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A Note on Quadratic Twists of an Elliptic Curve

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Let $E : y^2 = x^3 + Ax + B$, A, $B \in \mathbb{Z}$ be an elliptic curve over $\mathbb{Q}$, of conductor N and discriminant Δ. If D is any squarefree integer. we denote E_D the quadratic twist of E. The Weierstrass equation of E_D has the form:

$$E_D : y^2 = x^3 + D^2Ax + D^3B.$$

Also, denote N_D and Δ_D the conductor and discriminant of E_D. The Birch and Swinnerton-Dyer conjecture for E_D states that $L_D(s)$ has a zero of order $r_D = \text{rank}_{\mathbb{Z}} E_D(\mathbb{Q})$ and

$$(*) \qquad \frac{L^{(r)}(1)}{r!} = \frac{|\text{Ш}_D| R_D}{2^r |E_D(\mathbb{Q})_{\text{tors}}|^2} \pi_\infty \prod_{p|\Delta_D} \pi_p,$$

where

Ш_D is the Shafarevich-Tate group of E_D.

R_D is the regulator of E_D, with respect to the Neron-Tate height pairing.

$E_D(\mathbb{Q})_{\text{tors}}$ is the torsion part of the Mordell-Weil group $E_D(\mathbb{Q})$.

π_D is the real period of E_D, namely

$$\pi_\infty = \int_{E_D(\mathbb{R})} \omega_D$$

where ω_D is the Neron differential of E_D.

For each p dividing the discriminant Δ_D of E_D.

$$\pi_p = |E_D(\mathbb{Q}_p)/E_D^o(\mathbb{Q}_p)|$$

where $E_D^o(\mathbb{Q}_p)$ is the subgroup of points in $E_D(\mathbb{Q}_p)$ whose reduction modulo p are not singular points.

If one consider E_D's whose rank of the corresponding Mordell-Weil group is 1, $(*)$ can be stated as

$$(**) \qquad L'_D(1) = \frac{|\text{Ш}_D| R_D}{2 |E_D(\mathbb{Q})_{\text{tors}}|^2} \pi_\infty \prod_{p|\Delta_D} \pi_p.$$

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This is the final form of the paper.

THEOREM 1. *Let* $E : y^2 = x^3 + Ax + B$, $A, B \in \mathbb{Z}$ *be a fixed modular curve over* $\mathbb{Q}$. *Assuming the Birch and Swinnerton-Dyer conjecture for all quadratic twists* E_D, *where* D *are squarefree integers, then there exist infinitely many* D *such that* E_D *has rank* 1 *and*

$$|\text{Ш}_D| R_D \gg_\varepsilon (N_D)^{1/4-\varepsilon},$$

where the implied constant depends on ε.

PROOF. In [3], M. Ram Murty and V. Kumar Murty have shown that:

$$\sum_{\substack{0<-D\leq Y \\ D\equiv 1(4N)}} L'_D(1) = CY\log Y + o(Y\log Y)$$

as Y tends to infinity and here $C \neq 0$.

Therefore, there exist infinitely many D such that $L'_D(1) \gg 1$. For these D's, Kolyvagin (see [1]) proved that $\operatorname{rank}_D(\mathbb{Q}) = 1$. Assuming the Birch and Swinnerton-Dyer conjecture, one gets

$$(***) \qquad |\text{Ш}_D| R_D \gg \frac{2|E_D(\mathbb{Q})_{\text{tors}}|^2}{\pi_\infty \prod_{p|\Delta_D} \pi_p} \gg \frac{1}{\pi_\infty \prod_{p|\Delta_D} \pi_p}.$$

About the real period π_∞, we have

$$\begin{aligned}\pi_\infty &= \int_{E_D(\mathbb{R})} \omega_D \\ &= \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{x^3 + D^2Ax + D^3B}} \\ &= \frac{1}{D^{1/2}} \int_{-\infty}^{+\infty} \frac{du}{\sqrt{u^3 + Au + B}} \\ &\ll D^{-1/2}.\end{aligned}$$

To get an upper bound for $\prod_{p|\Delta_D} \pi_p$, we calculate

$$\begin{aligned}b_2 &= a_1^2 + 4a_2 = 0 \\ b_4 &= a_1a_3 + 2a_4 = 2D^2A \\ b_6 &= a_3^2 + 4a_6 = 4D^3B \\ b_8 &= a_1a_6 - a_1a_3a_4 + 4a_2a_6 + a_2a_3^2 - a_4^2 = -D^4A^2 \\ c_4 &= b_2^2 - 24b_4 = -48D^2A \\ c_6 &= -b_2^3 + 36b_2b_4 - 216b_6 = -864D^3B \\ \Delta_D &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 \\ &= -16D^6(4A^3 + 27B^2) = D^6\Delta \\ j_E &= \frac{(-48D^2A)^3}{\Delta_D} = j_E.\end{aligned}$$

For each $p \mid D$ and $p \nmid 16(4A^3 + 27B^2)$, using Tate's algorithm (see [5]), the corresponding Kodaira symbol is type I_0^* or I_ν^*, hence $\pi_p = |E_D(\mathbb{Q}_p)/E_D^o(\mathbb{Q}_p)| = 4$ and the exponent of the conductor is 2. For finitely many $p \mid 16(4A^3 + 27B^2)$, $|\pi_p| \leq 4$ and the exponent of the conductor is at most 2, except when $p = 2$ or 3, but even in this case, it is still bounded.

Therefore

$$\prod_{p|\Delta_D} \pi_p \ll 4^{\psi(D)}$$

where $\psi(D)$ is the number of prime divisors of D. Hence

$$\prod_{p|\Delta_p} \pi_p \ll 4^{\frac{\log D}{\log\log D}} = O(D^{\varepsilon}).$$

We also have

$$N_D \ll D^2.$$

Therefore, by $(***)$,

$$\begin{aligned} |\text{Ш}_D| R_D &\gg \frac{1}{\pi_\infty \prod_{p|\Delta_D} \pi_p} \\ &\gg D^{1/2} D^{-\varepsilon} \\ &\gg N_D^{1/4-\varepsilon}. \quad \square \end{aligned}$$

The regulator is 1 in the case that the Mordell-Weil rank is 0. More precisely, we prove that

THEOREM 2. *Assuming the Birch and Swinnerton-Dyer conjecture for the quadratic twists E_D, then there exist infinitely many D such that E_D has rank 0 and*

$$|\text{Ш}_D| \gg N_D^{1/4-\varepsilon}.$$

PROOF. V. K. Murty (see [4]) proved that

$$\frac{1}{Y} \int_1^Y \sum_{\substack{|D| \le t \\ D \equiv 1 (4N)}} L_D(1) dt = CY + O\left(\frac{Y}{(\log Y)^\nu}\right), \quad \nu > 0.$$

Therefore, there exist infinitely many D such that $L_D(1) \gg 1$. For these squarefree integers D, the rank is 0 and if we assume the Birch and Swinnerton-Dyer, we have

$$L_D(1) = \frac{|\text{Ш}_D|}{|E_D(\mathbb{Q})_{\text{tors}}|^2} \pi_\infty \prod_{p|\Delta_D} \pi_p.$$

Hence, similarly to the proof of Theorem 1. we get

$$|\text{Ш}_D| \gg N_D^{1/4-\varepsilon}. \quad \square$$

References

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