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Simultaneous non-vanishing and sign changes of Fourier coefficients of modular forms

Moni Kumari

School of Mathematical Sciences
National Institute of Science Education and Research
HBNI, Bhubaneswar, Via-Jatni, Khurda, Odisha 752050, India
moni.kumari@niser.ac.in

M. Ram Murty

Department of Mathematics and Statistics
Queen's University, Kingston, Ontario K7L 3N6, Canada
murty@queensu.ca

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In this paper, we give some results on simultaneous non-vanishing and simultaneous sign-changes for the Fourier coefficients of two modular forms. More precisely, given two modular forms f and g with Fourier coefficients a_n and b_n respectively, we consider the following questions: existence of infinitely many primes p such that $a_p b_p \neq 0$; simultaneous non-vanishing in the short intervals and in arithmetic progressions; simultaneous sign changes in short intervals.

Keywords: Modular forms; simultaneous sign changes; Fourier coefficients of cusp forms.

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1. Introduction

The vanishing or non-vanishing of an arithmetically defined analytic function is a recurring motif in mathematics. In recent times, such questions have arisen in the context of modular forms both of integral weight and half-integral weight. In this paper, we will study simultaneous non-vanishing of Fourier coefficients of distinct modular forms of integral weight.

Throughout, let k, N be positive integers and p be a prime. We write $S_k(\Gamma_0(N))$ for the space of cusp forms of weight k for the group $\Gamma_0(N)$. Let $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(N))$ and $g(z) = \sum_{n=1}^{\infty} b_n q^n \in S_k(\Gamma_0(N))$ be two nonzero cusp forms which are not a linear combination of CM forms. One of the goals of this paper is to study simultaneous non-vanishing of a_n, b_n partially inspired by a long-standing conjecture of Lehmer which predicts that $\tau(n) \neq 0,$ for all $n > 0$, where

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q := e^{2i\pi z}$$

is the unique normalized cusp form of weight 12 on $SL_2(\mathbb{Z})$. In relation to Lehmer's conjecture, Serre in his paper [21] motivated the general study of estimating the size of possible gaps in the Fourier expansion of modular forms via the gap function

$$i_f(n) := \min\{j \geq 0 : a_{n+j} \neq 0\}.$$

He proved that $i_f(n) \ll_f n$, where $f(z)$ is a cusp form which is not a linear combination of CM forms. In the same paper [21], he posed the question of whether one can prove an estimate of the form

$$i_f(n) \ll_f n^{\delta},$$

where $\delta < 1$. In his paper [14], Murty first pointed out that $i_f(n) \ll n^{3/5}$ follows immediately from the celebrated work of Rankin [19] and Selberg [20] done in 1939/40. After that many authors improved the value of δ (for more detail see [4]).

In the case of level 1 and f an eigenform, Das and Ganguly [4] discovered a clever argument to show $i_f(n) \ll n^{1/4}$ by combining a classical result of Bambah and Chowla [1] with a congruence of Hatada [6] along with a basic lemma of Murty and Murty [15]. Here is a synopsis of their elegant proof. In 1947, Bambah and Chowla showed using an elementary argument that in any interval of length $x^{1/4}$ there is a number n (say) which can be written as a sum of two squares. As f is an eigenform, a_n is multiplicative. Hatada's theorem [6] implies that $a_p \equiv 2 \pmod 4$, for $p \equiv 1 \pmod 4$ and $a_{p^r} \equiv 1 \pmod 4$ if r is even and $p \equiv 3 \pmod 4$. The lemma in [15] shows that $a_{p^r} \neq 0$ for $p \equiv 1 \pmod 4$ provided p is sufficiently large. These congruences combined with the classical theorem about factorization of natural numbers that can be written as a sum of two integral squares now imply $a_n \neq 0$ provided n is coprime to a given finite set of primes. Thus, one now needs the Bambah–Chowla theorem with n coprime to a finite set of primes. One can tweak the argument in [1] to accommodate this extra condition and thus deduce the non-vanishing result as done in [4]. Actually, the argument of Bambah and Chowla can be generalized with considerable latitude. We prove the following which is of independent interest.

