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General Section

A triple convolution sum of the divisor function

Bikram Misra [a], M. Ram Murty [b], Biswajyoti Saha [a,*]

[a] *Department of Mathematics, Indian Institute of Technology Delhi, New Delhi 110016, India*
[b] *Department of Mathematics and Statistics, Queen's University, Kingston K7L 3N6, Canada*

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ABSTRACT

We study the triple convolution sum of the divisor function given by

$$\sum_{n\le x} d(n)d(n-h)d(n+h)$$

for $h \neq 0$ where $d(n)$ denotes the number of positive divisors of n. Based on some algebraic and geometric considerations, Browning conjectured that the above sum is asymptotic to $c_h x(\log x)^3$, for a suitable constant $c_h \neq 0$, as $x \to \infty$. This conjecture is still unproved. Using sieve-theoretic results of Wolke and Nair (respectively), it is possible to derive the exact order of the sum. The lower bound of the correct order of magnitude can also be derived by very elementary arguments. In this article, using the Tauberian theory for multiple Dirichlet series, we prove an explicit lower bound and provide a new theoretical framework to predict Browning's conjectured constant c_h.

* Corresponding author.
E-mail addresses: bikram.misra@maths.iitd.ac.in (B. Misra), murty@queensu.ca (M.R. Murty), biswajyoti@maths.iitd.ac.in (B. Saha).

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1. Introduction

The study of convolution sums of arithmetic functions lies at the heart of analytic number theory. Ingham studied [9] the shifted and additive convolution sums of the divisor function. For any integer n, let $d(n)$ denote the number of positive divisors of n. Ingham showed that for a positive integer h,

$$\sum_{n\le N} d(n)d(n+h) = \frac{6}{\pi^2}\sigma_{-1}(h)N(\log N)^2 + O(N\log N) \tag{1}$$

as $N\to\infty$ and

$$\sum_{n<N} d(n)d(N-n) = \frac{6}{\pi^2}\sigma_1(N)(\log N)^2 + O(\sigma_1(N)\log N\log\log N) \tag{2}$$

as $N\to\infty$, where $\sigma_s(n) := \sum_{\substack{d|n\\ d>0}} d^s$ for a complex number s.

For a fixed positive integer h, the triple convolution sum of the divisor function is defined as

$$S(x;h) := \sum_{n\le x} d(n)d(n-h)d(n+h).$$

It is still an open problem to determine the asymptotic behaviour of this sum. In [3], Browning suggested using some algebraic and geometric methods that $S(x;h)\sim c_h x(\log x)^3$ as $x\to\infty$ for a precise constant $c_h>0$, defined as

$$c_h := \frac{11}{8}f(h)\prod_p\left(1-\frac{1}{p}\right)^2\left(1+\frac{2}{p}\right), \tag{3}$$

with an explicit function f defined multiplicatively by $f(1)=1$ and

$$f(p^\nu) = \begin{cases}\left(1+\frac{4}{p}+\frac{1}{p^2}-\frac{3\nu+4}{p^{\nu+1}}-\frac{4}{p^{\nu+2}}+\frac{3\nu+2}{p^{\nu+3}}\right)\left(1+\frac{2}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{-2} & \text{if } p>2,\\ \frac{52}{11}-\frac{41+15\nu}{11\times 2^\nu} & \text{if } p=2,\end{cases} \tag{4}$$

for $\nu\ge 1$. While this conjecture is still open, Browning [3] proved that for $\epsilon>0$ and $H\ge x^{\frac{3}{4}+\epsilon}$, as $x\to\infty$, one has

$$\sum_{h\le H}\left(S(x;h)-c_h x(\log x)^3\right) = o(Hx(\log x)^3). \tag{5}$$

There is a typo in formula (1.2) on page no. 580 in the published version of the article [3]. The factor $(1-\frac{1}{p})^{-1}$ should be $(1-\frac{1}{p})^{-2}$ for $p>2$ as we have written here in (4). This correction is consistent with our new heuristic derivation below using multiple Dirichlet series. In fact, the definition in the arXiv version of [3] agrees with (4).

Formula (5) suggests that the conjectured asymptotic formula for $S(x;h)$ is true "on average." Since divisor functions appear as the Fourier coefficients of Eisenstein series, spectral methods have been extensively used by several authors in the study of the convolution sums of the divisor functions. For example, Blomer [1] derived a related average result with power-saving error terms in this context using spectral tools. A far reaching generalisation of these average results for the higher convolutions of a fixed generalised divisor function was recently obtained by Matomäki, Radziwiłł, Shao, Tao and Teräväinen [11].

1.1. Statement of the theorems

First and foremost, we provide a new framework using the theory of Dirichlet series attached to the multivariable arithmetic functions to predict the constant c_h in Browing's conjecture. It is worth noting that a lower bound of the correct order of magnitude can be obtained using elementary arguments (given in Section 3). However, our result go further to capture an explicit constant in the lower bound. Using the theory of the multiple Dirichlet series, we prove the following unconditional lower bound of $S(x;h)$.

Theorem 1.1. *As $x \to \infty$, we have*

$$S(x;h) \geq c_h x(\log x)^3/27 + O_h(x(\log x)^2).$$

Our method of dealing with the triple convolution sum $S(x;h)$ leads to the constant c_h as in (3) (see Proposition 4.1). This provides a theoretical framework to get to the arithmetic constants appearing in several convolution questions related to divisor functions, which is studied in the forthcoming paper [12].

The study of the asymptotic behaviour of the divisor function at polynomial arguments has a classical origin. Using the theory of smooth numbers, Erdős [6] established in 1952 that as $x \to \infty$, $\sum_{n \leq x} d(F(n)) \asymp x \log x$, where $F(t)$ is an irreducible polynomial with integer coefficients. Wolke [20] generalised this result for a general multiplicative function evaluated at a sequence of natural numbers, under certain sieve-theoretic hypotheses. Applying inequality (7), it can be shown from Wolke's results that as $x \to \infty$, $S(x;h) \asymp_h x(\log x)^3$. Further generalisations of Wolke's work were obtained by Nair [14], and subsequently by Nair and Tenenbaum [15].

An upper bound of the correct order can also be derived from the main theorem of Nair [14]. For the sake of clarity of our exposition, we give a streamlined proof of the upper bound inequality by applying the theory of smooth numbers implicit in the method of Erdős. Such a streamlined proof will make the method available for more general problems in number theory.

Theorem 1.2. *As $x \to \infty$, we have*

$$S(x;h) \ll_h x(\log x)^3.$$

2. Preliminaries

In this section, we collect the results that are required to prove our theorems. Firstly, we need a variant of the Chinese remainder theorem. The key tools for the first theorem come from the theory of the multiple Dirichlet series and Tauberian theorems due to de la Bretèche [2]. For the second theorem, we need some results on smooth numbers, which is a standard chapter in sieve theory.

2.1. Elementary number theory results

The following variant of the Chinese remainder theorem can be found in [10, Theorem 3-12]. Apparently, this non-standard version of the Chinese remainder theorem for non-coprime moduli was first written down by the Buddhist monk Yih-hing in 717 CE (see pages 57-64 of [5]). The first formal proof seems to have been written down by Stieltjes (of Stieltjes integral fame) as late as 1890.

Lemma 2.1. *For positive integers $d_1, \dots, d_k$ and integers $a_1, \dots, a_k$, the system*

$$\begin{cases} x \equiv a_1 \bmod d_1 \\ \vdots \\ x \equiv a_k \bmod d_k \end{cases} \tag{6}$$

has a solution if and only if $\gcd(d_i, d_j) \mid (a_i - a_j)$ for all $1 \le i, j \le k$. Moreover, when a solution exists, it is unique modulo the least common multiple $[d_1, \dots, d_k]$.

