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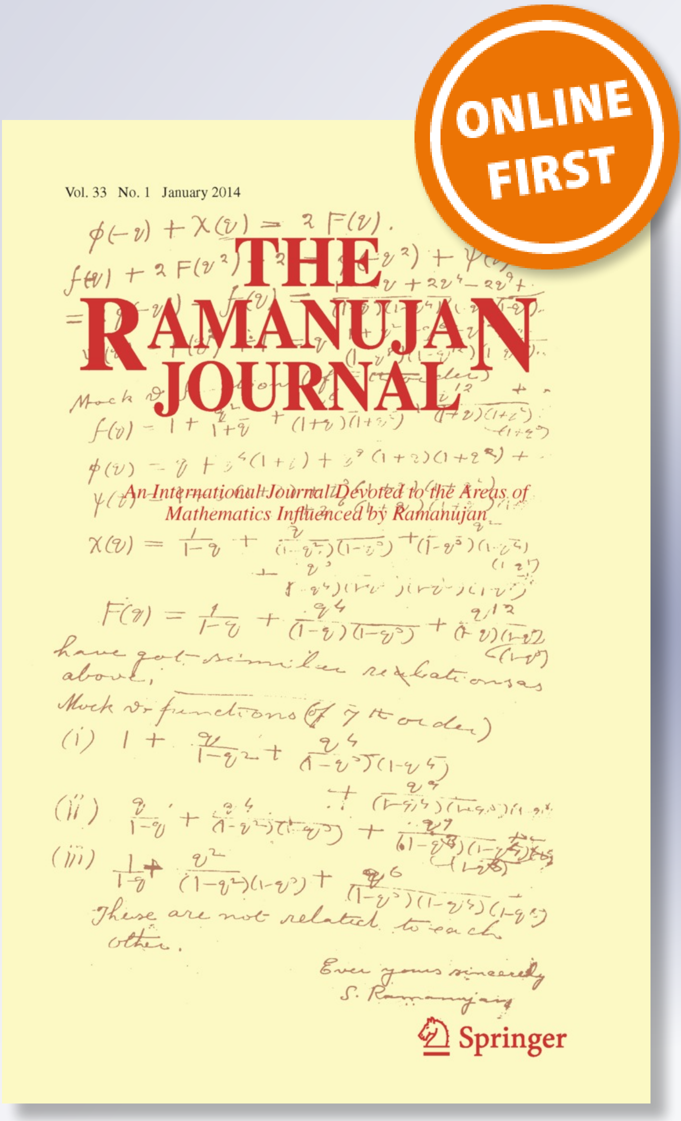


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# Special values of the Gamma function at CM points

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**Abstract** Little is known about the transcendence of certain values of the Gamma function, $\Gamma(z)$. In this article, we study values of $\Gamma(z)$ when $\mathbb{Q}(z)$ is an imaginary quadratic field. We also study special values of the digamma function, $\psi(z)$, and the polygamma functions, $\psi_t(z)$. As part of our analysis we will see that certain infinite products

$$\prod_{n=1}^{\infty} \frac{A(n)}{n^t}$$

can be evaluated explicitly and are transcendental for $A(z) \in \overline{\mathbb{Q}}[z]$ with degree $t$ and roots from an imaginary quadratic field. Special cases of these products were studied by Ramanujan. Additionally, we explore the implications that some conjectures of Gel'fond and Schneider have on these values and products.

**Keywords** Transcendence · Gamma function · Digamma function · Polygamma functions · Infinite products · Gel'fond–Schneider conjecture

**Mathematics Subject Classification** Primary 11J91 · Secondary 11J81

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## 1 Introduction

The Gamma function defined for $\Re(z) > 0$ as

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\, dt$$

is well-studied, though much is unknown about the transcendence of $\Gamma(z)$. It is known that $\Gamma(1/2) = \sqrt{\pi}$ is transcendental since $\pi$ is transcendental. In 1984, Chudnovsky [2] showed that $\Gamma(1/3), \Gamma(2/3), \Gamma(1/4), \Gamma(3/4), \Gamma(1/6)$, and $\Gamma(5/6)$ are transcendental, along with any integer translate of these fractions, by showing that firstly, $\pi$ and $\Gamma(1/3)$ are algebraically independent, and secondly, $\pi$ and $\Gamma(1/4)$ are algebraically independent. One gets the transcendence of $\Gamma(1/6)$ by relating it to $\Gamma(1/3)$ via the duplication formula