Theorem 1.1. *Let r and s be natural numbers and set $\alpha = (r-1)(s-1)/rs$. There is an effectively computable C (depending only on r and s) such that in any interval of the form $[n, n + Cn^{\alpha}]$, there is a number m which can be written as*

$$m = A^r + B^s,$$

with A and B integers.

We hasten to highlight that the argument of Das and Ganguly allows for simultaneous non-vanishing. In fact, if $f_1, \ldots, f_r$ are normalized eigenforms of level 1, with corresponding Fourier coefficients $a_n(f_j)$, then one can find an i with $i \ll n^{1/4}$ such that

$$a_{n+i}(f_j) \neq 0, \quad \forall\, 1 \leq j \leq r.$$

It has been suggested that perhaps $i_f(n) \ll n^{\epsilon}$ for any $\epsilon > 0$. Perhaps even the stronger conjecture $i_f(n) \ll 1$ is true (see for example, [14]).

It would be nice to extend these results to higher levels but as the authors in [4] remark, one needs to extend Hatada's result, which probably can be done, but would take us in a direction orthogonal to the methods of this paper. We expect to return to this question at a later time.

In this paper, we introduce the analogous concept of gap function $i_{f,g}$ for simultaneous non-vanishing and then we derive a bound for $i_{f,g}$ as small as possible, based on current knowledge. One can, of course, consider more general gap functions for several modular forms.

Theorem 1.2. *Let* $f(z) = \sum_{n=1}^{\infty} a_n n^{\frac{k-1}{2}} q^n \in S_k(\Gamma_0(N))$ *and* $g(z) = \sum_{n=1}^{\infty} b_n n^{\frac{k-1}{2}} q^n \in S_k(\Gamma_0(N))$ *be two newforms which are not CM forms, then there exist infinitely many primes p such that*

$$a_p b_p \neq 0.$$

Actually, as we show below, the theorem is true without the constraint that the forms are not CM. It should be remembered that the recent solution of the Sato–Tate conjecture for two distinct eigenforms (see [16] and the references therein), the theorem is immediate. This is because there is a positive density of primes p such that both a_p and b_p are simultaneously nonzero since the joint Sato–Tate distribution holds for two eigenforms f and g. But this is invoking a "sledgehammer" result and we underscore that our methods do not make use of this major advance. This comment amplifies that there are other ways of approaching such questions.

Now, for $n \in \mathbb{N}$ define

$$i_{f,g}(n) := \min\{m \geq 0 : a_{n+m} b_{n+m} \neq 0\},$$

which is well-defined from the above theorem. We are interested to find the growth of the function $i_{f,g}(n)$ as $n \to \infty$. In 2014, Lu [10] by using the result of Chandrasekharan and Narasimhan [2] proved the following.

$$\sum_{n \leq x} a_n^2 b_n^2 = cx + O(x^{\frac{7}{8}+\varepsilon}),$$

where c is a nonzero constant. It then follows that

$$i_{f,g}(n) \ll n^{\frac{7}{8}+\varepsilon}.$$

In this paper, we give a better estimate than above.

Theorem 1.3. *Suppose that* $f(z) = \sum_{n=1}^{\infty} a_n n^{\frac{k-1}{2}} q^n \in S_k(\Gamma_0(N))$ *and* $g(z) = \sum_{n=1}^{\infty} b_n n^{\frac{k-1}{2}} q^n \in S_k(\Gamma_0(N))$ *are two newforms with* $k > 2$ *which are not a linear* *combination of CM forms. Then the following results hold.*

(i) *For every* $\varepsilon > 0$, $x > x_0(f, g, \varepsilon)$ *and* $x^{\frac{7}{17}+\varepsilon} \le y$ *we have*

$$|\{x < n < x + y : a_n b_n \neq 0\}| \gg_{f,g,\varepsilon} y. \tag{1}$$

In particular, we get that $i_{f,g}(n) \ll_{f,g,\varepsilon} n^{\frac{7}{17}+\varepsilon}$.

