MATH 382: Assignment 2 (due: October 18, 2019)

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1. Prove that
$$2222^{5555} + 5555^{2222}$$
is divisible by 7.

2. Find the last three digits of 3^{2019}.

3. Let p be a prime number greater than 3. Show that the numerator of
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{1}{(p-1)^2}$$
(when expressed in lowest terms) is divisible by p

4. Show that a natural number is divisible by 9 if and only if the sum of its digits is divisible by 9.

5. If p is a prime number, show that $\sqrt{p}$ is irrational.

6. For any natural number m, show that
$$\sum_{\substack{j=1 \\ (j,m)=1}}^{m} j = \frac{1}{2} m\phi(m).$$
[Hint: first show that if j is coprime to m, then so is $m - j$.]

7. For $m \geq 4$, show that the numerator of
$$\sum_{\substack{j=1 \\ (j,m)=1}}^{m} \frac{1}{j}$$
(when expressed in lowest terms) is divisible by m.

8. Show that the number of $j \leq n$ with $(j, n) = d$ is precisely $\phi(n/d)$. Deduce that
$$\sum_{d|n} \phi(d) = n.$$

9. Let p be a prime number greater than 3. Show that the numerator of
$$S := \sum_{j=1}^{p-1} \frac{1}{j}$$
is divisible by p^2. [Hint: Note that $2S = \sum_{j=1}^{p-1} \left(\frac{1}{j} + \frac{1}{p-j}\right)$. and apply question 2.]

10. Let R be the ring $\{a + b\sqrt{-2} : a, b \in \mathbb{Z}\}$ and define $N(a + b\sqrt{-2}) := a^2 + 2b^2$. Given $\alpha, \beta \in R$, show that there exist $\gamma, \delta \in R$ such that $\alpha = \beta\gamma + \delta$ with $N(\delta) < N(\beta)$.