is at most

$$\frac{x}{(\log\log x)^{2\epsilon}}.$$

Remark 2 In other words, almost all numbers have $\log\log n$ prime factors (in the sense of natural density). Their proof was technically complicated.

Precisely, we say $\omega(n)$ has normal order $\log\log n$. In general, an arithmetic function $f(n)$ has normal order $g(n)$ (where $g(n)$ is a continuous monotone function) if given any $\epsilon > 0$, the number of $n \leqslant x$ such that

$$|f(n) - g(n)| > \epsilon g(n)$$

is $o(x)$ as $x \to \infty$.

Now, $\omega(n)$ and $\Omega(n)$ are examples of *additive functions*. In other words, $\Omega(mn) = \Omega(m) + \Omega(n)$ whenever m and n are coprime. Thus, one can write

$$\omega(n) = \sum_{p|n} \omega(p),$$

and it is suggestive to view $\omega(n)$ as a sum of "independent random variables" when looked at this way. However, this viewpoint was long in coming and had a roundabout emergence taking several decades.

In 1934, Paul Turán [30] gave an elementary proof of the Hardy–Ramanujan theorem which was extremely simple. He did this by showing that

$$\sum_{n \leqslant x} (\omega(n) - \log\log n)^2 = O(x \log\log x).$$

In retrospect, one can see that what Turán did was a number theoretic analogue of the celebrated Chebycheff inequality in probability theory. His two-page paper gave rise to a spectacular cascade of events and ultimately led to the development of probabilistic number theory. In a letter to Elliott (p. 18 of [15]) written in 1976, Turán recalls "When writing Hardy first in 1934 on my proof of Hardy–Ramanujan's theorem I did not know what Tchebycheff's inequality was and a fortiori of the central limit theorem. Erdős, to my best knowledge, was at that time not aware too. It was Mark Kac who wrote to me a few years later that he discovered when reading my proof in JLMS that this is basically probability and so was his interest turned to this subject." Apparently, Kac asked if Turán could prove similar estimates for the higher moments:

$$\sum_{n \leqslant x} (\omega(n) - \log\log n)^k.$$

Apparently, Kac hinted in his letter that if one could derive similar estimates, then there is a normal distribution law for $\omega(n)$. Turán continues that though he realized he could estimate the higher moments, he "found absolutely no interest to do it actually." His reasons, he confesses, for not doing so were that he saw no applications of such results! It was Mark Kac who picked up the sequence of ideas leading to the possibility of applying the central limit theorem to additive number theoretic functions. He recalls (see page 24 of [15]) that "If I remember it correctly I first stated (as a conjecture) the theorem on the normal distribution of the number of prime divisors during a lecture in Princeton in March 1939. Fortunately for me and possibly for Mathematics, Erdős was in the audience, and he immediately perked up. Before the lecture was over he had completed the proof, which I could not have done not having been versed in the number theoretic methods, especially those related to the sieve." He continues, "With Erdős's contribution it became clear that we have had a beginning of a nice chapter of Number Theory, bringing upon it to bear the concepts and methods of Probability Theory."

So in 1940, Erdős and Kac [14] proved that

$$P\left(n \leqslant N : \alpha \leqslant \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \leqslant \beta\right) \to \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-t^2/2} dt.$$

Although the metaphor of the central limit theorem suggests this result, it is not a corollary of it. In the 1950s Kubilius [22] generalized this result to arbitrary additive functions.

5 Goncharov's Theorem

Around 1942, unaware of the developments in number theory along the lines of the Erdős–Kac theorem, Goncharov considered the following problem related to the symmetric group on n letters, denoted Ω_n. For each $\sigma \in \Omega_n$, it is elementary that it can be written uniquely (unique up to ordering) as a product of disjoint cycles and one can define $\omega(\sigma)$ to be the number of cycles in its unique factorization, viewing it as a group-theoretic analogue of $\omega(n)$ of Hardy and Ramanujan. For suggestive reasons, we write $\omega(\sigma)$ and $\omega(n)$, it being clear from the discussion the context we are in. One can directly apply Theorem 3.1 to show that $\omega(\sigma)$, when appropriately normalized, is normally distributed (no pun intended).

The reader will recall from basic algebra that every permutation $\sigma \in \Omega_n$ can be written as a product of disjoint cycles in a canonical way as follows. We begin with 1 which is mapped to $\sigma(1)$ which in turn is mapped to $\sigma^2(1)$ and so forth. In other words, the canonical decomposition of σ as a product of disjoint cycles is written as an ordered sequence of orbits, beginning with the orbit of 1 under σ, then the orbit of the smallest number not in the orbit of 1 and so on.

To illustrate, let us consider the permutation σ of Ω_8 given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 4 & 1 & 6 & 8 & 7 & 2 \end{pmatrix}$$

This has the canonical disjoint cycle decomposition

$$\sigma = (134)(2568)(7), \tag{4}$$

so that $\omega(\sigma) = 3$.