(ii) *For every* $\varepsilon > 0$, $x \ge x_0(f, g, \varepsilon)$, $y \ge x^{\frac{17}{38}+100\varepsilon}$ *and* $1 \le a \le q \le x^{\varepsilon}$ *with* $(a, q) = 1$, *we have*

$$|\{x < n \le x + y : n \equiv a \pmod q \text{ and } a_n b_n \neq 0\}| \gg_{f,g,\varepsilon} y/q. \tag{2}$$

In 2009, Kohnen and Sengupta [7] considered a problem related with the simultaneous sign changes. They proved that, given two normalized cusp forms f and g of the same level and different weights with totally real algebraic Fourier coefficients, there exists a Galois automorphism σ such that f^{σ} and g^{σ} have infinitely many Fourier coefficients of the opposite sign. Recently Gun, Kohnen and Rath [5] removed the dependency on the Galois conjugacy. In fact, they extended their result to arbitrary cusp forms with arbitrary real Fourier coefficients but they assumed that both f and g should have first Fourier coefficient to be nonzero. More precisely, they proved the following.

Theorem 1.4. *Let*

$$f(z) = \sum_{n=1}^{\infty} a_n q^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n q^n$$

be nonzero cusp forms of level N *and weights* $1 < k_1 < k_2$, *respectively. Suppose that* a_n, b_n *are real numbers. If* $a_1 b_1 \neq 0$, *then there exist infinitely many* n *such that* $a_n b_n > 0$ *and infinitely many* n *such that* $a_n b_n < 0$.

In this paper, we extend the above result by removing the assumption $a_1 b_1 \neq 0$. We prove the following result.

Theorem 1.5. *Let*

$$f(z) = \sum_{n \ge 1} a_n q^n \quad \text{and} \quad g(z) = \sum_{n \ge 1} b_n q^n$$

be nonzero cusp forms of level N *and weights* $1 < k_1 < k_2$, *respectively. Further, let* a_n, b_n *be real numbers. Then there exist infinitely many* n *such that* $a_n b_n > 0$ *and infinitely many* n *such that* $a_n b_n < 0$.

If f and g are newforms then we have the following quantitative result for the simultaneous sign changes.

Theorem 1.6. *Let* $k \ge 2$ *be an integer. Assume that*

$$f(z) = \sum_{n \ge 1} a_n n^{\frac{k-1}{2}} q^n \quad \text{and} \quad g(z) = \sum_{n \ge 1} b_n n^{\frac{k-1}{2}} q^n$$

are two distinct newforms of weight k on $\Gamma_0(N)$. Further, let a_n, b_n be real numbers, then for any $\delta > \frac{7}{8}$, the sequence $\{a_n b_n\}_{n\in\mathbb{N}}$ has at least one sign change for $n \in (x, x + x^\delta]$ for sufficiently large x. In particular, the number of sign changes for $n \leq x$ is $\gg x^{1-\delta}$.

2. Preliminaries

In this section, we collect various results from the literature that will be needed in our proofs. In 1982, Serre [21, Corollary 2, p. 174] in his very famous paper, proved the following result.

Lemma 2.1. *Let $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(N))$ be a newform with weight $k \geq 2$ which does not have complex multiplication. For every $\epsilon > 0$ we have*

$$|\{p \leq x : a_p = 0\}| \ll_{f,\epsilon} \frac{x}{(\log x)^{\frac{3}{2}-\epsilon}}.$$

To prove Theorem 1.3, we shall use the concept of $\mathcal{B}$-free numbers which was introduced by Erdös in 1966 and later many authors studied the distribution of $\mathcal{B}$-free numbers.

Definition 2.2 ($\mathcal{B}$-free numbers). Let $\mathcal{B} = \{b_i : 1 < b_1 < b_2 < \cdots\}$ be a sequence of mutually coprime positive integers for which $\sum_{i=1}^{\infty} \frac{1}{b_i} < \infty$. A positive integer n is called $\mathcal{B}$-free if it is not divisible by any element in $\mathcal{B}$.

By using sieve theory and estimates for multiple exponential sums, Chen and Wu [3], studied the distribution of $\mathcal{B}$-free numbers in short intervals as well as in an arithmetic progression and they proved the following results.