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

For $z = 1/6$, one can show that $\Gamma(1/6) = 2^{-1/3}\sqrt{3/\pi}\,\Gamma^2(1/3)$. The remaining values $\Gamma(2/3)$, $\Gamma(3/4)$, and $\Gamma(5/6)$ are seen to be transcendental by the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

It is unknown whether or not $\Gamma(1/5)$ is algebraic or transcendental, though it is conjectured to be transcendental. In 1941, Schneider [10] proved that the beta function

$$B(a,b) := \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is transcendental whenever $a, b, a+b \in \mathbb{Q}\setminus\mathbb{Z}$. By choosing $a = b$, we see that Schneider's theorem implies that $\Gamma(a)$ or $\Gamma(2a)$ is transcendental for any $a, 2a \in \mathbb{Q}\setminus\mathbb{Z}$. In particular, $\Gamma(1/5)$ or $\Gamma(2/5)$ is transcendental.

More recently, Nesterenko [7] proved the following result which will be useful for our purposes.

**Theorem** (Nesterenko) *For any imaginary quadratic field with discriminant $-D$ and character $\varepsilon$, the numbers*

$$\pi, \quad e^{\pi\sqrt{D}}, \quad \prod_{a=1}^{D-1} \Gamma(a/D)^{\varepsilon(a)}$$

*are algebraically independent.*

Here the character $\varepsilon(n) = (\frac{-D}{n})$ is the Kronecker–Jacobi symbol. This shows that in general, $\pi$ and $e^{\pi\sqrt{D}}$ are algebraically independent. Additionally, if we have $D = 3$, then we obtain the algebraic independence of $\pi, e^{\pi\sqrt{3}}, \Gamma(1/3)$, and if $D = 4$, we see that $\pi, e^{\pi}, \Gamma(1/4)$ are algebraically independent.

In this article, we explore properties of the Gamma function at certain CM points (that is, $z \in \mathbb{C}$ such that $\mathbb{Q}(z)$ is an imaginary quadratic field). We show that all values $\Gamma(\alpha)$ and $\Gamma(\alpha + 1/2)$ are transcendental for $\alpha \notin \mathbb{Q}$ of the form $k + ia\sqrt{D}/q \in \mathbb{Q}(\sqrt{-D})$ for integers $k, a, q$. In particular, we will evaluate $|\Gamma(ia/q)|^2$ explicitly for $a, q \in \mathbb{Z}$ and $(a, q) = 1$. Applying Nesterenko's theorem allows us to deduce that $\Gamma(ia/q)$ is always transcendental.

The digamma function $\psi(z)$ is the logarithmic derivative of $\Gamma(z)$ while the polygamma functions $\psi_t(z)$ are defined as the $t$th derivatives of $\psi(z)$ with $\psi_0(z) = \psi(z)$. Transcendence of these values at rational values has been studied by Bundschuh [1], the first author with Saradha [4, 5], the second author [13], among others. Here we focus on values of $\psi_t(z)$ at irrational values.

As a result of the analysis of $\Gamma(z)$, we also obtain a method for evaluating certain infinite products

$$\prod_{n=1}^{\infty} \frac{A(n)}{n^t} \tag{1}$$

for $A(z) \in \overline{\mathbb{Q}}[z]$ monic with degree $t$. We are able to relate these products to values of $\Gamma(z)$ and derive results from there by specifying the roots of $A(z)$.

We point out that some of these results on infinite products of rational functions were known to Ramanujan [8] 100 years ago. At the time, many values which appeared in his calculations were not known to be transcendental. Some of the results in [8] are now implied by a more general method shown here with the addition of showing transcendence of these products. Indeed, in [8], we find some elegant formulas of the following kind:

$$\prod_{n=1}^{\infty}\left(1 + \left(\frac{2\alpha}{\alpha+n}\right)^3\right) = \frac{\Gamma(1+\alpha)^3 \sinh(\pi\alpha\sqrt{3})}{\Gamma(1+3\alpha)\pi\alpha\sqrt{3}}$$

and

$$\prod_{n=1}^{\infty}\left(1 + \left(\frac{2\alpha+1}{\alpha+n}\right)^3\right) = \frac{\Gamma(1+\alpha)^3 \cosh(\pi(\frac{1}{2}+\alpha)\sqrt{3})}{\Gamma(2+3\alpha)\pi}.$$