Related to the above lemma, we will need various facts regarding the number of solutions of the congruence

$$f(n) \equiv 0 \bmod a,$$

for a given polynomial $f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_m \in \mathbb{Z}[x]$. Such a polynomial is called an integral polynomial. It is said to be primitive if the greatest common divisor of a_i's is equal to 1. We quote the following from Nagell [13, Theorems 52 and 54, page no. 90].

Lemma 2.2. *Let $f(x)$ be a primitive integral polynomial of degree m with non-zero discriminant D. If p is a prime divisor of D, then the congruence*

$$f(x) \equiv 0 \bmod p^a$$

has at most mD^2 many solutions. If p is coprime to D, then the number of solutions is at most m.

Combining this with the Chinese remainder theorem and specialising to the polynomial $f(x) = x(x+h)(x-k)$, immediately we get the following:

Lemma 2.3. *For $hk(h+k) \neq 0$, the number of solutions of the congruence*

$$n(n+h)(n-k) \equiv 0 \bmod a$$

is bounded by $3^{\omega(a)}(hk)^4(h+k)^4$, where $\omega(a)$ is the number of distinct prime divisors of a.

Lemma 2.2 has since been improved by Huxley [8], who gave a bound of $m|D|^{1/2}$ instead of mD^2 in the case $p \mid D$. Thus, the bound in the above Lemma 2.3 can be improved to

$$3^{\omega(a)}hk(h+k).$$

We will not need the sharper result, but we merely record it here for academic interest.

We will need to use the elementary results recorded in the next lemma. They follow from standard analytic number theory (see [17, Ex. 4, Chap. I.3]).

Lemma 2.4. *For an integer $a \geq 1$, let $\omega(a)$ be the number of distinct prime divisors of a. Then as $x \to \infty$, we have*

$$\sum_{a \leq x} 3^{\omega(a)} \ll x(\log x)^2 \qquad \textit{and} \qquad \sum_{a \leq x} \frac{3^{\omega(a)}}{a} \ll (\log x)^3.$$

We will also need the following result from elementary number theory.

Lemma 2.5. *For the divisor function $d(n)$, we have for $hk(h+k) \neq 0$,*

$$d(n)d(n+h)d(n-k) \leq d(h)d(k)d(h+k)d(n(n+h)(n-k)). \tag{7}$$

Proof. As usual, we denote by (u, v) and $[u, v]$, the gcd and lcm respectively of the integers u, v and by (u, v, w) and $[u, v, w]$, the gcd and lcm respectively of the integers u, v, w. Given three non-negative integers a, b, c, it is evident that

$$\max\{a, b, c\} = a + b + c - \min\{a, b\} - \min\{a, c\} - \min\{b, c\} + \min\{a, b, c\}. \tag{8}$$

The easiest way to see this is by setting $S_r = \{1, \ldots, r\}$ so that the assertion is equivalent to

$$|S_a \cup S_b \cup S_c| = |S_a| + |S_b| + |S_c| - |S_a \cap S_b| - |S_a \cap S_c| - |S_b \cap S_c| + |S_a \cap S_b \cap S_c|,$$

which follows from the inclusion-exclusion principle. The equality (8) implies that for any multiplicative arithmetic function f, we have

$$f(u)f(v)f(w) = \frac{f([u,v,w])f((u,v))f((u,w))f((v,w))}{f((u,v,w))}, \tag{9}$$

generalising the familiar identity for two variables, $f([u,v])f(u,v) = f(u)f(v)$, which can be derived using the fact that $\max\{a,b\} = a + b - \min\{a,b\}$. Applying (9) to the divisor function gives

$$d(n)d(n+h)d(n-k) = \frac{d([n,n+h,n-k])d((n,n+h))d((n,n-k))d((n+h,n-k))}{d((n,n+h,n-k))}.$$

Thus,

$$d(n)d(n+h)d(n-k) \le d(h)d(k)d(h+k)d(n(n+h)(n-k)),$$

as claimed. □

2.2. Results from multiple Dirichlet series theory

We now recall the setup of arithmetic functions of several variables and the multiple Dirichlet series. It seems that the arithmetic functions of several variables were first discussed by Vaidyanathaswamy [19] in 1931. He defined a multiplicative function of several variables as a function $f : \mathbb{N}^k \to \mathbb{C}$ that satisfies the property

$$f(m_1n_1, \ldots, m_kn_k) = f(m_1, \ldots, m_k)f(n_1, \ldots, n_k)$$

when $\gcd(m_1 \cdots m_k, n_1 \cdots n_k) = 1$. In the one variable case, we know that the Dirichlet convolution of two multiplicative functions is again a multiplicative function. The same can be found true if we define the Dirichlet convolution of two arithmetic functions of several variables f and g as

$$(f \star g)(n_1, \ldots, n_k) := \sum_{\substack{d_i | n_i \\ 1 \le i \le k}} f(d_1, \ldots, d_k)g(n_1/d_1, \ldots, n_k/d_k).$$

In the context of determining the average order of arithmetic functions of several variables, de la Bretèche [2] derived a multi-variable version of the classical Tauberian theorems. To state this version, we need the following notations.

Notations

Let $\mathbb{R}^+$ denote the set of all non-negative real numbers and $\mathbb{R}^+_*$ denote the set of all positive real numbers. For a positive integer m, we denote an m-tuple $(s_1, \ldots, s_m)$ of complex numbers by $\boldsymbol{s}$. Let $\tau_j = \Im(s_j)$ and $\mathcal{L}_m(\mathbb{C})$ be the space of all linear forms on $\mathbb{C}^m$ over $\mathbb{C}$. We denote by $\{e_j\}_{j=1}^m$, the canonical basis of $\mathbb{C}^m$ and $\left\{e_j^*\right\}_{j=1}^m$, the dual basis in $\mathcal{L}_m(\mathbb{C})$. Let $\mathcal{L}\mathbb{R}_m(\mathbb{C})$ (respectively $\mathcal{L}\mathbb{R}_m^+(\mathbb{C})$) denote the set of linear forms in $\mathcal{L}_m(\mathbb{C})$ whose restriction to $\mathbb{R}^m$ (respectively to $(\mathbb{R}^+)^m$) has values in $\mathbb{R}$ (respectively, in $\mathbb{R}^+$). Let $\beta_j > 0$ for all $j = 1, \ldots, m$. Then we denote by $\mathcal{B}$ the linear form $\sum_{j=1}^m \beta_j e_j^*$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$ be the associated row matrix. We define $X^{\boldsymbol{\beta}} := (X^{\beta_1}, \ldots, X^{\beta_m})$. Let $\mathcal{L}$ be a family of linear forms and for this we define $\operatorname{conv}(\mathcal{L}) := \sum_{\ell\in\mathcal{L}} \mathbb{R}^+\ell$ and $\operatorname{conv}^*(\mathcal{L}) := \sum_{\ell\in\mathcal{L}} (\mathbb{R}^+_*)\ell$. With these notations in place, [2, Théorème 1] reads as follows:

Theorem 2.1. *Let f be an arithmetic function on $\mathbb{N}^m$ taking positive values and F be the associated Dirichlet series*

$$F(\boldsymbol{s}) = \sum_{d_1=1}^{\infty} \cdots \sum_{d_m=1}^{\infty} \frac{f(d_1, \ldots, d_m)}{d_1^{s_1} \ldots d_m^{s_m}}.$$

Suppose that there exists $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{R}^+)^m$ such that F satisfies the following three properties:

