1. Use the Euclidean algorithm to find all integers \(x\) and \(y\) such that
\[
42823x + 6409y = 17.
\]
2. If \(m = p_1^{a_1} \cdots p_k^{a_k}\) and \(n = p_1^{b_1} \cdots p_k^{b_k}\) are the respective unique factorizations, show that
\[
\gcd(m, n) = p_1^\min\{a_1, b_1\} \cdots p_k^\min\{a_k, b_k\},
\]
and
\[
\lcm(m, n) = p_1^\max\{a_1, b_1\} \cdots p_k^\max\{a_k, b_k\}.
\]
3. Show that for all natural numbers \(n \geq 1, \)
\[
1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1.
\]
4. Find all integers \(x\) such that
\[
11111x \equiv 10 \pmod{78787}.
\]
5. Show that for all natural numbers \(N \geq 1, \)
\[
\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}.
\]
6. Let \(p_n\) denote the \(n\)-th prime. Show that for \(n \geq 1, \)
\[
p_n < 2^{2^n}.
\]
7. Show that if \(n|2^n - 1,\) then \(n = 1.\)
8. Show that for any natural number \(n > 1,\) the number
\[
S := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}
\]
is not a natural number. [Hint: consider \(k\) such that \(n/2 < 2^k \leq n\) and let \(d\) be the \(\lcm\) of all the numbers \(1, 2, \ldots, n\) except for \(2^k\) and analyze \(dS\).]
9. Let \(d = \gcd(m, n).\) Show that
\[
\gcd(a^m - 1, a^n - 1) = a^d - 1,
\]
for any natural number \(a > 1.\)
10. The Fibonacci sequence is defined as follows. \(F_1 = 1, F_2 = 1\) and for \(n \geq 3, F_n = F_{n-1} + F_{n-2}.\) Show that
(a) the \(\gcd\) of \(F_n\) and \(F_{n-1}\) is 1 for \(n \geq 2;\)
(b) \(F_{n+m} = F_{n-1}F_m + F_nF_{m+1}\) for \(n \geq 2\) and \(m \geq 1;\) [Hint: induct on \(m.\)]
(c) \(\gcd(F_n, F_m) = F_{\gcd(n, m)}.\)