If we put $\alpha = 0$ in the second formula, we deduce the strikingly beautiful evaluation

$$\prod_{n=1}^{\infty}\left(1 + \frac{1}{n^3}\right) = \frac{\cosh(\pi\sqrt{3}/2)}{\pi}, \tag{2}$$

which is transcendental by Nesterenko's theorem. One would expect these products to be transcendental for every rational $\alpha$, but this deduction can only be made (at present) for a limited number of rational values of $\alpha$ like $\alpha = 1/3, 1/4$ or $1/6$.

Analogous products appear in Ramanujan's notebooks (see in particular Chaps. 13 and 14) and one finds formulas like (see [9])

$$\prod_{n=1}^{\infty}\left(1 + \left(\frac{x}{\alpha+n}\right)^2\right) = \frac{|\Gamma(\alpha)|^2}{|\Gamma(\alpha+ix)|^2}.$$

These formulas will be special cases of our results.

In the final section, we explore implications of conjectures of Schneider and Gel'fond on the algebraic independence of certain algebraic powers of algebraic numbers. In some cases, we are able to prove conditional results on the transcendence of values of $\Gamma(z)$, $\psi_t(z)$ as well as infinite products (1).

## 2 Transcendental values of $\Gamma(z)$ and infinite products

The Gamma function can be written as an infinite product

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}, \tag{3}$$

where $\gamma$ is Euler's constant. The sine function has infinite product

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right). \tag{4}$$

Putting (3) and (4) together gives the reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}. \tag{5}$$

Also recall the functional equation

$$\Gamma(z+1) = z\Gamma(z), \tag{6}$$

which can be extended for a positive integer $n$ to

$$\Gamma(z+n) = (z)_n \Gamma(z) \quad \text{and} \quad \Gamma(z-n) = \frac{\Gamma(z)}{(z-n)_n}, \tag{7}$$

where we use the rising factorial or Pochhammer symbol

$$(z)_n := z(z+1)\cdots(z+n-1).$$

From these elementary observations, we are able characterize certain special values of the Gamma function.

**Theorem 1** *Let $\alpha = k + \frac{ai\sqrt{D}}{q} \in (\mathbb{Z} + \mathbb{Q}\sqrt{-D}) \setminus \mathbb{Z}$ for a positive integer $D$. For $k > 0$,*

$$\left|\Gamma(\alpha)\right|^2 = \frac{\pi(1-\alpha)_{2k-1}}{\sin(\pi\alpha)} \quad \textit{and} \quad \left|\Gamma\left(\alpha + \frac{1}{2}\right)\right|^2 = \frac{\pi(\frac{1}{2}-\alpha)_{2k}}{\cos(\pi\alpha)},$$

*while for $k \leq 0$,*

$$\left|\Gamma(\alpha)\right|^2 = \frac{\pi}{\sin(\pi\alpha)(\overline{\alpha})_{2|k|+1}} \quad \textit{and} \quad \left|\Gamma\left(\alpha + \frac{1}{2}\right)\right|^2 = \frac{\pi}{\cos(\pi\alpha)(\frac{1}{2}+\overline{\alpha})_{2|k|}}.$$

*All of $|\Gamma(\alpha)|^2$, $|\Gamma(\alpha+\frac{1}{2})|^2$, $\Gamma(\alpha)$, and $\Gamma(\alpha+\frac{1}{2})$ are transcendental.*

*Proof* Note that it is easily seen that $\Gamma(\overline{z}) = \overline{\Gamma(z)}$ for any $z \in \mathbb{C}$. Additionally, we *shift* values as in Eq. (7) so that for any non-integral algebraic number $z$, $\Gamma(\pm n + z)$ is a nonzero algebraic multiple of $\Gamma(z)$. Using these two observations as well as the reflection formula (5), for $k$ positive we have that

$$\left|\Gamma(\alpha)\right|^2 = \Gamma(\alpha)\Gamma(\overline{\alpha}) = (1-\alpha)_{2k-1}\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi(1-\alpha)_{2k-1}}{\sin(\pi\alpha)}$$

and

$$\begin{aligned}\left|\Gamma\left(\alpha+\frac{1}{2}\right)\right|^2 &= \Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\overline{\alpha}+\frac{1}{2}\right)\\ &= \left(\frac{1}{2}-\alpha\right)_{2k}\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(1-\left(\alpha+\frac{1}{2}\right)\right) = \frac{\pi(\frac{1}{2}-\alpha)_{2k}}{\cos(\pi\alpha)}.\end{aligned}$$