We define random variables X_k $(1 \leqslant k \leqslant n)$ on Ω_n by setting $X_k = 1$ if a cycle is completed at the kth step and otherwise we set $X_k = 0$. For example, with σ as in (4), we have

$$X_3(\sigma) = X_7(\sigma) = X_8(\sigma) = 1,$$

$$X_1(\sigma) = X_2(\sigma) = X_4(\sigma) = X_5(\sigma) = X_6(\sigma) = 0.$$

Clearly, $X_1(\sigma) = 1$ if and only if $\sigma(1) = 1$ and the number of such permutations is $(n-1)!$ so that $P(X_1 = 1) = \dfrac{1}{n}$.

More generally, we see that $P(X_k = 1)$ is given by $\frac{1}{(n-(k-1))}$ since there are $n - (k-1)$ choices at the kth step to complete a cycle. Indeed, the permutations $\sigma \in \Omega_n$

for which $X_k(\sigma) = 1$ have a disjoint cycle decomposition where at the kth position, we also have a "close bracket."

Since the outcome at the kth step does not depend on the earlier outcomes, the X_k's are independent. We will now write X_{nk} to denote X_k for obvious reasons. This example and formulation are discussed on p. 242 of [17] and p. 78 of [4].

Clearly $S_n(\sigma) = X_{n1}(\sigma) + X_{n2}(\sigma) + \cdots + X_{nn}(\sigma)$ is the number of cycles in the cycle representation of σ. The mean m_{nk} of X_{nk} is the probability that X_{nk} equals 1, which is $\dfrac{1}{n-k+1}$ and variance is $\sigma_{nk}^2 = m_{nk}(1 - m_{nk})$. If $L_n = \sum_{k=1}^{n} \dfrac{1}{k}$, then S_n has mean

$$\sum_{k=1}^{n} m_k = \sum_{k=1}^{n} \frac{1}{n-k+1} = L_n$$

and variance

$$\sum_{k=1}^{n} m_{nk}(1 - m_{nk}) = L_n + O(1).$$

Applying Chebychev's inequality and the fact that

$$L_n = \log n + O(1),$$

one can prove an analogue of Theorem 4.1 (as outlined in the next section) concluding that most permutations on n letters have about $\log n$ cycles in their unique disjoint cycle decomposition.

We can now apply the central limit theorem to deduce Goncharov's theorem because the Lindeberg condition is vacuously satisfied. Indeed, the X_{nk}'s are bounded random variables and $\sigma_n \to \infty$ so that for n large, the sequence in Lindeberg's limit condition eventually becomes the zero sequence.

Thus, unlike the Erdős–Kac theorem, Goncharov's theorem can be deduced via the central limit theorem. Goncharov's paper is long and complicated. The underlying conceptual fabric is missing. Harper [21] is less sympathetic. He writes, "Goncharov …by brute force tortuously manipulates the characteristic functions of the distributions until they approach $\exp(-x^2/c)$, c a positive constant."

6 The Connection to Stirling Numbers of the First Kind

One can also approach the problem in the previous section from a different combinatorial perspective. Following Turán, we can consider

$$\sum_{\sigma \in \Omega_n} \omega(\sigma), \qquad \text{and} \qquad \sum_{\sigma \in \Omega_n} \omega^2(\sigma),$$

although this is not the way Goncharov thought about the problem. However, interpreted this way, the question becomes an elementary problem in basic combinatorics.

Denoting $s(n, k)$ to be the signed Stirling number of the first kind, and recalling that $|s(n, k)|$ is the number of permutations in Ω_n with exactly k disjoint cycles in its factorization (see pg. 26 of [10]), we immediately see that

$$\sum_{\sigma\in\Omega_n} \omega(\sigma) = \sum_{k=1}^{n} k|s(n,k)|, \qquad \text{and} \qquad \sum_{\sigma\in\Omega_n} \omega^2(\sigma) = \sum_{k=1}^{n} k^2\, s(n,k)|.$$

Now it is elementary that

$$x(x+1)\cdots(x+n-1) = \sum_{k=1}^{n} |s(n,k)|x^k. \tag{5}$$

That is, the unsigned Stirling numbers $|s(n, k)|$ are defined algebraically as the coefficients of the rising factorial. Differentiating both sides of this polynomial identity and setting $x = 1$, we deduce

$$\begin{aligned}\sum_{\sigma\in\Omega_n} \omega(\sigma) &= n!\sum_{j=1}^{n} \frac{1}{j} \\ &= n!H_n,\end{aligned}$$

where H_n denotes the nth harmonic number. Since $\int_1^n \frac{1}{x}dx = \log n$, we get (by an application of the method of proof for the integral test)

$$\log n + \frac{1}{n} \leqslant H_n \leqslant \log n + 1.$$

More precisely,

$$H_n = \log n + \gamma + \frac{1}{2n} + \frac{1}{12n^2} + \frac{\epsilon(n)}{120n^4},$$

where γ denotes Euler's constant and $0 < \epsilon(n) < 1$ for all n. So, we get

$$\sum_{\sigma\in\Omega_n} \omega(\sigma) = n!\left(\log n + \gamma + O\left(\frac{1}{n}\right)\right).$$

