Math 892: Assignment 2 (due: March 15, 2019)

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1. Let $w \in \mathbb{C}$ be a fixed complex number with $|w| < 1$. Let
$$f(z) = \frac{z - w}{1 - \overline{w}z}.$$
Show that f is regular in $|z| \leq 1$ and calculate $f(w)$ and $f'(w)$.

2. Find the radius of convergence of
$$\sum_{n=1}^{\infty} n! \frac{z^n}{n^n}.$$

3. Prove the following discrete version of integration by parts: For any two sequences of numbers a_n, b_n,
$$S_N := \sum_{n=1}^{N} a_n b_n = a_N B_N - \sum_{n=1}^{N-1} B_n (a_{n+1} - a_n),$$
where
$$B_n = \sum_{j=1}^{n} b_j.$$
Using this result, show that the power series
$$\sum_{n=1}^{\infty} \frac{z^n}{n}$$
converges for every complex number z with $|z| = 1$, and $z \neq 1$.

4. Show that the series
$$\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$$
converges for all $|z| \leq 1$. What is its radius of convergence? Does the series
$$\sum_{n=1}^{\infty} z^n$$
converge for any z with $|z| = 1$?

5. Let n be a non-negative integer. Define for $\alpha \in \mathbb{C}$, the binomial coefficient
$$\binom{\alpha}{n} = \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}.$$
Show that the series
$$\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n$$
converges for $|z| < 1$.