If $k=-m$ is non-positive, then

$$\left|\Gamma(\alpha)\right|^2 = \Gamma(\alpha)\Gamma(\overline{\alpha}) = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{(\overline{\alpha})_{2m+1}} = \frac{\pi}{\sin(\pi\alpha)(\overline{\alpha})_{2m+1}}$$

and

$$\begin{aligned}\left|\Gamma\left(\alpha+\frac{1}{2}\right)\right|^2 &= \Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\overline{\alpha}+\frac{1}{2}\right) = \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(1-(\alpha+\frac{1}{2}))}{(\frac{1}{2}+\overline{\alpha})_{2m}}\\ &= \frac{\pi}{\cos(\pi\alpha)(\frac{1}{2}+\overline{\alpha})_{2m}}.\end{aligned}$$

Writing

$$\cos(\pi\alpha) = \frac{e^{\pi ik}e^{-\pi a\sqrt{D}/q} + e^{-\pi ik}e^{\pi a\sqrt{D}/q}}{2}$$

and

$$\sin(\pi\alpha) = \frac{e^{\pi ik}e^{-\pi a\sqrt{D}/q} - e^{-\pi ik}e^{\pi a\sqrt{D}/q}}{2i},$$

we see that $|\Gamma(\alpha)|^2$ and $|\Gamma(\alpha+1/2)|^2$ are in $\pi\overline{\mathbb{Q}}(e^{\pi\sqrt{D}/q})\setminus\{0\}$ and are therefore transcendental by Nesterenko's Theorem. Thus, $\Gamma(\alpha)$ and $\Gamma(\alpha+1/2)$ are transcendental as well. □

Theorem 1 shows that many values of $\Gamma(z)$ are transcendental, including $\Gamma(ia/q)$ and $\Gamma(ia\sqrt{D}/q)$ and $\Gamma(\alpha)$ when $\alpha$ is an algebraic integer of an imaginary quadratic field. Note that for $k \in \mathbb{Z}$, the argument above can be used to prove that $\Gamma(k+1/2)$ is transcendental. We unfortunately do not get a fully characterizing result for any $\alpha$ from an imaginary quadratic field, however, from the reflection formula (5) we see that for any $\alpha \in \mathbb{Q}(\sqrt{-D})\setminus\mathbb{Q}$ at least one of $\Gamma(\alpha)$ and $\Gamma(-\alpha)$ is transcendental since the product is transcendental by Nesterenko's Theorem.

The simple observations made above can be used to analyze infinite products of rational functions (1). In some cases, we are able to relate the product to values of the Gamma function. In particular, this can be done for any *depressed* polynomial $A(z) \in \mathbb{C}[z]$ with any roots which are not natural numbers. By a *depressed* polynomial

$$a_n z^n + \cdots + a_1 z + a_0$$

we mean a polynomial in which $a_{n-1} = 0$. This is equivalent to the sum of the roots of the polynomial being 0. In order to see the relation between these products and values of $\Gamma$, we list the roots $\alpha_1, \ldots, \alpha_t \in \mathbb{C} \setminus \mathbb{N}$ and write the product

$$\prod_{n=1}^{\infty} \frac{A(n)}{n^t} = \prod_{n=1}^{\infty} \prod_{j=1}^{t} \left(1 - \frac{\alpha_j}{n}\right).$$

For depressed $A(z)$, since $\sum_{j=1}^{t} \alpha_j = 0$, we can insert exponentials into the product yielding

$$e^{\gamma(-\alpha_1 - \cdots - \alpha_t)} \prod_{n=1}^{\infty} \prod_{j=1}^{t} \left(1 - \frac{\alpha_j}{n}\right) e^{\alpha_j/n}.$$