In other words, the average number of disjoint cycles of a random permutation of Ω_n is $\log n$. Similarly, in order to calculate the second moment of $\omega(\sigma)$, we differentiate (5) twice and set $x = 1$ to get,

$$\sum_{k=0}^{n} |s(n,k)|k(k-1) = n!\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{ij} - \sum_{i=1}^{n}\frac{1}{i^2}\right)$$
$$\implies \sum_{k=0}^{n} |s(n,k)|k^2 = \sum_{k=0}^{n} k|s(n,k)| + n!\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{ij} - \sum_{i=1}^{n}\frac{1}{i^2}\right).$$

Now,

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{ij} &= \left(\sum_{i=1}^{n}\frac{1}{i}\right)^2\\ &= \left(\log n + \gamma + O\left(\frac{1}{n}\right)\right)^2\\ &= (\log n)^2 + 2\gamma\log n + O\left(\frac{\log n}{n}\right).\end{aligned}$$

Also, $\sum_{i=1}^{n}\frac{1}{i^2} = O(1)$. Putting these together, we get that the second moment is

$$\sum_{\sigma\in\Omega_n}\omega(\sigma)^2 = n!\left((2\gamma+1)\log n + \gamma + (\log n)^2 + O\left(\frac{\log n}{n}\right)\right).$$

Then the variance

$$\frac{1}{n!}\sum_{\sigma\in\Omega_n}(\omega(\sigma)-\log n)^2 = \log n + \gamma + O\left(\frac{\log n}{n}\right). \tag{6}$$

From here, one can now derive an analogue of Turán's theorem for symmetric groups. To show $\omega(\sigma)$ has normal order $\log n$, one needs to show that given any $\epsilon > 0$, the number of $\sigma \in \Omega_n$ such that

$$|\omega(\sigma) - \log n| > \epsilon \log n$$

is $o(n!)$. Using Chebychev's inequality, this is

$$O\left(\frac{n!}{\epsilon^2\log n}\right)$$

which is $o(n!)$ as $n \to \infty$ for any $\epsilon > 0$.

More precisely, it is worth noting here that given a fixed $A > 0$, the approach of Turán applied to this problem gives a nice estimate of $O\left(\frac{n!}{A\log n}\right)$ for the exceptional set, that is, the number of permutations for which $\omega(\sigma)$ deviates from its mean value $\log n$ by more than $A \log n$.

In order to understand the distribution of $\omega(\sigma)$, one can now consider the higher moments

$$\sum_{\sigma \in \Omega_n} (\omega(\sigma) - \log n)^k$$

for $k \geqslant 2$. Using the binomial theorem, one can show that this is

$$\begin{aligned} &\sum_{j=0}^{k} \binom{k}{j} (\log n)^{k-j} \sum_{\sigma \in \Omega_n} \omega(\sigma)^j \\ &= \sum_{j=0}^{k} \binom{k}{j} (\log n)^{k-j} \sum_{m=0}^{n} m^j |s(n, m)|. \end{aligned}$$

Denoted by M_j the inner sum

$$M_j := \sum_{m=0}^{n} m^j |s(n, m)|. \tag{7}$$

Estimation of this sum for a general j seems quite complicated and hence we do not pursue this method in this paper any further.

One can, however, apply

$$P\left(\sigma : \alpha \leqslant \frac{\omega(\sigma) - \log n}{\sqrt{\log n}} \leqslant \beta\right) \to \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2} dt,$$

as n tends to infinity, to deduce the behaviour of (7). Goncharov in [18] takes another approach to prove the above result. Though the Erdős–Kac theorem cannot be deduced from the central limit theorem, it is possible to derive a modified version of the central limit theorem to do so. We refer the reader to the papers of Billingsley [5, 6] for further amplification of this discussion.

The polynomial identity (5) can be given a probabilistic interpretation. Indeed, if we write

$$\sum_{k=1}^{n} \frac{|s(n, k)| t^k}{k!} = \prod_{k=1}^{n} \left(\frac{t + n - k}{n - k + 1} \right), \tag{8}$$

we can interpret each factor on the right as the probability generating function of X_k. Since the X_k's are independent and

$$\omega(\sigma) = \sum_k X_k(\sigma),$$

we have

$$E(t^{\omega}) = \prod_k E(t^{X_k}).$$

In other words, we can deduce (8) as a consequence of the independence of the X_k's from purely probabilistic considerations.

There is a well-known duality principle in combinatorics that relates certain theorems about the symmetric group Ω_n to theorems about partitions of sets of n elements. This is not a rigid principle but only a metaphor. As such, the analogue of Goncharov's theorem has been worked out by Harper, and we refer the reader to [21] for further details.

7 Normal Number of Prime Factors of Fourier Coefficients of Modular Forms

The generalization of the Erdős–Kac theorem to study the normal number of prime factors of Fourier coefficients of modular forms was initiated in 1984 by the second author and Kumar Murty in [25, 26]. We now describe their work and very briefly indicate future directions.

