MATH 382: Hints and Solutions to Assignment 2 (due: October 18, 2019)

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1. Prove that
$$2222^{5555} + 5555^{2222}$$
is divisible by 7.

Solution: By Euler's theorem, $a^6 \equiv 1$ (mod 6) if a is coprime to 6. Now $5555 \equiv 5$ (mod 6) and $2222 \equiv 2$ (mod 6). Also, $2222 \equiv 3$ (mod 7) and $5555 \equiv 4$ (mod 7). So our number is
$$2222^{5555} + 5555^{2222} \equiv 3^5 + 4^2 \equiv 12 + 16 \equiv 0 (\bmod 7).$$

2. Find the last three digits of 3^{2019}.

Solution: We need to find the reduced residue class mod 1000. By Euler's theorem, $\phi(1000) = 400$ so that
$$3^{2019} \equiv 3^{19} (\bmod 1000).$$
Now $3^2 = 9$, so we can rewrite this as:
$$3^{19} = 3(-1+10)^9 = 3\left[(-1)^9 + \binom{9}{1}(-1)^8(10) + \binom{9}{2}(-1)^7 10^2\right] (\bmod 1000).$$
This is now easily computed by hand to be -533, which mod 1000 is the same as 467. So these are the last three digits.

3. Let p be a prime number greater than 3. Show that the numerator of
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{1}{(p-1)^2}$$
(when expressed in lowest terms) is divisible by p.

Solution: Working modulo p, we see that the sum is congruent to
$$1 + 2^2 + 3^2 + \cdots + (p-1)^2 \quad (\bmod p).$$
Now, it is easily proved by induction that
$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$
As $p > 3$, the denominator is coprime to p and the numerator is for $n = p-1$ clearly dvisible by p.

4. Show that a natural number is divisible by 9 if and only if the sum of its digits is divisible by 9.

Solution: Let n be written in base 10:
$$n = a_k 10^k + a_{k-1} 10^{k-1} + \cdots + a_1 10 + a_0$$

and looking at n (mod 10) shows

$$n \equiv a_k + a_{k-1} + \cdots + a_1 + a_0 \pmod{10}.$$

Thus, n is zero mod 10 if and only if the sum of the digits is zero mod 10.

5. If p is a prime number, show that $\sqrt{p}$ is irrational.

   Solution: If $\sqrt{p} = a/b$, with a, b coprime, we get $pb^2 = a^2$. Thus $p|a$, But then $p^2|a^2$ so that p^2 divides pb^2 and so p divides b, contrary to coprimaility of a and b.

6. For any natural number m, show that

$$\sum_{\substack{j=1 \\ (j,m)=1}}^{m} j = \frac{1}{2} m\phi(m).$$

   [Hint: first show that if j is coprime to m, then so is $m - j$.]

   Solution: Let S be the sum. Then applying the hint,

$$2S = \sum_{\substack{j=1 \\ (j,m)=1}}^{m} [j + (m - j)] = m\phi(m).$$

7. For $m \geq 4$, show that the numerator of

$$\sum_{\substack{j=1 \\ (j,m)=1}}^{m} \frac{1}{j}$$

   (when expressed in lowest terms) is divisible by m.

   Solution: As j ranges over the coprime residues mod m, so does $1/j$. Thus the sum (mod m) is

$$\equiv \sum_{\substack{j=1 \\ (j,m)=1}}^{m} j = \frac{1}{2} m\phi(m),$$

   by the previous question. Since $m \geq 4$, $\phi(m)$ is even so that the whole sum is zero mod m.

8. Show that the number of $j \leq n$ with $(j, n) = d$ is precisely $\phi(n/d)$. Deduce that

$$\sum_{d|n} \phi(d) = n.$$

   Solution: We partition the numbers $j \leq n$ according to their gcd:

$$\{j \leq n\} = \cup_{d|n} \{j \leq n : (j, n) = d\}.$$

   Now $(j, n) = d$ with $j \leq n$ if and only if $(j/d, n/d) = 1$ with $j/d \leq n/d$. But this is precisely $\phi(n/d)$. Thus

$$n = \sum_{d|n} \phi(n/d) = \sum_{d|n} \phi(d),$$

because as d runs over the divisors of n so does n/d.

9. Let p be a prime number greater than 3. Show that the numerator of

$$S := \sum_{j=1}^{p-1} \frac{1}{j}$$

is divisible by p^2. [Hint: Note that $2S = \sum_{j=1}^{p-1} \left(\frac{1}{j} + \frac{1}{p-j} \right)$. and apply question 3.] By the hint,

$$2S = p \sum_{j=1}^{p-1} \frac{1}{j(p-j)}.$$

Thus, we need to study the sum

$$\sum_{j=1}^{p-1} \frac{1}{j(p-j)}$$

mod p and show that it is divisible by p. But mod p, the sum is

$$-\sum_{j=1}^{p-1} \frac{1}{j^2}$$

which by question 3 has numerator divisible by p.

10. Let R be the ring $\{a + b\sqrt{-2} : a, b \in \mathbb{Z}\}$ and define $N(a + b\sqrt{-2}) := a^2 + 2b^2$. Given $\alpha, \beta \in R$, show that there exist $\gamma, \delta \in R$ such that $\alpha = \beta\gamma + \delta$ with $N(\delta) < N(\beta)$.

Solution: We follow the method used in class to show that $\mathbb{Z}[\sqrt{-1}]$ has a division algorithm. Indeed, given $\alpha = a + b\sqrt{-2}$, $\beta = c + d\sqrt{-2}$, we have upon rationalizing the denominator

$$\frac{\alpha}{\beta} = \frac{a + b\sqrt{-2}}{c + d\sqrt{-2}} = x + y\sqrt{-2},$$

for some rational numbers x, y. Now choose integers m, n such that

$$|x - m| \le \frac{1}{2} \quad |y - n| \le \frac{1}{2}.$$

Then,

$$\frac{\alpha}{\beta} - (m + n\sqrt{-2}) = (x - m) + (y - n)\sqrt{-2}$$

has norm

$$(x - m)^2 + 2(y - n)^2 \le \frac{1}{4} + \frac{2}{4} = \frac{3}{4} < 1.$$

Set $\gamma = m + n\sqrt{-2}$ and $\delta = \alpha - \beta\gamma$ which has the required property.