Each product $\prod_{n=1}^{\infty}(1 - \alpha_j/n)e^{\alpha_j/n}$ converges, so we can interchange the order of multiplication

$$e^{\gamma(-\alpha_1 - \cdots - \alpha_t)} \prod_{j=1}^{t} \prod_{n=1}^{\infty} \left(1 - \frac{\alpha_j}{n}\right) e^{\alpha_j/n}, \tag{8}$$

and (3) implies that

$$\prod_{n=1}^{\infty} \frac{A(n)}{n^t} = \frac{(-1)^t (\alpha_1 \cdots \alpha_t)^{-1}}{\Gamma(-\alpha_1) \cdots \Gamma(-\alpha_t)}. \tag{9}$$

When $A(z)$ is not depressed, $\prod_{n=1}^{\infty} A(n)/n^t$ diverges, however, when the real part of the sum of the roots of $A(z)$ is zero, then the product $A(z)\overline{A(z)}$ is depressed with degree $2t$. By (9), we have

$$\prod_{n=1}^{\infty} \frac{A(n)\overline{A(n)}}{n^{2t}} = \frac{|\alpha_1|^{-2} \cdots |\alpha_t|^{-2}}{|\Gamma(-\alpha_1)|^2 \cdots |\Gamma(-\alpha_t)|^2}. \tag{10}$$

The left side of (10) arises naturally as the product of the convergent products

$$P = \prod_{n=1}^{\infty} \frac{A(n)}{n^t} e^{s/n} \quad \text{and} \quad \overline{P} = \prod_{n=1}^{\infty} \frac{\overline{A(n)}}{n^t} e^{-s/n},$$

where, and from now on, we write $s = \alpha_1 + \cdots + \alpha_t$ to be the sum of the roots of $A(z)$. We use the relationship (10) to study the case when $A(z)$ has roots similar to the values $\alpha$ and $\alpha + 1/2$ which were studied in Theorem 1. To simplify the notation we define the set

$$\mathcal{S}_1 := (\mathbb{Z} + \mathbb{Q}\sqrt{-D}) \cup \left(\mathbb{Z} + \frac{1}{2} + \mathbb{Q}\sqrt{-D}\right).$$

**Theorem 2** *Let $A(z) \in \overline{\mathbb{Q}}[z]$ be monic with roots $\alpha_1, \ldots, \alpha_t$ satisfying $\Re(s) = \Re(\sum_{j=1}^{t} \alpha_j) = 0$. If $\alpha_1, \ldots, \alpha_t \in \mathcal{S}_1 \setminus \mathbb{N}$ with at least one not an integer, then*

$$|P|^2 = \prod_{n=1}^{\infty} \frac{A(n)\overline{A(n)}}{n^{2t}} \quad \text{and therefore} \quad P = \prod_{n=1}^{\infty} \frac{A(n)}{n^t} e^{s/n}$$

*are transcendental. If $D = 3$ or $4$ with $\alpha_1, \ldots, \alpha_t \in \mathcal{S}_1 \cup (\mathbb{Z} \pm \frac{1}{D}) \setminus \mathbb{N}$ with at least one not an integer, then $|P|^2$ and therefore $P$ are transcendental.*

*Proof* We keep track of contributions to the finite product in (10). For any negative integer $\alpha_j = -m$, $\Gamma(-\alpha_j) = (m-1)!$. For each non-integer root $\alpha_j \in \mathcal{S}_1$, Theorem 1 implies that $|\Gamma(-\alpha_j)|^2$ contributes $\pi$ multiplied by some nonzero number from $\overline{\mathbb{Q}}(e^{\pi\sqrt{D}/q_j})$ for some positive integer $q_j$. By Nesterenko's Theorem, $|P|^2$ is transcendental and therefore $P$ is transcendental as well, which proves the first assertion.