To expedite our exposition, we only discuss the case of the Ramanujan τ-function and refer the reader to [25, 26] for the general case of modular forms. Recall that the Ramanujan τ-function is defined via the infinite product:

$$q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n, \qquad q = e^{2\pi i z}$$

with $\Im(z) > 0$. Ramanujan conjectured that $\tau(n)$ is a multiplicative function of n and that $|\tau(p)| \leqslant 2p^{11/2}$. As is well-known, the Ramanujan conjecture was proved by Deligne [12] as a culmination of his work on the Weil conjectures.

In [25], the second author and K. Murty show subject to a generalized quasi-Riemann hypothesis (more precisely, there exists a $1/2 < \delta < 1$ such that all Artin L-functions have no zeros in $\Re(s) > \delta$) that

$$\sum_{\substack{p\leqslant x \\ \tau(p)\neq 0}} (\omega(\tau(p)) - \log\log p)^2 = O(\pi(x)\log\log x),$$

where the summation is over primes p. In other words, the normal order of $\omega(\tau(p))$ is $\log\log p$. In [26], they extend this work and establish the analogue of the Erdős–Kac theorem, again subject to the same generalized quasi-Riemann hypothesis. More generally, they studied $\omega(\tau(n))$ and showed that

$$P\left(n : \alpha \leqslant \frac{\omega(\tau(n)) - \frac{1}{2}(\log\log n)^2}{(\log\log n)^{3/2}/\sqrt{3}} \leqslant \beta\right) \to \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-t^2/2} dt$$

In current work in progress, the authors hope to extend these studies to shifts of primes. We expect that, for example, $\omega(\tau(p + a))$ with $a \neq 0$ has normal order $(\log\log p)^2/2$. We also expect an analogue of the Erdős–Kac theorem to hold for these shifts.

8 Probabilistic Connections to the Riemann Hypothesis

The Riemann zeta function $\zeta(s)$, originally defined as a Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\Re(s) > 1$ can be analytically continued to the entire complex plane except for $s = 1$ where it has a simple pole. The celebrated Riemann hypothesis is the statement that the real part of all the non-trivial zeroes of $\zeta(s)$ is $\frac{1}{2}$.

This fugitive Riemann hypothesis has been both a source of inspiration and frustration for many generations of mathematicians. It is said that Hilbert and Pólya were the first to suggest that if we could interpret the non-trivial zeroes of $\zeta(s)$ as related to eigenvalues of some Hermitian operator, the Riemann hypothesis would follow. But the hypothesized Hermitian operator has not been found yet. Probability theory may offer us a window into interpreting the zeroes of $\zeta(s)$ and give them new meaning. Indeed, in a paper of Biane, Pitman and Yor [3], we find the following exposition. Let

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi t}$$

be the classical Jacobi theta function. It is well known that it satisfies the modular transformation

$$\sqrt{t}\theta(t) = \theta(1/t), \qquad t > 0. \tag{9}$$

By means of this transformation, one can show that if we define

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

then

$$\frac{4\xi(s)}{s(s-1)} = \int_0^\infty (\theta(t) - 1)t^{s/2}\frac{dt}{t}, \tag{10}$$

which is valid for $\Re(s) > 1$, can be extended analytically to the entire complex plane and deduce that

$$\xi(s) = \xi(1-s),$$

which is the celebrated functional equation of the Riemann zeta function. Now, if we define

$$G(y) := \theta(y^2) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi y^2},$$

then (9) becomes

$$G(1/y) = yG(y). \tag{11}$$

Set

$$\begin{aligned} H(y) &= \frac{d}{dy}\left(y^2 \frac{d}{dy} G(y)\right) \\ &= 2yG'(y) + y^2 G''(y), \end{aligned}$$

which by definition of $G(y)$ becomes

$$H(y) = 4y^2 \left(\sum_{n=1}^{\infty} (2\pi^2 n^4 y^2 - 3\pi n^2) e^{-\pi n^2 y^2}\right)$$

and $H(y)$ satisfies the same functional equation as G, namely

$$yH(y) = H(y^{-1}), \qquad y > 0.$$

Then Riemann's formula (10) for $\xi(s)$ becomes

$$2\xi(s) = \int_0^{\infty} H(y) y^s \frac{dy}{y}.$$

Since

$$2\pi^2 n^4 y^2 > 3\pi n^2$$

for $y \geqslant 1$, we see $H(y) > 0$ in this region. But then, by the functional equation, we have $H(y) > 0$ for $y > 0$ also. A routine computation shows that

$$2\xi(0) = 2\xi(1) = 1$$

and so

$$\int_0^{\infty} \frac{H(y)}{y} dy = \int_0^{\infty} H(y) dy = 1.$$

Thus, the function $H(y)$ can be viewed as the density function of a probability distribution on $(0, \infty)$ with mean 1. If Y is a random variable with this distribution, the functional equation for H translates as

$$E(f(1/Y)) = E(Yf(Y)).$$

Also, one can view $2\xi(s)$ as $E(Y^s)$.