(1) the series $F(\boldsymbol{s})$ is absolutely convergent for $\boldsymbol{s} \in \mathbb{C}^m$ such that $\Re(s_i) > \alpha_i$;
(2) there exists a family $\mathcal{L}$ of n many non-zero linear forms $\mathcal{L} := \left\{\ell^{(i)}\right\}_{i=1}^n$ in $\mathcal{L}\mathbb{R}_m^+(\mathbb{C})$ and a family of finitely many linear forms $\left\{h^{(k)}\right\}_{k\in\mathcal{K}}$ in $\mathcal{L}\mathbb{R}_m^+(\mathbb{C})$, such that the function H from $\mathbb{C}^m$ to $\mathbb{C}$ defined by

$$H(\boldsymbol{s}) := F(\boldsymbol{s} + \boldsymbol{\alpha}) \prod_{i=1}^{n} \ell^{(i)}(\boldsymbol{s})$$

can be extended to a holomorphic function in the domain

$$\mathcal{D}(\delta_1, \delta_3) := \left\{\boldsymbol{s} \in \mathbb{C}^m : \Re\left(\ell^{(i)}(\boldsymbol{s})\right) > -\delta_1 \text{ for all } i \text{ and } \Re\left(h^{(k)}(\boldsymbol{s})\right) > -\delta_3 \text{ for all } k \in \mathcal{K}\right\}$$

for some $\delta_1, \delta_3 > 0$;
(3) there exists $\delta_2 > 0$ such that for every $\epsilon, \epsilon' > 0$, the upper bound

$$|H(\boldsymbol{s})| \ll (1 + ||\Im(\boldsymbol{s})||_1^{\epsilon}) \prod_{i=1}^{n} \left(|\Im\left(\ell^{(i)}(\boldsymbol{s})\right)| + 1\right)^{1-\delta_2 \min\left\{0, \Re\left(\ell^{(i)}(\boldsymbol{s})\right)\right\}}$$

is uniformly valid in the domain $\mathcal{D}(\delta_1 - \epsilon', \delta_3 - \epsilon')$.

Let $J(\boldsymbol{\alpha}) := \{j \in \{1, \ldots, m\} : \alpha_j = 0\}$. *Let* $r := |J(\boldsymbol{\alpha})|$ *and* $\ell^{(n+1)}, \ldots, \ell^{(n+r)}$ *be the linear forms* e_j^* *for* $j \in J(\boldsymbol{\alpha})$. *Then for* $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m) \in (\mathbb{R}^+)^m$, *there exists a polynomial* $Q_{\boldsymbol{\beta}} \in \mathbb{R}[X]$ *of degree less than or equal to* $n + r - \mathit{rank}\left(\left\{\ell^{(i)}\right\}_{i=1}^{n+r}\right)$ *and a real number* $\theta = \theta\left(\mathcal{L}, \left\{h^{(k)}\right\}_{k\in\mathcal{K}}, \delta_1, \delta_2, \delta_3, \boldsymbol{\alpha}, \boldsymbol{\beta}\right) > 0$ *such that we have, for* $X \geq 1$,

$$S(X^{\boldsymbol{\beta}}) := \sum_{1\leq d_1\leq X^{\beta_1}} \cdots \sum_{1\leq d_m\leq X^{\beta_m}} f(d_1, \ldots, d_m) = X^{\langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle} \left(Q_{\boldsymbol{\beta}}(\log X) + O(X^{-\theta})\right).$$

We also need [2, Théorème 2].

Theorem 2.2. *Let the notations be as in Theorem 2.1. If we have that* $\mathcal{B}$ *is not in the span of* $\left\{\ell^{(i)}\right\}_{i=1}^{n+r}$, *then* $Q_{\boldsymbol{\beta}} = 0$. *Next suppose, we have the following two conditions:*

(1) there exists a function G *such that* $H(\boldsymbol{s}) = G(\ell^{(1)}(\boldsymbol{s}), \ldots, \ell^{(n+r)}(\boldsymbol{s}))$;
(2) $\mathcal{B}$ *is in the span of* $\left\{\ell^{(i)}\right\}_{i=1}^{n+r}$ *and there exists no strict subfamily* $\mathcal{L}'$ *of* $\left\{\ell^{(i)}\right\}_{i=1}^{n+r}$ *such that* $\mathcal{B}$ *is in the span of* $\mathcal{L}'$ *with*

$$\mathit{card}(\mathcal{L}') - \mathit{rank}(\mathcal{L}') = \mathit{card}\left(\left\{\ell^{(i)}\right\}_{i=1}^{n+r}\right) - \mathit{rank}\left(\left\{\ell^{(i)}\right\}_{i=1}^{n+r}\right).$$

Then, for $X \geq 3$, *the polynomial* $Q_{\boldsymbol{\beta}}$ *satisfies the relation*

$$Q_{\boldsymbol{\beta}}(\log X) = C_0 X^{-\langle \boldsymbol{\alpha}, \boldsymbol{\beta}\rangle} I(X^{\boldsymbol{\beta}}) + O((\log X)^{\rho-1}),$$

where $C_0 := H(0, \ldots, 0)$, $\rho := n + r - \mathit{rank}\left(\left\{\ell^{(i)}\right\}_{i=1}^{n+r}\right)$, *and*

$$I(X^{\boldsymbol{\beta}}) := \iint \cdots \int_{A(X^{\boldsymbol{\beta}})} \frac{dy_1 \ldots dy_n}{\prod_{i=1}^n y_i^{1-\ell^{(i)}(\boldsymbol{\alpha})}}$$

with

$$A(X^{\boldsymbol{\beta}}) := \left\{\boldsymbol{y} \in [1, \infty)^n : \prod_{i=1}^n y_i^{\ell^{(i)}(e_j)} \leq X^{\beta_j} \text{ for all } j\right\}.$$

2.3. Results from the theory of smooth numbers

We now recall some key results on smooth numbers. Let $\mathcal{S}(x, y)$ be the set of natural numbers less than x that have all their prime factors less than y. Such numbers are called y-smooth. The size of $\mathcal{S}(x, y)$ is traditionally denoted by $\Psi(x, y)$. The results we record here can all be found in Chapter III.5 of Tenenbaum's book [17]. The first assertion was first proved by Canfield, Erdős and Pomerance in [4] (see Corollary on page 15). The second assertion originates in a 1938 paper of Rankin [16] and his method of proof is often dubbed as "Rankin's trick." The elementary nature of this trick is explained well in Granville's survey article [7].

Proposition 2.1. *Letting*

$$u = \frac{\log x}{\log y},$$

we have

$$\Psi(x, y) \ll x \exp\left(-\frac{1}{2} u \log u\right),$$

as $x \to \infty$, provided that for any fixed $\epsilon > 0$, we have

$$1 \leq u \leq (1 - \epsilon)\frac{\log x}{\log\log x}.$$

Also, for any $A > 0$,

$$\Psi(x, (\log x)^A) \ll x^{1-\frac{1}{A}} \exp\left(\frac{c \log x}{\log\log x}\right),$$

as $x \to \infty$, for a positive constant c.

These results are very useful. In particular, we can apply them to derive the following variants of Lemma 2.4.

Lemma 2.6. *For a fixed $\delta \in (0, 1)$, x large enough, we have*

$$\sum_{\substack{x^\delta < d \leq x \\ d \in \mathcal{S}(x; (\log x)^2)}} \frac{3^{\omega(d)}}{d} \ll x^{-\frac{\delta}{2}+\epsilon},$$

for any $\epsilon > 0$. Moreover, for a positive integer t such that $x^{1/t} \geq (\log x)^2$, we have

$$\sum_{\substack{x^\delta < d \leq x \\ d \in \mathcal{S}(x; x^{1/t})}} \frac{3^{\omega(d)}}{d} \ll (\log x)^3 \exp(-ct \log t),$$

for some positive constant c.

Proof. This can be seen as an application of Proposition 2.1 along with partial summation. □

3. Setting up the proof of Theorem 1.1

Before embarking on the proof proper, we exhibit an elementary proof of the fact that $S(x;h) \gg_h x(\log x)^3$.