Proposition 2.3. *Let $\mathcal{B}$ be a sequence of positive integers satisfying the conditions in the definition of $\mathcal{B}$-free numbers. Then,*

(i) *for any $\varepsilon > 0$, $x > x_o(\mathcal{B}, \varepsilon)$ and $y \geq x^{\frac{7}{17}+\varepsilon}$, we have*

$$|\{x < n \leq x + y : n \text{ is } \mathcal{B}\text{-free}\}| \gg_{\mathcal{B},\varepsilon} y, \tag{3}$$

(ii) *for any $\varepsilon > 0$, $x > x_o(\mathcal{B}, \varepsilon)$ and $y \geq x^{\frac{17}{38}+100\varepsilon}$, $1 \leq a \leq q \leq x^\varepsilon$ with $((a, q), b) = 1$, for all $b \in \mathcal{B}$, we have*

$$|\{x < n \leq x + y : n \equiv a \pmod{q} \text{ and } n \text{ is } \mathcal{B}\text{-free}\}| \gg_{\mathcal{B},\varepsilon} y/q. \tag{4}$$

Here the implied constants depend only on $\mathcal{B}$ and ε.

In the proof of Theorem 1.5 we use the following theorem of Pribitkin [17].

Theorem 2.4. *Let $F(s) = \sum_{n=1}^{\infty} a_n e^{-s\lambda_n}$ be a non-trivial general Dirichlet series which converges somewhere, where the sequence $\{a_n\}_{n=1}^{\infty}$ is complex, and the exponent sequence $\{\lambda_n\}_{n=1}^{\infty}$ is real and strictly increasing to ∞. If the function F is holomorphic on the whole real line and has infinitely many real zeros, then there exist infinitely many $n \in \mathbb{N}$ such that $a_n > 0$ and there exist infinitely many $n \in \mathbb{N}$ such that $a_n < 0$.*

3. Proof of Theorem 1.1

We essentially follow Bambah and Chowla [1] and modify their argument to our setting. Let $t = [n^{1/s}] = n^{1/s} - \theta$ with $0 \leq \theta < 1$. Let x_1, x_2 be positive real numbers such that

$$x_1^r + t^s = n,$$

$$x_2^r + t^s = n + Cn^\alpha$$

with C to be chosen later. Thus, $x_2^r - x_1^r = Cn^\alpha$. Now,

$$x_1 = (n - t^s)^{1/r} \ll n^{(s-1)/rs}, \quad x_2 \ll n^{(s-1)/rs},$$

by a simple application of the binomial theorem. Hence,

$$x_2^{r-1} + x_2^{r-2}x_1 + \cdots + x_1^{r-1} \ll n^{(s-1)(r-1)/rs} = n^\alpha.$$

Now writing

$$(x_2 - x_1)(x_2^{r-1} + \cdots + x_1^{r-1}) = x_2^r - x_1^r = Cn^\alpha,$$

we immediately see that

$$x_2 - x_1 > 1,$$

for a suitable choice of C. (In fact, $C = 2^{rs}rs$ will work.) Therefore, there is a natural number N in the interval $[x_1, x_2]$ so that

$$n = x_1^r + t^s < N^r + t^s < x_2^r + t^s = n + Cn^\alpha,$$

as desired. This completes the proof of Theorem 1.1.

We remark that there are several variations of this theorem that can be derived from this proof. For example, if $f(x)$ is a monotonic, continuous function for x sufficiently large, and $f(x) \asymp x^r$, then there is a natural number m such that $m = f(A) + B^s$ for some natural numbers A, B and with $m \in [n, n + Cn^\alpha]$. In particular, this can be applied to the norm form $a^2 + Db^2$, with D squarefree. We record these remarks with the view that the result may have potential applications in other contexts.

4. Proof of Theorem 1.2

From Lemma 2.1, we have

$$|\{p \leq x : a_p = 0\}| \ll_{f,\epsilon} \frac{x}{(\log x)^{\frac{3}{2}-\epsilon}}$$

and

$$|\{p \leq x : b_p = 0\}| \ll_{g,\epsilon} \frac{x}{(\log x)^{\frac{3}{2}-\epsilon}}.$$

Since $a_p b_p = 0$, we have either $a_p = 0$ or $b_p = 0$. Hence

$$|\{p \leq x : a_p b_p = 0\}| \ll_{f,g,\epsilon} \frac{x}{(\log x)^{\frac{3}{2}-\epsilon}}.$$