In 1997, Xian-Jin Li [23] derived a remarkable criterion for the truth of the Riemann hypothesis. To state Li's criterion, we define for each natural number n,

$$\lambda_n = \sum_{\rho} \left(1 - \left(1 - \frac{1}{\rho}\right)^n\right),$$

where the sum is over the non-trivial zeros of the Riemann zeta function. Then, Li's criterion is that the Riemann hypothesis is true if and only if

$$\lambda_n \geqslant 0,$$

for all natural numbers n. It is not difficult to see that

$$\lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left(s^{n-1} \log \xi(s)\right)\Big|_{s=1}.$$

Using Leibniz's rule,

$$\lambda_n = n \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{k_{n-j}}{(n-j)!}, \tag{12}$$

where

$$k_n = \frac{d^n}{ds^n} (\log \xi(s))\Big|_{s=1}.$$

If all the k_n's were positive, then by (12), the Riemann hypothesis follows. However, this is not always the case. For example,

$$k_3 = -0.000222316, \;\; k_4 = -0.0000441763$$

according to the table on pg. 441 of [3]. In fact, one can interpret the k_n's as cumulants of the random variable $\log(1/Y)$ with Y as before.

Recall that given a random variable X, the moment generating function is $E(e^{tX})$ and the cumulant generating function is $\log E(e^{tX})$. If we let

$$L := \log(1/Y),$$

then as

$$2\xi(s) = E(Y^s) = E(Y^{1-s}) = E(e^{(s-1)L}),$$

we see that

$$\log(e^{(s-1)L}) = \log 2\xi(s) = \sum_{n=1}^{\infty} k_n \frac{(s-1)^n}{n!}.$$

The cumulants k_n's are thus related to the moments of L as follows

$$\mu_n = E(L^n) = \int_0^{\infty} (-\log y)^n \frac{H(y)}{y} dy \tag{13}$$

through the formula

$$\mu_n = \sum_{j=0}^{n-1} \binom{n-1}{j} \mu_j k_{n-j}. \tag{14}$$

To see this, observe that taking the derivative with respect to t of the equation

$$\log E(e^{tX}) = \sum_{n=1}^{\infty} k_n \frac{t^n}{n!},$$

we see (upon setting $X = \log(1/Y)$) that (14) follows.

The positivity of the first cumulant k_1 is assured by Jensen's inequality since

$$k_1 = \mu_1 = -E(\log Y) \geqslant -\log E(Y) = 0. \tag{15}$$

Recall that for any convex function ϕ defined on the range of a random variable X, Jensen's inequality states that

$$E(\phi(X)) \geqslant \phi(E(X)) \tag{16}$$

so that (15) follows on applying (16) to $\phi(x) = -\log x$ which is convex (as it has a positive second derivative).

The positivity of the second cumulant can also be deduced through considerations of probability. Indeed,

$$k_2 = \mu_2 - \mu_1^2$$

is the variance of L, and therefore positive. However, as stated earlier, k_3, k_4 are negative.

Let us record here the following curious related fact. In [19], it was shown that the Riemann hypothesis is true if and only if

$$\sum_{\rho} \frac{1}{|\rho|^2} = 2 + \gamma - \log 4\pi,$$

where γ is Euler's constant and the sum is over non-trivial zeroes of $\zeta(s)$.

The cumulants k_n are related to the Stieltjes constants γ_n which are defined as the coefficients of the Laurent expansion of $\zeta(s)$ about $s = 1$,

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \gamma_n (s-1)^n,$$

where $\gamma_0 = \gamma$ is Euler's constant and one views the γ_n's as generalizations of this. In fact, it is not difficult to show that γ_n are given by the limits

$$\gamma_n = \lim_{m\to\infty} \left(\sum_{k=1}^{m} \frac{\log^n k}{k} - \frac{\log^{n+1} m}{n+1} \right).$$

We are more interested in the generalized Stieltjes constants η_n given by the Laurent expansion of $\frac{\zeta'}{\zeta}(s)$ about $s = 1$. Thus,

$$-\frac{\zeta'}{\zeta}(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \eta_n (s-1)^n.$$

By basic algebra, it is easy to express the η_n's as polynomials in the constants γ_n with rational coefficients. For example,

$$\eta_0 = -\gamma_0, \ \eta_1 = -\gamma_1 + \frac{1}{2}\gamma_0^2.$$

The significance of these constants lies in an important arithmetic formula for the λ_n's derived by Bombieri and Lagarias [7]:

$$\lambda_n = 1 - \frac{n}{2}(\gamma + \log 4\pi) + S_1(n) + S_2(n), \tag{17}$$

where

$$S_1(n) = \sum_{j=2}^{n} \binom{n}{j} (-1)^j \left(1 - \frac{1}{2^j}\right) \zeta(j)$$

and

$$S_2(n) = \sum_{j=1}^{n} \binom{n}{j} \eta_{j-1} = -n\gamma_0 + \sum_{j=2}^{n} \binom{n}{j} \eta_{j-1}. \tag{18}$$

They also showed that the condition of positivity can be considerably weakened to deduce the Riemann hypothesis. In fact, they show that if for any $\epsilon > 0$, there is a constant $c(\epsilon) > 0$ such that for all $n \geqslant 1$,

$$\lambda_n \geqslant -c(\epsilon)e^{\epsilon n}$$

then the Riemann hypothesis follows. Clearly, this is a substantial weakening of Li's criterion.