3.1. Elementary proof of the lower bound

We first assume that h is even. Note that if n is coprime to h, then the numbers $n, n \pm h$ are mutually coprime. Therefore, in this case

$$S(x;h) \geq \sum_{\substack{n\leq x\\(n,h)=1}} d(n)d(n-h)d(n+h) = \sum_{\substack{n\leq x\\(n,h)=1}} d(n(n^2-h^2)).$$

Any divisor of $n(n^2-h^2)$ factors uniquely as $d_1d_2d_3$ with $d_1 \mid n$, $d_2 \mid (n+h)$ and $d_3 \mid (n-h)$. In counting the divisors of $n(n^2-h^2)$, we can restrict to counting only divisors d_1, d_2, d_3 less than $x^{1/3}$ to obtain a lower bound for our sum under consideration. By the coprimality of the terms, we must have n belonging to a unique progression $\bmod d_1d_2d_3$. Thus,

$$S(x;h) \geq \sideset{}{'}\sum_{d_1,d_2,d_3\leq x^{1/3}} \left(\frac{x}{d_1d_2d_3} + O(1)\right),$$

where the prime on the sum indicates that d_1, d_2, d_3 are coprime to h. This shows that $S(x;h) \gg_h x(\log x)^3$.

Now if h is odd, we consider only even integers n which are coprime to h. Then the numbers $n, n\pm h$ are mutually coprime. Therefore, in this case

$$S(x;h) \geq \sum_{\substack{\text{even } n\leq x\\(n,h)=1}} d(n)d(n-h)d(n+h) = \sum_{\substack{\text{even } n\leq x\\(n,h)=1}} d(n(n^2-h^2)).$$

We can then get a similar estimate as above by choosing the divisor d_1 of n to be even. To derive the finer result stated in Theorem 1.1, we proceed as follows.

3.2. Estimating $S(x;h)$

Without loss of generality, we can assume that $h \leq x/2$. We write the triple convolution sum of the divisor function as follows

$$S(x;h) = \sum_{n\leq x} d(n)d(n-h)d(n+h) = \sum_{n\leq x}\left(\sum_{u\mid n+h} 1\right)\left(\sum_{v\mid n} 1\right)\left(\sum_{w\mid n-h} 1\right)$$

$$= \sum_{\substack{u \le x+h \\ v \le x \\ w \le x-h}} \sideset{}{'}\sum_{n \le x} 1,$$

where the primed sum denotes the sum for $n \le x$ where $n + h \equiv 0 \bmod u, n \equiv 0 \bmod v$ and $n - h \equiv 0 \bmod w$. Note that the outer sum is over all the integer triples (u, v, w) in the set $[1, x + h] \times [1, x] \times [1, x - h]$. We split the sum to write

$$S(x; h) = S_1(x; h) + S_2(x; h) - S_3(x; h),$$

where

$$S_1(x; h) := \sum_{u,v,w \le x} \sideset{}{'}\sum_{n \le x} 1, \ S_2(x; h) := \sum_{\substack{x < u \le x+h \\ v,w \le x}} \sideset{}{'}\sum_{n \le x} 1 \ \text{ and}$$

$$S_3(x; h) := \sum_{\substack{u \le x+h, v \le x \\ x-h < w \le x}} \sideset{}{'}\sum_{n \le x} 1.$$

It can be seen that $S_2(x; h) = O(\sigma_1(h) \log^2 x)$. However, for a lower bound of $S(x; h)$, we can just write $S(x; h) \ge S_1(x; h) - S_3(x; h)$. For $S_3(x; h)$, we prove the following lemma.

Lemma 3.1. *For a fixed positive integer h, $S_3(x; h) \le hd(h)d(2h)$.*

Proof. Note that $1 - h \le n - h \le x - h$ and $w \mid (n - h)$. If $n - h = mw$ for some $m \ge 1$, then $x - h < w \le mw = n - h \le x - h$, a contradiction. Hence, $n - h = mw$ for $m \le 0$. Now suppose $n - h = -tw$ for some $t \ge 1$. Since $x - h < w \le x$, we have $-tx \le -tw < -tx + th$ and $1 - h \le n - h \le x - h$. Therefore, we must have $-tx + th > 1 - h$, which implies that $h > (1 + tx)/(1 + t) > x/2$, a contradiction. Hence $n = h$ is the only possibility and thus

$$S_3(x; h) = \sum_{\substack{u \le x+h \\ v \le x \\ x-h < w \le x}} \sum_{\substack{2h \equiv 0 \bmod u \\ h \equiv 0 \bmod v}} 1 \le hd(h)d(2h). \quad \square$$

This means that to determine the asymptotic behaviour of $S(x; h)$, it suffices to study $S_1(x; h)$.

4. Proof of Theorem 1.1

As mentioned, we begin by estimating $S_1(x; h)$.

4.1. Estimating $S_1(x;h)$

Applying Lemma 2.1, we can write the inner sum in the definition of $S_1(x;h)$ as

$$\sum_{n\le x}{}' 1 = \frac{xg(u,v,w)}{[u,v,w]} + E(x;u,v,w),$$

where the error term $E := E(x;u,v,w)$ is bounded by 1, the term $[u,v,w]$ denotes the least common multiple of u,v,w and $g(u,v,w)$ is a multiplicative function taking the value 1 if and only if the system $n \equiv -h \bmod u, n \equiv 0 \bmod v, n \equiv h \bmod w$ has a solution, else it is 0. Therefore,

$$S_1(x;h) = \sum_{u,v,w\le x} g(u,v,w)\left\{\frac{x}{[u,v,w]} + E\right\}.$$

In order to understand the sum $\sum_{u,v,w\le x} g(u,v,w)/[u,v,w]$, we analyse the multiple Dirichlet series

$$F(s_1,s_2,s_3) := \sum_{u,v,w\ge 1} \frac{g(u,v,w)}{[u,v,w]}\frac{1}{u^{s_1}}\frac{1}{v^{s_2}}\frac{1}{w^{s_3}},$$

defined for $\Re(s_1),\Re(s_2),\Re(s_3) > 1$. As g is a multiplicative function (which can be seen using Lemma 2.1), the function F has a convergent Euler product over the prime numbers, for $\Re(s_1),\Re(s_2),\Re(s_3) > 1$, namely,

$$F(s_1,s_2,s_3) = \prod_p \left(\sum_{\nu_1,\nu_2,\nu_3\ge 0} \frac{g(p^{\nu_1},p^{\nu_2},p^{\nu_3})}{[p^{\nu_1},p^{\nu_2},p^{\nu_3}]}\frac{1}{p^{\nu_1 s_1+\nu_2 s_2+\nu_3 s_3}}\right).$$

If at least two of ν_1,ν_2,ν_3 are ≥ 1 and $g(p^{\nu_1},p^{\nu_2},p^{\nu_3}) = 1$, then $p \mid 2h$. So we split the Euler product into two sub-products, one for $p \mid 2h$ and the other for $p \nmid 2h$. We first consider the infinite product

$$\prod_{p\,\nmid\, 2h} \left(\sum_{\nu_1,\nu_2,\nu_3\ge 0} \frac{g(p^{\nu_1},p^{\nu_2},p^{\nu_3})}{[p^{\nu_1},p^{\nu_2},p^{\nu_3}]}\frac{1}{p^{\nu_1 s_1+\nu_2 s_2+\nu_3 s_3}}\right).$$