By the prime number theorem, we have

$$\pi(x) := |\{p \leq x\}| \sim \frac{x}{\log x}.$$

Hence

$$|\{p \leq x : a_p b_p \neq 0\}| = \pi(x) - |\{p \leq x : a_p b_p = 0\}| \sim \frac{x}{\log x}.$$

Thus there exist infinitely many primes p such that

$$a_p b_p \neq 0.$$

We make some remarks in the case that either f or g is of CM type. Suppose first that f has CM by an order in an imaginary quadratic field K and g does not. Then, for primes p coprime to the level of f, $a_p = 0$ if and only if p is inert in K. The density of such primes is $1/2$ and so

$$|\{p \leq x : a_p b_p \neq 0\}| = \pi(x) - |\{p \leq x : a_p b_p = 0\}| \gtrsim \frac{x}{2\log x}.$$

Hence, in this case also, there are infinitely many primes p such that $a_p b_p \neq 0$. If both f and g have CM by two imaginary quadratic fields K_1, K_2 (say, respectively), then we need only choose primes p coprime to the level which split in K_1 and K_2. This density is either $1/2$ (if $K_1 = K_2$) or $1/4$ (if $K_1 \neq K_2$). Thus, in all cases, Theorem 1.2 is valid in general.

5. Proof of Theorem 1.3

Let $S = \{p : a_p b_p = 0\} \cup \{p|N\}$. Put $\mathcal{B} = S \cup \{p^2 : p \notin S\}$. Clearly $\mathcal{B}$ is a sequence of mutually coprime integers and if n is $\mathcal{B}$-free, then n is square-free and $a_n b_n \neq 0$ by using the multiplicative properties of a_n and b_n. Thus (3) and (4) imply the first and second assertions of Theorem 1.2 respectively, if we can show that $\sum_{p\in\mathcal{B}} \frac{1}{p} < \infty$. Since $\sum_p 1/p^2 < \infty$, it suffices to show that

$$\sum_{p\in S} \frac{1}{p} < \infty.$$

We know, from Lemma 2.1 that

$$\sum_{\substack{p\leq x\\ p\in S}} 1 \ll_{f,g} \frac{x}{(\log x)^{1+\delta}}, \quad \text{for some } \delta > 0.$$

Hence, by partial summation formula, we have

$$\sum_{\substack{p\leq x\\ p\in S}} \frac{1}{p} = \frac{1}{x}\sum_{\substack{p\leq x\\ p\in S}} 1 + \int_2^x \frac{1}{t^2}\left(\sum_{\substack{p\leq t\\ p\in S}} 1\right) dt \ll_{f,g} \frac{1}{(\log x)^{1+\delta}} + \int_2^x \frac{dt}{t(\log t)^{1+\delta}} \ll_{f,g} 1.$$

This completes the proof of Theorem 1.3.

6. Proof of Theorem 1.5

We assume that either $a_1 = 0$ or $b_1 = 0$, since otherwise by using Theorem 1.4, we get the result. We will show that there exists infinitely many $n \in \mathbb{N}$ such that $a_n b_n < 0$ the other case being similar. Suppose not, then there exist $n_0 \in \mathbb{N}$ such that

$$a_n b_n \geq 0,$$

for all $n > n_0$. Set $M = \prod_{p \leq n_0} p$. Clearly, $a_n b_n \geq 0$ whenever $(n, M) = 1$ by our assumption. Let

$$f_1(z) := \sum_{\substack{n \geq 1 \\ (n,M)=1}} a_n q^n \quad \text{and} \quad g_1(z) := \sum_{\substack{n \geq 1 \\ (n,M)=1}} b_n q^n.$$

