1. Use the Euclidean algorithm to find all integers $x$ and $y$ such that $42823x + 6409y = 17$.

2. If $m = p_1^{a_1} \cdots p_k^{a_k}$ and $n = p_1^{b_1} \cdots p_k^{b_k}$ are the respective unique factorizations, show that
   \[ \text{gcd}(m, n) = p_1^{\min\{a_1, b_1\}} \cdots p_k^{\min\{a_k, b_k\}} , \]
   and
   \[ \text{lcm}(m, n) = p_1^{\max\{a_1, b_1\}} \cdots p_k^{\max\{a_k, b_k\}} . \]
3. Show that for all natural numbers $n \geq 1$,
   \[ 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1. \]
4. Find all integers $x$ such that $11111x \equiv 10 \pmod{78787}$.
5. Show that for all natural numbers $N \geq 1$,
   \[ \sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1} . \]
6. Let $p_n$ denote the $n$-th prime. Show that for $n \geq 1$,
   \[ p_n < 2^{2^n} . \]
7. Show that if $n|2^n - 1$, then $n = 1$.
8. Show that for any natural number $n > 1$, the number
   \[ S := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \]
   is not a natural number. [Hint: consider $k$ such that $n/2 < 2^k \leq n$ and let $d$ be the lcm of all the numbers $1, 2, \ldots, n$ except for $2^k$ and analyze $dS$.]
9. Let $d = \text{gcd}(m, n)$. Show that
   \[ \text{gcd}(a^m - 1, a^n - 1) = a^d - 1 , \]
   for any natural number $a > 1$.
10. The Fibonacci sequence is defined as follows. $F_1 = 1$, $F_2 = 1$ and for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$. Show that
    (a) the gcd of $F_n$ and $F_{n-1}$ is 1 for $n \geq 2$;
    (b) $F_{n+m} = F_{n-1}F_m + F_n F_{m+1}$ for $n \geq 2$ and $m \geq 1$; [Hint: induct on $m$.]
    (c) $\text{gcd}(F_n, F_m) = F_{\text{gcd}(n, m)}$.