Hence in this product if at least two of ν_1,ν_2,ν_3 are ≥ 1, then $g(p^{\nu_1},p^{\nu_2},p^{\nu_3}) = 0$. Therefore, we need to consider the cases when at most one of them is positive and we will get non-zero contributions from the triples $(0,0,0),(\nu_1,0,0),(0,\nu_2,0),(0,0,\nu_3)$, where $\nu_i \ge 1$ in the respective cases. For $(\nu_1,0,0)$, the contribution is

$$\sum_{\nu_1\ge 1} \frac{1}{[p^{\nu_1},1,1]}\frac{1}{p^{\nu_1 s_1}} = \sum_{\nu_1\ge 1}\frac{1}{p^{(1+s_1)\nu_1}} = \frac{1}{p^{1+s_1}-1}.$$

Similarly, for $(0, \nu_2, 0)$ and $(0, 0, \nu_3)$, the contributions are

$$\sum_{\nu_2 \geq 1} \frac{1}{p^{(1+s_2)\nu_2}} = \frac{1}{p^{1+s_2}-1} \quad \text{and} \quad \sum_{\nu_3 \geq 1} \frac{1}{p^{(1+s_3)\nu_3}} = \frac{1}{p^{1+s_3}-1},$$

respectively. Therefore, for $p \nmid 2h$ the corresponding Euler product is

$$\prod_{p \nmid 2h} \left(1 + \frac{1}{p^{1+s_1}-1} + \frac{1}{p^{1+s_2}-1} + \frac{1}{p^{1+s_3}-1}\right).$$

We claim that this is same as $\zeta(1+s_1)\zeta(1+s_2)\zeta(1+s_3)$ up to an infinite product which is convergent on the domain

$$\left\{(s_1, s_2, s_3) \in \mathbb{C}^3 : \Re(s_i) > -\frac{1}{2} \text{ for all } i = 1, 2, 3\right\}.$$

To see this, we study the product

$$\prod_{p \nmid 2h} \left(1 + \frac{1}{p^{1+s_1}-1} + \frac{1}{p^{1+s_2}-1} + \frac{1}{p^{1+s_3}-1}\right)\left(1 - \frac{1}{p^{1+s_1}}\right)\left(1 - \frac{1}{p^{1+s_2}}\right)\left(1 - \frac{1}{p^{1+s_3}}\right).$$

For $X, Y, Z \neq 0, 1$, note that

$$\begin{aligned}&\left(1 + \frac{1}{X-1} + \frac{1}{Y-1} + \frac{1}{Z-1}\right)\left(1 - \frac{1}{X}\right)\left(1 - \frac{1}{Y}\right)\left(1 - \frac{1}{Z}\right)\\ &= \left(1 - \frac{1}{XY} - \frac{1}{YZ} - \frac{1}{ZX} + \frac{2}{XYZ}\right).\end{aligned}$$

Clearly, the infinite product

$$A_h(s_1, s_2, s_3) := \prod_{p \nmid 2h} \left(1 - \frac{1}{p^{s_1+s_2+2}} - \frac{1}{p^{s_2+s_3+2}} - \frac{1}{p^{s_3+s_1+2}} + \frac{2}{p^{s_1+s_2+s_3+3}}\right)$$

converges absolutely if $\Re(1+s_i) > 1/2$ for all $i = 1, 2, 3$. In this context, the Tauberian theorems of de la Bretèche, Theorem 2.1 and Theorem 2.2, allow us to derive the following proposition.

Proposition 4.1. *As $x \to \infty$, we have*

$$\sum_{u,v,w \leq x} \frac{g(u,v,w)}{[u,v,w]} = \Delta(h) \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p}\right) (\log x)^3 + O_h((\log x)^2),$$

where $\Delta(h)$ is a non-zero constant given by

$$\Delta(h) = \prod_{p|2h} \left(1 - \frac{1}{p}\right) \left(1 + \frac{2}{p}\right)^{-1} \left(\sum_{\nu_1,\nu_2,\nu_3 \ge 0} \frac{g(p^{\nu_1}, p^{\nu_2}, p^{\nu_3})}{[p^{\nu_1}, p^{\nu_2}, p^{\nu_3}]} \right). \tag{10}$$

Moreover, we have $\Delta(h) = 11f(h)/8$*, where* f *is the multiplicative function given by* (4)*.*

Assuming Proposition 4.1, we first complete our proof of Theorem 1.1.

Proof of Theorem 1.1 assuming Proposition 4.1. Note that

$$S_1(x;h) = \sum_{u,v,w \le x} g(u,v,w) \left\{ \frac{x}{[u,v,w]} + E \right\} \ge \sum_{u,v,w \le x^{1/3}} g(u,v,w) \left\{ \frac{x}{[u,v,w]} + O(1) \right\}.$$

Applying Proposition 4.1, we therefore have

$$S_1(x;h) \ge \sum_{u,v,w \le x^{1/3}} g(u,v,w) \frac{x}{[u,v,w]} + O(x) \gtrsim C_h x (\log x)^3/27 + O_h(x(\log x)^2),$$

where

$$C_h = \Delta(h) \prod_p \left(1 - \frac{1}{p}\right)^2 \left(1 + \frac{2}{p}\right).$$

Since $\Delta(h) = 11f(h)/8$, we get $C_h = c_h$ (as in (3)) and hence we are done. □

So we are left to prove Proposition 4.1.

4.2. Proof of Proposition 4.1

We need to apply both Theorem 2.1 and Theorem 2.2. Recall that

$$\begin{aligned} F(s_1,s_2,s_3) &= \prod_{p \nmid 2h} \left(\sum_{\nu_1,\nu_2,\nu_3 \ge 0} \frac{g(p^{\nu_1}, p^{\nu_2}, p^{\nu_3})}{[p^{\nu_1}, p^{\nu_2}, p^{\nu_3}]} \frac{1}{p^{\nu_1 s_1 + \nu_2 s_2 + \nu_3 s_3}} \right) \times \\ &\quad \prod_{p \mid 2h} \left(\sum_{\nu_1,\nu_2,\nu_3 \ge 0} \frac{g(p^{\nu_1}, p^{\nu_2}, p^{\nu_3})}{[p^{\nu_1}, p^{\nu_2}, p^{\nu_3}]} \frac{1}{p^{\nu_1 s_1 + \nu_2 s_2 + \nu_3 s_3}} \right) \\ &= A_h(s_1,s_2,s_3) B_h(s_1,s_2,s_3) \zeta(1+s_1)\zeta(1+s_2)\zeta(1+s_3) \times \\ &\quad \prod_{p \mid 2h} \left(\sum_{\nu_1,\nu_2,\nu_3 \ge 0} \frac{g(p^{\nu_1}, p^{\nu_2}, p^{\nu_3})}{[p^{\nu_1}, p^{\nu_2}, p^{\nu_3}]} \frac{1}{p^{\nu_1 s_1 + \nu_2 s_2 + \nu_3 s_3}} \right), \end{aligned}$$

where

$$B_h(s_1,s_2,s_3) := \prod_{p|2h}\left(1-\frac{1}{p^{1+s_1}}\right)\left(1-\frac{1}{p^{1+s_2}}\right)\left(1-\frac{1}{p^{1+s_3}}\right).$$