Coffey [11] has shown that for $n \geqslant 2$,

$$\frac{1}{2}(n(\log n + \gamma - 1) + 1) \leqslant S_1(n) \leqslant (n(\log n + \gamma + 1) - 1).$$

In particular, $S_1(n)$ is non-negative for all $n \geqslant 2$ so that apart from $S_2(n)$, the contributions of the other terms to λ_n in (17) is $O(n \log n)$. Thus the truth of Riemann hypothesis hinges on $S_2(n)$ and the growth of the generalized Stieltjes constants. Omar and Bouanani [8] extend Li's criterion to the function field setting.

Clearly, we should, therefore, focus our attention on η_j's. As noted in [7], one can show that

$$\eta_j = \frac{(-1)^j}{j!} \lim_{x\to\infty} \left(\sum_{m \leqslant x} \frac{\Lambda(m) \log^j m}{m} - \frac{(\log x)^{j+1}}{j+1} \right),$$

where

$$\Lambda(m) = \begin{cases} \log p, \ m = p^\alpha, \alpha \geqslant 1, \\ 0, \ \text{otherwise}. \end{cases}$$

This expression is unwieldly. We offer an alternate expression via Laguerre polynomials.

For $\psi(x) = \sum_{n \leqslant x} \Lambda(n)$, we know

$$\begin{aligned} -\frac{\zeta'}{\zeta}(s) &= s \int_1^\infty \frac{\psi(x)}{x^{s+1}} dx \\ &= \frac{s}{s-1} + s \int_1^\infty \frac{\psi(x) - x}{x^{s+1}} dx \\ &= \frac{1}{s-1} + 1 + s \int_1^\infty \frac{\psi(x) - x}{x^{s+1}} dx. \end{aligned}$$

Here, we can write the integral as

$$\begin{aligned} &(s-1+1) \int_1^\infty \frac{(\psi(x) - x)}{x^2} e^{-(s-1)\log x} dx \\ =&(s-1+1) \sum_{j=0}^\infty \frac{(-1)^j (s-1)^j}{j!} \int_1^\infty \frac{\Delta(x) \log^j x}{x^2} dx, \end{aligned}$$

where $\Delta(x) = \psi(x) - x$. If we let

$$\delta_j = \int_1^\infty \frac{\Delta(x)\log^j x}{x^2}dx, \tag{19}$$

then

$$\eta_j = \frac{(-1)^j \delta_j}{j!} + \frac{(-1)^{j-1}\delta_{j-1}}{(j-1)!}. \tag{20}$$

It is possible to derive estimates for η_j, δ_j and γ_j but all of these estimates have exponential growth. So we need to explore cancellations.

In this context, we have the Laguerre polynomials:

$$L_n(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{(-x)^j}{j}.$$

The generalized Laguerre polynomials which can be defined by the following generating function (see [1, 9, 24]):

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = \frac{1}{(1-t)^{\alpha+1}} \exp\left(-\frac{tx}{1-t}\right), \ (|t| < 1)$$

can also be defined by the closed formula for $n \geqslant 0$,

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!} \tag{21}$$

where $\binom{r}{k}$ denote the binomial coefficients given by

$$\binom{r}{0} = 1 \text{ and } \binom{r}{k} = \frac{r(r-1)(r-2)\cdots(r-k+1)}{k(k-1)\cdots 1}$$

for positive integer k and any complex number r.

In order to understand $S_2(n)$, we can try to get the expression of η_{j-1} in (18) in terms of integrals of generalized Laguerre polynomials. Using (18) and (20), we get

$$\begin{aligned} S_2(n) &= -n\gamma_0 + \sum_{j=2}^{n} \binom{n}{j} \left(\frac{(-1)^{j-1}\delta_{j-1}}{(j-1)!} + \frac{(-1)^{j-2}\delta_{j-2}}{(j-2)!} \right) \\ &= -n\gamma_0 + T_1(n) + T_2(n) \end{aligned}$$

where using (19)

$$
\begin{aligned}
T_1(n) &= \sum_{j=2}^{n} \binom{n}{j} \frac{(-1)^{j-1}\delta_{j-1}}{(j-1)!} \\
&= \int_1^{\infty} \frac{\Delta(x)}{x^2} \left(\sum_{j=2}^{n} \binom{n}{j} \frac{(-\log x)^{j-1}}{(j-1)!} \right) dx
\end{aligned}
$$

and

$$
\begin{aligned}
T_2(n) &= \sum_{j=2}^{n} \binom{n}{j} \frac{(-1)^{j-2}\delta_{j-2}}{(j-2)!} \\
&= \int_1^{\infty} \frac{\Delta(x)}{x^2} \left(\sum_{j=2}^{n} \binom{n}{j} \frac{(-\log x)^{j-2}}{(j-2)!} \right) dx.
\end{aligned}
$$