Now suppose that $D = 3$ or $D = 4$. For any $\alpha_j \in \mathbb{Z} \pm 1/D$, by (7) $\Gamma(-\alpha_j) = \xi_j \Gamma(\frac{1}{D})$ or $\Gamma(-\alpha_j) = \xi_j \Gamma(\frac{D-1}{D})$ for some algebraic number $\xi_j$. Note that by (5)

$$\Gamma\left(\frac{D-1}{D}\right) = \frac{\pi}{\sin(\pi/D)\Gamma(\frac{1}{D})}$$

so that in either case, $|\Gamma(-\alpha_j)|^2$ contributes an algebraic number multiplied by either $\Gamma^2(1/D)$ or $\pi^2/\Gamma^2(1/D)$ which are both algebraically independent from $\pi$ and $e^{\pi\sqrt{D}}$ by Nesterenko's Theorem. With each type of contributing value of the Gamma function, the product (10) will be non-zero of the form

$$|P|^2 = \frac{T}{\pi^{s_1}\Gamma^{s_2}(1/D)}$$

for some integers $s_1 \geq 0$, $s_2 \in \mathbb{Z}$, with $(s_1, s_2) \neq (0, 0)$ and some $T \in \overline{\mathbb{Q}}(e^{\pi\sqrt{D}/q})$. By Nesterenko's Theorem, $|P|^2$ and therefore $P$ are transcendental. □

For the final assertion we restrict $D$ to be 3 or 4, but in fact, the explicit value of the product works for a general value of $D$. Without having an algebraic independence result as we do in the cases $D = 3$ or 4, we cannot conclude transcendence of the product as stated in Theorem 2. However, if we restrict the polynomial $A(z)$ to have roots which are either negative integers or of the form $\alpha_j = k_j + 1/6$, then we can conclude transcendence since $\Gamma(1/6)$ is transcendental. Similarly, if all roots are negative integers or of the form $\alpha_j = k + 5/6$, then the product is transcendental since $\Gamma(5/6)$ is transcendental.

In some cases, we can remove the condition that the polynomial $A(z)$ is depressed, or partially depressed ($\Re(s) = 0$), and also allow for any complex roots from an imaginary quadratic field. Doing so prevents us from relating the product (1) to values of the Gamma function, however, by examining a particular product we can show that (at least) half of the following products are transcendental.

**Theorem 3** *Let $A(z) \in \overline{\mathbb{Q}}[z]$ be monic with roots $\alpha_1, \ldots, \alpha_t \in \mathbb{Q}(\sqrt{-D}) \setminus \mathbb{Z}$ and write $s = \sum_{j=1}^{t} \alpha_j$. The product*

$$\prod_{n=1}^{\infty} \frac{A(n)A(-n)}{n^{2t}} = \frac{\prod_{j=1}^{t} \sin(\pi\alpha_j)}{(-\pi)^t A(0)}$$

*is transcendental and therefore at least one of*

$$\prod_{n=1}^{\infty} \frac{A(n)}{n^t} e^{s/n} \quad or \quad \prod_{n=1}^{\infty} \frac{A(-n)}{n^t} e^{-s/n}$$

*is transcendental.*

*Proof* The product $A(z)A(-z)$ is depressed, so by collecting factors in the natural way, we have

$$\prod_{n=1}^{\infty} \left(\frac{A(n)}{n^t}\right)\left(\frac{A(-n)}{n^t}\right) = \prod_{j=1}^{t} \prod_{n=1}^{\infty} \left(1 - \frac{\alpha_j}{n}\right)\left(1 + \frac{\alpha_j}{n}\right).$$

By Eq. (4), the product equals

$$\frac{\prod_{j=1}^{t} \sin(\pi\alpha_j)}{\pi^t \alpha_1 \cdots \alpha_t} = \frac{\prod_{j=1}^{t} \sin(\pi\alpha_j)}{(-\pi)^t A(0)}$$

since for any monic polynomial $P(z)$ of degree $t$, the product of all of its roots is $(-1)^t P(0)$. Similar to the conclusion of Theorem 1, this final expression can be rewritten as $\pi^{-t}$ times a nonzero number from $\overline{\mathbb{Q}}(e^{\pi\sqrt{D}/q})$ for some positive integer $q$. By Nesterenko's Theorem, this product is transcendental and the final assertion follows immediately. □

A natural corollary to Theorem 3 comes from studying the case that $A(-n) = A(n)$. When $A(z)$ is an even polynomial we have $s = 0$ and we obtain transcendence of the product $\prod A(n)/n^t$ immediately by examining $(\prod A(n)/n^t)^2$ as in Theorem 3. We state the result and leave the details to the reader.