Note that from the expressions of $A_h(s_1,s_2,s_3)$ and $F(s_1,s_2,s_3)$, we clearly see that $F(s_1,s_2,s_3)$ converges if $\Re(s_i) > 0$ for all $i = 1,2,3$. So we have $\boldsymbol{\alpha} = (0,0,0)$. Moreover, choosing the family $\mathcal{L} = \left\{\ell^{(1)},\ell^{(2)},\ell^{(3)}\right\}$ of non-zero linear forms defined as $\ell^{(i)}(s_1,s_2,s_3) = s_i$ for $i = 1,2,3$, we get that the function $H(s_1,s_2,s_3)$, defined by

$$H(\boldsymbol{s}) := F(\boldsymbol{s}+\boldsymbol{\alpha})\prod_{i=1}^{3}\ell^{(i)}(\boldsymbol{s}) = F(s_1,s_2,s_3)s_1s_2s_3,$$

can be extended to the domain $\left\{(s_1,s_2,s_3)\in\mathbb{C}^3 : \Re(\ell^{(i)}(s)) > -1/2 \text{ for all } i = 1,2,3\right\}$. So we can choose $\delta_1 = 1/2$, to apply Theorem 2.1. Moreover, for the choice of δ_2, we note the estimate $|s\zeta(s+1)| \ll (1+|t|)^{1-\frac{\sigma}{2}+\epsilon}$ for $-1/2 < \sigma < 0$, where $s = \sigma + \iota t$, giving us $\delta_2 = 1/2$. This uses the standard convexity principle (see Titchmarsh [18, Chapter 5, eq. (5.1.4)]). Since $\boldsymbol{\alpha} = (0,0,0)$, $r = |J(\boldsymbol{\alpha})| = |\{j\in\{1,2,3\} : \alpha_j = 0\}| = 3$. We consider the linear forms $\{\ell^{(4)},\ell^{(5)},\ell^{(6)}\}$ given by $\ell^{(3+i)}(s_1,s_2,s_3) = e_i^*(s_1,s_2,s_3) = s_i$ for $i = 1,2,3$. Therefore, using Theorem 2.1, we conclude that

$$\sum_{u\le x^{\beta_1}}\sum_{v\le x^{\beta_2}}\sum_{w\le x^{\beta_3}}\frac{g(u,v,w)}{[u,v,w]} \sim x^{\langle\boldsymbol{\alpha},\boldsymbol{\beta}\rangle}Q_{\boldsymbol{\beta}}(\log x).$$

Now we will apply Theorem 2.2 for $\boldsymbol{\beta} = (\beta_1,\beta_2,\beta_3) = (1,1,1)$. The concerned linear form is $\mathcal{B}(s_1,s_2,s_3) := \left(\sum_{j=1}^{3}\beta_j e_j^*\right)(s_1,s_2,s_3) = s_1+s_2+s_3$. The polynomial $Q_{\boldsymbol{\beta}}$ satisfies the relation

$$Q_{\boldsymbol{\beta}}(\log x) = C_0 x^{-\langle\boldsymbol{\alpha},\boldsymbol{\beta}\rangle}I(x^{\boldsymbol{\beta}}) + O((\log x)^{\rho-1}),$$

where $C_0 := H(0,0,0)$, $\rho := n + r - \operatorname{rank}\left(\left\{\ell^{(i)}\right\}_{i=1}^{6}\right) = 3$ and

$$I(x^{\boldsymbol{\beta}}) := \iiint\limits_{A(x^{\boldsymbol{\beta}})}\frac{dy_1dy_2dy_3}{\prod_{i=1}^{3}y_i^{1-\ell^{(i)}(\boldsymbol{\alpha})}},$$

with

$$\begin{aligned}A(x^{\boldsymbol{\beta}}) &:= \left\{\boldsymbol{y}\in[1,\infty)^3 : \prod_{i=1}^{3}y_i^{\ell^{(i)}(e_j)} \le x^{\beta_j} \text{ for all } j\right\}\\ &= \{(y_1,y_2,y_3) : 1\le y_i\le x \text{ for all } i\}.\end{aligned}$$

Therefore,

$$I(x^{\boldsymbol{\beta}}) = \int_{y_1=1}^{x}\int_{y_2=1}^{x}\int_{y_3=1}^{x} \frac{dy_1 dy_2 dy_3}{y_1 y_2 y_3} = (\log x)^3.$$

Further,

$$\begin{aligned} H(0,0,0) &= \prod_{p \nmid 2h}\left(1-\frac{3}{p^2}+\frac{2}{p^3}\right)\prod_{p \mid 2h}\left(1-\frac{1}{p}\right)^3 \prod_{p \mid 2h}\left(\sum_{\nu_1,\nu_2,\nu_3\geq 0}\frac{g(p^{\nu_1},p^{\nu_2},p^{\nu_3})}{[p^{\nu_1},p^{\nu_2},p^{\nu_3}]}\right) \\ &= \prod_{p \nmid 2h}\left(1-\frac{1}{p}\right)^2\left(1+\frac{2}{p}\right)\prod_{p \mid 2h}\left(1-\frac{1}{p}\right)^3 \prod_{p \mid 2h}\left(\sum_{\nu_1,\nu_2,\nu_3\geq 0}\frac{g(p^{\nu_1},p^{\nu_2},p^{\nu_3})}{[p^{\nu_1},p^{\nu_2},p^{\nu_3}]}\right) \\ &= \prod_{p}\left(1-\frac{1}{p}\right)^2\left(1+\frac{2}{p}\right)\prod_{p \mid 2h}\left(1-\frac{1}{p}\right)\left(1+\frac{2}{p}\right)^{-1} \\ &\quad\times\left(\sum_{\nu_1,\nu_2,\nu_3\geq 0}\frac{g(p^{\nu_1},p^{\nu_2},p^{\nu_3})}{[p^{\nu_1},p^{\nu_2},p^{\nu_3}]}\right) \\ &= \Delta(h)\prod_{p}\left(1-\frac{1}{p}\right)^2\left(1+\frac{2}{p}\right). \end{aligned}$$

4.2.1. Verification of $\Delta(h) = 11f(h)/8$

We begin with an odd prime factor p of h. We have to evaluate the sum

$$s_p := \sum_{\nu_1,\nu_2,\nu_3\geq 0}\frac{g(p^{\nu_1},p^{\nu_2},p^{\nu_3})}{[p^{\nu_1},p^{\nu_2},p^{\nu_3}]}.$$

Recall that $g(p^{\nu_1},p^{\nu_2},p^{\nu_3}) = 1$ if and only if $(p^{\nu_1},p^{\nu_2}) \mid h$, $(p^{\nu_2},p^{\nu_3}) \mid h$ and $(p^{\nu_1},p^{\nu_3}) \mid 2h$. Let r be the highest power of p dividing h. Clearly, if $g(p^{\nu_1},p^{\nu_2},p^{\nu_3}) = 1$, then at most one of the ν_i can be $\geq r+1$. The contribution in each case to the sum s_p is

$$(r+1)^2\sum_{a\geq r+1}\frac{1}{p^a}.$$

Hence the total contribution here would be $3(r+1)^2\sum_{a\geq r+1}1/p^a$. Now we can assume $0 \leq \nu_i \leq r$ for all $i = 1,2,3$. If all these indices are equal, we get the contribution

$$\sum_{a=0}^{r}\frac{1}{p^a}.$$

If exactly two of the indices are equal, then the remaining index has 3 possibilities and that index can be bigger or smaller than the other two indices. In any case, we get the contribution

$$6\sum_{a=1}^{r}\sum_{b=0}^{a-1}\frac{1}{p^a} = 6\sum_{a=1}^{r}\frac{a}{p^a}.$$

Now if all three indices are different, we can have 6 possible ordering among them and in this case, we get the contribution

$$6\sum_{a=2}^{r}\sum_{b=1}^{a-1}\sum_{c=0}^{b-1}\frac{1}{p^a} = 6\sum_{a=2}^{r}\sum_{b=1}^{a-1}\frac{b}{p^a} = 3\sum_{a=2}^{r}\frac{a(a-1)}{p^a},$$

and hence

$$s_p = 3(r+1)^2\sum_{a\geq r+1}\frac{1}{p^a} + \sum_{a=0}^{r}\frac{1}{p^a} + 6\sum_{a=1}^{r}\frac{a}{p^a} + 3\sum_{a=2}^{r}\frac{a(a-1)}{p^a}.$$