Writing $j' = j - 1$ and using $\binom{n}{k} = \binom{n}{n-k}$ for $0 \leqslant k \leqslant n$, we can rewrite the sum in expression for $T_1(n)$ as

$$
\begin{aligned}
&\sum_{j'=1}^{n-1} \binom{n}{j'+1} (-1)^{j'} \frac{(\log x)^{j'}}{(j')!} \\
=&\sum_{j'=1}^{n-1} \binom{(n-1)+1}{(n-1)-j'} (-1)^{j'} \frac{(\log x)^{j'}}{(j')!} \\
=&-n + L_{n-1}^{(1)}(\log x).
\end{aligned}
$$

A similar calculation for the sum in the expression for $T_2(n)$ gives

$$
\begin{aligned}
&\sum_{j=2}^{n} \binom{n}{j} \frac{(-\log x)^{j-2}}{(j-2)!} \\
=&\sum_{j'=0}^{n-2} \binom{(n-2)+2}{(n-2)-j'} \frac{(-\log x)^{j'}}{(j')!} \\
=&L_{n-2}^{(2)}(\log x).
\end{aligned}
$$

Thus, we get the following expression for $S_2(n)$ in terms of the generalized Laguerre polynomials

$$
S_2(n) = -n\gamma_0 - n \int_1^{\infty} \frac{\Delta(x)}{x^2} dx + \int_1^{\infty} \frac{\Delta(x)}{x^2} \left[L_{n-1}^{(1)}(\log x) + L_{n-2}^{(2)}(\log x) \right] dx. \tag{22}
$$

The integral in the last term of (22) can be simplified as follows:

$$\begin{aligned}
&L^{(1)}_{n-1}(\log x) + L^{(2)}_{n-2}(\log x)\\
=&\sum_{k=0}^{n-1}\binom{n}{n-1-k}(-1)^k\frac{(\log x)^k}{k!} + \sum_{k=0}^{n-2}\binom{n}{n-2-k}(-1)^k\frac{(\log x)^k}{k!}\\
=&\frac{(\log x)^{n-1}}{(n-1)!} + \sum_{k=0}^{n-2}\left(\binom{n}{n-1-k} + \binom{n}{n-2-k}\right)(-1)^k\frac{(\log x)^k}{k!}\\
=&\frac{(\log x)^{n-1}}{(n-1)!} + \sum_{k=0}^{n-2}\binom{n+1}{n-1-k}\frac{(-\log x)^k}{k!}\\
=&\frac{(\log x)^{n-1}}{(n-1)!} - \binom{n-1+2}{n-1-(n-1)}\frac{(\log x)^{n-1}}{(n-1)!} + \sum_{k=0}^{n-1}\binom{n-1+2}{n-1-k}\frac{(-\log x)^k}{k!}\\
=&\sum_{k=0}^{n-1}\binom{n-1+2}{n-1-k}\frac{(-\log x)^k}{k!}\\
=&L^{(2)}_{n-1}(\log x).
\end{aligned}$$

Thus, we get

$$S_2(n) = -n\gamma_0 + \int_1^\infty \frac{\Delta(x)}{x^2}L^{(2)}_{n-1}(\log x)dx - n\int_1^\infty \frac{\Delta(x)}{x^2}dx. \tag{23}$$

The last term in the above equation is actually $O(n)$ by a simple application of the unconditional error term in the prime number theorem. So, to prove the Riemann hypothesis, we need to focus on the integral in the second term in view of the results of Bombieri and Lagarias.

9 Concluding Remarks

The discussion of the preceding section can also be applied to study the generalized Riemann hypothesis from a probabilistic perspective. In fact, this perspective is the origin of the large sieve method.

The probabilistic model can give us some idea of what we can expect. Indeed, treating for example, the Legendre symbols $\chi_p(n) := (n/p)$ with p prime, as a random variable, we can prove the following: let $\tilde{\pi}(x)$ denote the number of primes between x and $2x$. Let $z = z(x)$ be such that

$$\frac{\log z}{\log x} \to \infty$$

as $x \to \infty$. Then, for any continuous real-valued function h, we have

$$\lim_{x\to\infty} \frac{1}{z} \sum_{n\leqslant z} h\left(\frac{\sum_{x\leqslant p\leqslant 2x} \chi_p(n)}{\sqrt{\tilde{\pi}(x)}}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t)e^{-t^2/2}dt.$$

In particular, if $h(x) = |x|$, this says that GRH holds "on average." We refer the reader to the forthcoming paper [27].

To keep our survey succinct, we have refrained from giving an encyclopedic treatment of this topic. However, there is one result from probability theory that requires highlighting.

In 1918, Hardy and Ramanujan developed their celebrated circle method to study the partition function. The reader will recall that the number of partitions of n is denoted $p(n)$ and has the generating function

$$\sum_{n=0}^{\infty} p(n)t^n = \prod_{n=1}^{\infty}(1-t^n)^{-1}.$$

Thus, $p(4) = 5$ since $1+1+1+1$, $1+1+1+2$, $1+3$, $2+2$, 4 are the five partitions of 4. Hardy and Ramanujan proved that

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{2n/3}}. \tag{24}$$

The second author, in joint work with Dewar [13] showed how one can derive this using arithmetic. Báez-Duarte [2] showed that a local central limit theorem can be used to establish (24). These remarkable symbiotic developments will be studied in a later paper.