**Corollary 4** *Let $A(z) \in \overline{\mathbb{Q}}[z]$ be monic and even with roots $\pm\alpha_1, \ldots, \pm\alpha_t \in \mathbb{Q}(\sqrt{-D}) \setminus \mathbb{Z}$. The product*

$$\prod_{n=1}^{\infty} \frac{A(n)}{n^{2t}} = \pi^{-t} \prod_{j=1}^{t} \frac{\sin(\pi\alpha_j)}{\alpha_j}$$

*is transcendental.*

## 3 Transcendental values of $\psi_t(z)$

Methods which were developed in the previous section can be applied to gain insight into values of the digamma and polygamma functions. We recall some facts about the digamma function which follow from logarithmically differentiating various identities involving the Gamma function. From (3) we see that

$$-\psi(z) = \gamma + \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n+z} - \frac{1}{n}\right), \tag{11}$$

which shows that, similar to the Gamma function, we have an identity for complex conjugation of values of $\psi(z)$. Namely, $\psi(\overline{z}) = \overline{\psi(z)}$. From (5) we have a reflection formula

$$\psi(z) - \psi(1-z) = -\pi\cot(\pi z) \tag{12}$$

and from (6) we obtain the functional equation

$$\psi(z+1) = \psi(z) + \frac{1}{z}, \tag{13}$$

which is extended for a positive integer $n$ to

$$\psi(z+n) = \psi(z) + \sum_{j=0}^{n-1}\frac{1}{z+j} \quad \text{and} \quad \psi(z-n) = \psi(z) + \sum_{j=1}^{n}\frac{1}{j-z}. \tag{14}$$

Similarly, for $t \geq 1$ we have

$$\psi_t(z) = (-1)^{t+1}t!\sum_{n=0}^{\infty}\frac{1}{(n+z)^{t+1}}, \tag{15}$$

which shows $\psi_t(\overline{z}) = \overline{\psi_t(z)}$. Additionally, differentiating each of (12) and (13) $t$ times gives

$$\psi_t(z) + (-1)^{t+1}\psi_t(1-z) = -\frac{d^t}{dz^t}\big(\pi\cot(\pi z)\big) \tag{16}$$

and

$$\psi_t(z+1) = \psi_t(z) + \frac{(-1)^t t!}{z^{t+1}}, \tag{17}$$

respectively, while the latter extends for positive integers $n$ to

$$\psi_t(z+n) = \psi_t(z) + (-1)^t t!\sum_{j=0}^{n-1}\frac{1}{(z+j)^{t+1}} \tag{18a}$$

and

$$\psi_t(z-n) = \psi_t(z) + t!\sum_{j=1}^{n}\frac{1}{(j-z)^{t+1}}. \tag{18b}$$

Using these identities, we are able to partially characterize values of $\psi$ and $\psi_t$ at certain numbers from an imaginary quadratic field. In the case of $\Gamma(z)$ in the previous section, we were able to use identities involving the product $\Gamma(\alpha)\Gamma(\overline{\alpha})$ in order to characterize the transcendence of values at imaginary quadratic numbers. Here, we have additive identities for $\psi$ and $\psi_t$ and we isolate either the real or imaginary parts of these numbers.

**Theorem 5** *Let $\alpha = k + \frac{ai\sqrt{D}}{q} \in (\mathbb{Z} + \mathbb{Q}\sqrt{-D}) \setminus \mathbb{Z}$ for a positive square-free integer $D$. For $k > 0$,*

$$\Im\big(\psi(\alpha)\big) = -\frac{\pi\cot(\pi\alpha)}{2i} + \sum_{j=1}^{2k-1} \frac{i/2}{j-\alpha},$$

$$\Im\left(\psi\left(\alpha + \frac{1}{2}\right)\right) = \frac{\pi\tan(\pi\alpha)}{2i} + \sum_{j=0}^{2k-1} \frac{i/2}{j-\alpha+\frac{1}{2}},$$

*while for $k = -m \le 0$,*

$$\Im\big(\psi(\alpha)\big) = -\frac{\pi\cot(\pi\alpha)}{2i} + \sum_{j=0}^{2m} \frac{i/2}{j+\alpha},$$

$$\Im\left(\psi\left(\alpha + \frac{1}{2}\right)\right) = \frac{\pi\tan(\pi\alpha)}{2i} + \sum_{j=1}^{2m} \frac{i/2}{j+\alpha-\frac{1}{2}}.$$