It is easy to check that

$$(r+1)^2\sum_{a\geq r+1}\frac{1}{p^a} = \frac{(r+1)^2}{p^{r+1}}\left(1-\frac{1}{p}\right)^{-1} \text{ and } \sum_{a=0}^{r}\frac{1}{p^a} = \left(1-\frac{1}{p^{r+1}}\right)\left(1-\frac{1}{p}\right)^{-1}.$$

One can also check that

$$\sum_{a=1}^{r}\frac{a}{p^a} = \left(\frac{1}{p} - \frac{r+1}{p^{r+1}} + \frac{r}{p^{r+2}}\right)\left(1-\frac{1}{p}\right)^{-2},$$

and

$$\sum_{a=2}^{r}\frac{a(a-1)}{p^a} = \left(\frac{2}{p^2} - \frac{r^2+r}{p^{r+1}} + \frac{2r^2-2}{p^{r+2}} - \frac{r^2-r}{p^{r+3}}\right)\left(1-\frac{1}{p}\right)^{-3}.$$

So,

$$\begin{aligned}\left(1-\frac{1}{p}\right)^3 s_p = &\left(\frac{3r^2+6r+2}{p^{r+1}}+1\right)\left(1-\frac{2}{p}+\frac{1}{p^2}\right) + 6\left(\frac{1}{p}-\frac{r+1}{p^{r+1}}+\frac{r}{p^{r+2}}\right)\left(1-\frac{1}{p}\right)\\ &+3\left(\frac{2}{p^2}-\frac{r^2+r}{p^{r+1}}+\frac{2r^2-2}{p^{r+2}}-\frac{r^2-r}{p^{r+3}}\right)\\ =&1+\frac{4}{p}+\frac{1}{p^2}-\frac{3r+4}{p^{r+1}}-\frac{4}{p^{r+2}}+\frac{3r+2}{p^{r+3}}.\end{aligned}$$

This completes the verification of the factors corresponding to the odd prime divisors of h, using (4). For the prime $p = 2$, we show that if r is the highest power of 2 dividing h, then the corresponding factor in $\Delta(h)$ is

$$\frac{13}{2} - \frac{41+15r}{2^{r+3}} = \frac{11}{8}\left(\frac{52}{11} - \frac{41+15r}{2^r\times 11}\right).$$

This will complete the proof.

We evaluate the sum

$$s_2 := \sum_{\nu_1,\nu_2,\nu_3\geq 0} \frac{g(2^{\nu_1},2^{\nu_2},2^{\nu_3})}{[2^{\nu_1},2^{\nu_2},2^{\nu_3}]}.$$

If $0 \leq \nu_i \leq r$ for all $i = 1,2,3$, then as before the contribution to s_2 in this case is

$$\begin{aligned}
&\sum_{a=0}^{r} \frac{1}{2^a} + 6\sum_{a=1}^{r} \frac{a}{2^a} + 3\sum_{a=2}^{r} \frac{a(a-1)}{2^a} \\
&= 2\left(1 - \frac{1}{2^{r+1}}\right) + 24\left(\frac{1}{2} - \frac{r+1}{2^{r+1}} + \frac{r}{2^{r+2}}\right) + 24\left(\frac{1}{2} - \frac{r^2+r}{2^{r+1}} + \frac{2r^2-2}{2^{r+2}} - \frac{r^2-r}{2^{r+3}}\right) \\
&= 26 - \frac{25}{2^r} - \frac{15r}{2^r} - \frac{3r^2}{2^r}.
\end{aligned}$$

Now we assume that $\nu_i \geq r+1$ for at least one of $i = 1,2,3$. Here the counting is different. We note that if $g(2^{\nu_1},2^{\nu_2},2^{\nu_3}) = 1$, then we have the following conditions: at most one of $\nu_1,\nu_2 > r$, at most one of $\nu_2,\nu_3 > r$ and at most one of $\nu_1,\nu_3 > r+1$. We then have the following four possible choices:

i) if $\nu_1 = r+1$, then $\nu_2 < r+1$ and $\nu_3 \geq 0$;
ii) if $\nu_1 > r+1$, then $\nu_2 < r+1$ and $\nu_3 \leq r+1$;
iii) if $\nu_1 < r+1$ and $\nu_2 \geq r+1$, then $\nu_3 < r+1$;
iv) if $\nu_1 < r+1$ and $\nu_2 < r+1$, then $\nu_3 \geq r+1$.

The contribution of case i) to s_2 is

$$\frac{(r+1)^2}{2^{r+1}} + (r+1)\sum_{\nu_3\geq r+1} \frac{1}{2^{\nu_3}} = \frac{(r+1)^2}{2^{r+1}} + \frac{(r+1)}{2^r}.$$

The contribution of case ii) to s_2 is

$$(r+1)(r+2)\sum_{\nu_1>r+1} \frac{1}{2^{\nu_1}} = \frac{(r+1)(r+2)}{2^{r+1}} = \frac{(r+1)^2}{2^{r+1}} + \frac{(r+1)}{2^{r+1}}.$$

The contribution of case iii) to s_2 is

$$(r+1)^2\sum_{\nu_2\geq r+1} \frac{1}{2^{\nu_2}} = \frac{(r+1)^2}{2^r}.$$

The contribution of case iv) to s_2 is

$$(r+1)^2 \sum_{\nu_3 \geq r+1} \frac{1}{2^{\nu_3}} = \frac{(r+1)^2}{2^r}.$$

Hence the total contribution of these above four cases is

$$\frac{3(r+1)^2}{2^r} + \frac{3(r+1)}{2^{r+1}},$$

and thus

$$s_2 = 26 - \frac{41+15r}{2^{r+1}} = 4\left(\frac{13}{2} - \frac{41+15r}{2^{r+3}}\right).$$

This completes the proof.

5. Proof of Theorem 1.2

In 1952, Erdős proved that if $f(x)$ is any irreducible polynomial with integer coefficients, then there are positive constants c_1 and c_2 such that

$$c_1 x \log x \leq \sum_{n \leq x} d(f(n)) \leq c_2 x \log x,$$

for $x \geq 2$. The result is false if $f(x)$ is not irreducible as can be seen from Ingham's theorem (1). However, the argument of Erdős can be suitably adapted to show that in our case, as $x \to \infty$,

$$S(x;h) \ll_h x(\log x)^3.$$

Though it is difficult to identify them in Erdős's paper [6], his method has three steps and it proceeds as follows:

We consider $f(x) = x(x-h)(x+h)$. By Lemma 2.5, we first note that as $x \to \infty$,

$$S(x;h) \ll_h \sum_{n \leq x} d(f(n)).$$

Hence our goal is to bound the latter sum applying the method of Erdős. Suppose that $f(n)$ has r many prime factors, counted with multiplicity. We write

$$|f(n)| = (p_1 p_2 \cdots p_j)(p_{j+1} \cdots p_r), \tag{11}$$

with $p_1 \leq p_2 \leq \cdots \leq p_j \leq p_{j+1} \leq \cdots \leq p_r$ and the index j is the largest such that

$$m := p_1 p_2 \cdots p_j \leq x.$$

The strategy is that the bulk of the divisors of $f(n)$ are actually coming from the factor $m = (p_1 p_2 \cdots p_j)$. Indeed, by the sub-multiplicativity of the divisor function, we have

$$d(f(n)) \le 2^{r-j} d(p_1 \cdots p_j).$$

(1) **Step 1:** If n is such that $r-j$ is bounded by 11 (say), then the contribution to the sum $S(x;h)$ from such n is bounded by

$$\ll \sum_{m \le x} \sum_{\substack{a | f(m) \\ a \le x}} 1 = \sum_{a \le x} \sum_{\substack{a | f(m) \\ m \le x}} 1.$$

By the Chinese remainder theorem as in Lemma 2.3, the inner sum is easily seen to be

$$\ll \frac{3^{\omega(a)} x}{a}.$$

Summing this over $a \le x$ and using Lemma 2.4, we get the desired bound of $O(x \log^3 x)$ coming from these n's.