Then f_1 and g_1 are cusp forms of level NM^2 and weights k_1 and k_2 respectively. For $s \in \mathbb{C}$ with Re(s) $\gg 1$, the Rankin–Selberg L-function attached to f_1 and g_1 is defined by

$$R_{f_1,g_1}(s) := \sum_{\substack{n \geq 1 \\ (n,M)=1}} \frac{a_n b_n}{n^s}.$$

For Re(s) $\gg 1$, set

$$L_{f_1,g_1}(s) := \prod_{p \mid NM^2} (1 - p^{-(2s-(k_1+k_2)+2)})\zeta(2s - (k_1 + k_2) + 2) R_{f_1,g_1}(s) \tag{5}$$

$$:= \sum_{n=1}^{\infty} \frac{c_n}{n^s}. \tag{6}$$

Li [9] proved that $(2\pi)^{-2s}\Gamma(s)\Gamma(s - k_1 + 1)L_{f_1,g_1}(s)$ is entire and we also know that $\Gamma(s)\Gamma(s-k_1+1)$ does not have any zeros. Hence $L_{f_1,g_1}(s)$ is an entire function on $\mathbb{C}$. Let us observe that the coefficients of this Dirichlet series are non-negative because the term

$$\prod_{p \mid NM^2} (1 - p^{-(2s-(k_1+k_2)+2)})\zeta(2s - (k_1 + k_2) + 2)$$

is the Riemann zeta function with the Euler factors at primes $p \mid NM^2$ removed and so is a Dirichlet series with non-negative coefficients. Hence $\sum_{n=1}^{\infty} \frac{c_n}{n^s}$ is entire with $c_n \geq 0$ for all n. Now $\sum_{n=1}^{\infty} \frac{c_n}{n^s}$ has infinitely many real zeros coming from the real simple poles of the Γ-function. Then by Theorem 2.4, there exist infinitely many n such that $c_n > 0$ and there exist infinitely many n such that $c_n < 0$ which is a contradiction because $c_n \geq 0$ for all $n \in \mathbb{N}$. This completes our proof.

7. Proof of Theorem 1.6

Recently Meher and the second author [11], gave a general criteria for the sign changes of any sequence of real numbers $\{a_n\}_{n \in \mathbb{N}}$. More precisely, they proved the following.

Theorem 7.1. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that*

(i) $a_n = O(n^\alpha)$,
(ii) $\sum_{n\le x} a_n = O(n^\beta)$,
(iii) $\sum_{n\le x} a_n^2 = cx + O(n^\gamma)$,

with $\alpha, \beta, \gamma, c \ge 0$. If $\alpha+\beta<1$, then for any δ satisfying

$$\max\{\alpha+\beta, \gamma\} < \delta < 1,$$

the sequence $\{a_n\}_{n\in\mathbb{N}}$ has at least one sign change for $n \in [x, x+x^\delta]$. Consequently, the number of sign changes of a_n for $n \le x$ is $\gg x^{1-\delta}$ for sufficiently large x.

We will prove Theorem 1.6, as an application of the above theorem, for which we have to analyze the stated conditions for the sequence $\{a_n b_n\}_{n\in\mathbb{N}}$.

(i) Ramanujan–Deligne:

$$a_n b_n = O(n^\varepsilon) \quad \text{for all } \varepsilon > 0. \tag{7}$$

(ii) From the paper of Lu [10], one can deduce the following results

$$\sum_{n\le x} a_n b_n \ll x^{\frac{3}{5}} (\log x)^{-\frac{2\theta}{3}}, \tag{8}$$

where $\theta = 0.1512\ldots$

(iii)

$$\sum_{n\le x} a_n^2 b_n^2 = cx + O(x^{\frac{7}{8}+\varepsilon}).$$

Hence from Theorem 7.1, we immediately deduce Theorem 1.6.

8. Concluding Remarks

As mentioned earlier, it would be interesting to extend Hatada's congruence to modular forms of higher level. This is a research problem of independent interest and is accessible since there have been significant advances in the theory of congruences of modular forms. If one assumes standard conjectures about distribution of primes such as Cramér's conjecture, then it is easy to deduce that $i_f(n) = O(\log^2 n)$. The other problem that suggests itself is to obtain estimates with their dependence on level and weight made explicit. An initiation into such an enterprise can be found in the methods of [12, 13].

The analogues of these questions for modular forms of half-integral weight takes us into a parallel universe of ideas. There is, of course, a link between these two worlds provided by Waldspurger's theorem and the question is equivalent to the simultaneous non-vanishing of quadratic twists of L-series attached to modular forms. A modest beginning in this line of research was initiated in [8].

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