Acknowledgements We would like to thank the referee and Brundaban Sahu for their detailed comments and helpful suggestions on a previous version of the paper.

References

1. Arfken, G.B., Weber, H.J., Harris, F.E.: Mathematical Methods for Physicists: A Comprehensive Guide. Academic, Oxford (2011)
2. Báez-Duarte, L.: Hardy-Ramanujan's asymptotic formula for partitions and the central limit theorem. Adv. Math. **125**(1), 114–120 (1997)
3. Biane, P., Pitman, J., Yor, M.: Probability laws related to the Jacobi theta and Riemann zeta functions and Brownian excursions. Bull. Amer. Math. Soc. **38**, 435–465 (2001)
4. Billingsley, P.: Probability and Measure. Wiley Series in Probability and Statistics, Anniversary edn. Wiley, Hoboken (2012)
5. Billingsley, P.: On the central limit theorem for the prime divisor functions. Amer. Math. Monthly **76**, 132–139 (1969)
6. Billingsley, P.: Prime numbers and Brownian motion. Amer. Math. Monthly **80**, 1099–1115 (1973)
7. Bombieri, E., Lagarias, J.: Complements to Li's criterion for the Riemann hypothesis. J. Number Theory **77**(2), 274–287 (1999)

8. Omar, S., Bouanani, S.: Li's criterion and the Riemann hypothesis for function fields II. Finite Fields Appl. **45**, 53–58 (2017)
9. Carlitz, L.: Some generating functions for Laguerre polynomials. Duke Math. J. **35**, 825–827 (1968)
10. Cioabă, S.M., Murty, M.R.: A first course in graph theory and combinatorics. Texts and Readings in Mathematics, vol. 55. Hindustan Book Agency, New Delhi (2009)
11. Coffey, M.W.: Toward verification of the Riemann hypothesis: application of the Li criterion. Math. Phys. Anal. Geom. **8**, 211–255 (2005)
12. Deligne, P.: La conjecture de Weil. I. Publications Mathématiques de l'IHÉS **43**, 273–307 (1974)
13. Dewar, M., Murty, M.R.: A derivation of the Hardy-Ramanujan formula from an arithmetic formula. Proc. Amer. Math. Soc. **141**(6), 1903–1911 (2013)
14. Erdős, P., Kac, M.: The Gaussian law of errors in the theory of additive number theoretic functions. Amer. J. Math. **62**, 738–742 (1940)
15. Elliott, P.D.T.A.: Probabilistic Number Theory II. Central Limit Theorems. Springer, Berlin (1980)
16. Feller, W.: The fundamental limit theorems in probability. Bull. Amer. Math. Soc. **51**, 800–832 (1945)
17. Feller, W.: An Introduction to Probability Theory and Its Applications, 1st, 2nd edn. Wiley, New York (1957)
18. Goncharov, V.: On the field of combinatory analysis. Amer. Math. Soc. Transl. **2**(19), 1–46 (1962)
19. Gun, S., Murty, M.R., Rath, P.: Transcendental sums related to the zeros of zeta functions. Mathematika **64**(3), 875–897 (2018)
20. Hardy, G.H., Ramanujan, S.: The normal number of prime factors of a number n. Q. J. Math. **48**, 76–92 (1917)
21. Harper, L.H.: Stirling behaviour is asymptotically normal. Ann. Math. Stat. **38**, 410–414 (1967)
22. Kubilius, J.: Probabilistic methods in the theory of numbers. Translations of Mathematical Monographs, vol. 11. American Mathematical Society, Providence (1964)
23. Li, X.-J.: The positivity of a sequence of numbers and the Riemann hypothesis. J. Number Theory **65**, 325–333 (1997)
24. Magnus, W., Oberhettinger, F., Soni, R.P.: Formulas and Theorems for the Special Functions of Mathematical Physics. Springer, New York (1966)
25. Murty, M.R., Murty, V.K.: Prime divisors of Fourier coefficients of modular forms. Duke Math. J. **51**, 57–77 (1984)
26. Murty, M.R., Murty, V.K.: An analogue of the Erdős-Kac theorem for Fourier coefficients of modular forms. Indian J. Pure Appl. Math. **15**, 1090–1101 (1984)
27. Murty, M.R., Prabhu, N.: Central limit theorems for sums of quadratic characters, modular forms and elliptic curves. To appear in Proceedings of the American Mathematical Society
28. Pólya, G.: Verschiedene Bemerkungen zur Zahlentheorie. Jahresber. Deutsch. Math.-Verein. **28**, 31–40 (1919)
29. Ramanujan, S.: Collected Papers, Chelsea (1962)
30. Turan, P.: On a theorem of Hardy and Ramanujan. J. Lond. Math. Soc. **9**, 274–276 (1934)
31. Zabell, S.L.: Alan turing and the central limit theorem. Amer. Math. Monthly **102**, 483–494 (1995)