(2) **Step 2:** Suppose now that n is such that for $f(n)$, $r-j$ is greater than 11. Choosing x large enough, we get that $p_{j+1} < x^{4/11}$, for otherwise $|f(n)| > x^4$, a contradiction, since in our case f has degree 3 and $f(n) = O(n^3)$. By the definition of j, we have

$$(p_1 \cdots p_j) p_{j+1} > x$$

so that

$$x^{7/11} < p_1 \cdots p_j < x. \tag{12}$$

Now, $p_j \le p_{j+1} < x^{4/11}$ and so, there is a positive integer t such that

$$x^{1/(t+1)} \le p_j \le x^{1/t}.$$

Step 2a. We first suppose that $x^{1/t} > (\log x)^2$. Note that $p_1 \cdots p_j$ is $x^{1/t}$-smooth. The plan is to estimate the contribution from each possible value of t and then sum over the t. Thus, fixing t, we see that

$$p_{j+1} \cdots p_r > x^{(r-j)/(t+1)}.$$

For our cubic polynomial, we must have $r - j \le 3(t+1)$. Thus, for such n under consideration, we use the elementary inequality

$$d(n) \le 2 \sum_{\substack{a | n \\ a \ge \sqrt{n}}} 1, \tag{13}$$

to get

$$d(f(n)) < 2^{3t+3} d(p_1 \cdots p_j) \le 2^{3t+4} \sum_{\substack{d \mid p_1 \cdots p_j \\ x \ge d \ge \sqrt{p_1 \cdots p_j}}} 1,$$

with $p_1 \cdots p_j$ being $x^{1/t}$-smooth and the divisor $d > x^{7/22}$ by virtue of (12). By Lemma 2.3, we have that the total contribution from such numbers is bounded above by

$$\sum_{t \ge 1} 2^{3t+4} \sum_{\substack{d \in \mathcal{S}(x, x^{1/t}) \\ d > x^{7/22}}} \sum_{\substack{m \le x \\ d \mid f(m)}} 1 \ll \sum_{t \ge 1} 2^{3t} \sum_{\substack{d \in \mathcal{S}(x, x^{1/t}) \\ d > x^{7/22}}} \frac{x 3^{\omega(d)}}{d}.$$

As $x \to \infty$, the innermost sum is

$$\ll x (\log x)^3 \exp\left(-ct \log t\right),$$

for some suitable positive constant c, by Lemma 2.6. Since

$$\sum_{t \ge 1} 2^{3t} \exp\left(-ct \log t\right)$$

converges, we are done in this case.
Step 2b. It remains to consider the case $x^{1/t} \le (\log x)^2$. Using the familiar estimate $d(n) \ll n^{\epsilon}$, for any $\epsilon > 0$, we first write

$$d(f(n)) \ll n^{\epsilon} d(p_1 \cdots p_j) \ll x^{\epsilon} \sum_{\substack{d \mid p_1 \cdots p_j \\ x \ge d \ge \sqrt{p_1 \cdots p_j}}} 1,$$

with $p_1 \cdots p_j$ being $(\log x)^2$-smooth and the divisor $d > x^{7/22}$. Now summing for all these n's (indicated by a prime for the n's under consideration), we get

$$\sum_{n \le x}{}' d(f(n)) \ll x^{\epsilon} \sum_{\substack{d \in \mathcal{S}(x, (\log x)^2) \\ d > x^{7/22}}} \frac{x 3^{\omega(d)}}{d} \ll x^{1 + 2\epsilon - 7/44},$$

by Lemma 2.3 and Lemma 2.6. This is negligible for $0 < \epsilon < 7/88$ and hence completes the proof. □

We remark that in the above computations, the implicit constants can be made explicit.

6. A more general triple convolution sum

For fixed positive integers h, k, we can also consider the triple convolution sum

$$T(x;h,k) := \sum_{n\le x} d(n)d(n+h)d(n-k).$$

As before, we can write

$$\begin{aligned}T(x;h,k) &= \sum_{n\le x} d(n)d(n+h)d(n-k) = \sum_{n\le x}\left(\sum_{u|n+h} 1\right)\left(\sum_{v|n} 1\right)\left(\sum_{w|n-k} 1\right)\\ &= \sum_{\substack{u\le x+h\\ v\le x\\ w\le x-k}} \sum_{n\le x}{}' 1,\end{aligned}$$

where the primed sum denotes the sum for $n \le x$ where $n+h \equiv 0 \bmod u, n \equiv 0 \bmod v$ and $n-k \equiv 0 \bmod w$. Note that the outer sum is over all the integer triples (u,v,w) in the set $[1, x+h] \times [1, x] \times [1, x-k]$. Again we split the sum to write $T(x;h,k) = T_1(x;h,k) + T_2(x;h,k) - T_3(x;h,k)$, where

$$T_1(x;h,k) := \sum_{u,v,w\le x} \sum_{n\le x}{}' 1,\ T_2(x;h,k) := \sum_{\substack{x<u\le x+h\\ v,w\le x}} \sum_{n\le x}{}' 1 \ \text{ and}$$

$$T_3(x;h,k) := \sum_{\substack{u\le x+h, v\le x\\ x-k<w\le x}} \sum_{n\le x}{}' 1.$$

As earlier, the main contribution comes from $T_1(x;h,k)$, which can be studied using the multiple Dirichlet series

$$\sum_{u,v,w\ge 1} \frac{\tilde{g}(u,v,w)}{[u,v,w]} \frac{1}{u^{s_1}} \frac{1}{v^{s_2}} \frac{1}{w^{s_3}},$$

where $\tilde{g}(u,v,w)$ is a multiplicative function which takes the value 1 if and only if the system $n \equiv -h \bmod u, n \equiv 0 \bmod v, n \equiv k \bmod w$ has a solution, else it is 0. Again we write this series as a product and split that product into two parts, one for the primes dividing the least common multiple $[h, k, (h+k)]$ and the other one for the primes not dividing $[h, k, (h+k)]$. Applying Theorem 2.1 and Theorem 2.2, we can then derive the following proposition.

Proposition 6.1. *As $x \to \infty$, we have*

$$\sum_{u,v,w\le x} \frac{\tilde{g}(u,v,w)}{[u,v,w]} \sim \Delta(h,k) \prod_p \left(1-\frac{1}{p}\right)^2 \left(1+\frac{2}{p}\right) (\log x)^3,$$

where $\Delta(h,k)$ *is a non-zero constant given by*

$$\Delta(h,k) = \prod_{p\mid[h,k,(h+k)]} \left(1-\frac{1}{p}\right)\left(1+\frac{2}{p}\right)^{-1} \left(\sum_{\nu_1,\nu_2,\nu_3\geq 0} \frac{\tilde{g}(p^{\nu_1},p^{\nu_2},p^{\nu_3})}{[p^{\nu_1},p^{\nu_2},p^{\nu_3}]} \right). \tag{14}$$

As before, the lower bound and upper bound of $T(x;h,k)$ follows from Wolke's theorems [20]. We can also derive an explicit lower bound for $T(x;h,k)$ along the lines of the proof of Theorem 1.1.

7. Concluding remarks

Our methods have further applications. For example, we can investigate triple and higher convolution sums of the generalised divisor functions $d_k(n)$, which is the number of ways of writing n as a product of k factors ($k \geq 2$). This is to appear in the forthcoming article [12]. A further variation arises if instead of summing over $n \leq x$, we sum over primes $p \leq x$. A more involved sum

$$\sum_{n\leq x} d(f(n)),$$

where $f(t)$ is a polynomial with integer coefficients having a specified number of irreducible factors can also be dealt using these methods.

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Data availability

No data was used for the research described in the article.

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