*Both $\Im(\psi(\alpha))$ and $\Im(\psi(\alpha + \frac{1}{2}))$ are transcendental.*

*Proof* For $k \ge 1$ we have

$$2i\Im\big(\psi(\alpha)\big) = \psi(\alpha) - \psi(\overline{\alpha}) = \psi(\alpha) - \psi(1-\alpha+2k-1),$$

which, after shifting via Eq. (14), is equal to

$$\psi(\alpha) - \psi(1-\alpha) - \sum_{j=0}^{2k-2} \frac{1}{1-\alpha+j} = -\pi\cot(\pi\alpha) - \sum_{j=0}^{2k-2} \frac{1}{1-\alpha+j}.$$

Similarly,

$$\begin{aligned} 2i\Im\left(\psi\left(\alpha+\frac{1}{2}\right)\right) &= \psi\left(\alpha+\frac{1}{2}\right) - \psi\left(1-\left(\alpha+\frac{1}{2}\right)+2k\right) \\ &= \pi\tan(\pi\alpha) - \sum_{j=0}^{2k-1} \frac{1}{\frac{1}{2}-\alpha+j}. \end{aligned}$$

Similarly, for $k = -m \leq 0$ we have

$$2i\Im\big(\psi(\alpha)\big) = -\pi \cot(\pi\alpha) - \sum_{j=1}^{2m+1} \frac{1}{j+\alpha-1}$$

and

$$2i\Im\left(\psi\left(\alpha+\frac{1}{2}\right)\right) = \pi \tan(\pi\alpha) - \sum_{j=1}^{2m} \frac{1}{j+\alpha-\frac{1}{2}}.$$

In all cases, transcendence is asserted by Nesterenko's Theorem. $\square$

Analogously, following a similar shifting technique employed in the proof of Theorem 5, we obtain explicit values for (part of) the polygamma functions at certain CM points.

**Theorem 6** *Let* $\alpha = k + \frac{ai\sqrt{D}}{q} \in (\mathbb{Z} + \mathbb{Q}\sqrt{-D}) \setminus \mathbb{Z}$ *for a positive square-free integer* $D$*. For* $k > 0$,

$$\psi_t(\alpha) + (-1)^{t+1}\psi_t(\overline{\alpha}) = -\frac{d^t}{dz^t}\big(\pi\cot(\pi z)\big)\Big|_{z=\alpha} - t!\sum_{j=1}^{2k-1}\frac{1}{(j-\alpha)^{t+1}},$$

$$\psi_t\left(\alpha+\frac{1}{2}\right) + (-1)^{t+1}\psi_t\left(\overline{\alpha}+\frac{1}{2}\right)$$
$$= \frac{d^t}{dz^t}\big(\pi\tan(\pi z)\big)\Big|_{z=\alpha} - t!\sum_{j=0}^{2k-1}\frac{1}{(j+\frac{1}{2}-\alpha)^{t+1}},$$

*while for* $k = -m \leq 0$,

$$\psi_t(\alpha) + (-1)^{t+1}\psi(\overline{\alpha}) = -\frac{d^t}{dz^t}\big(\pi\cot(\pi z)\big)\Big|_{z=\alpha} + (-1)^{t+1}t!\sum_{j=0}^{2m}\frac{1}{(j+\alpha)^{t+1}},$$

$$\psi_t\left(\alpha+\frac{1}{2}\right) + (-1)^{t+1}\psi\left(\overline{\alpha}+\frac{1}{2}\right)$$
$$= \frac{d^t}{dz^t}\big(\pi\tan(\pi z)\big)\Big|_{z=\alpha} + (-1)^{t+1}t!\sum_{j=1}^{2m}\frac{1}{(j-\frac{1}{2}+\alpha)^{t+1}}.$$

*For* $t$ *even,* $\Im(\psi_t(\alpha))$ *and* $\Im(\psi_t(\alpha+\frac{1}{2}))$ *are transcendental and for* $t$ *odd,* $\Re(\psi_t(\alpha))$ *and* $\Re(\psi_t(\alpha+\frac{1}{2}))$ *are transcendental.*

*Proof* By (18a)–(18b) and (16) we can compute the explicit values of each sum. We leave the calculation to the reader and note that the sums and differences here are all transcendental by Nesterenko's Theorem. For the final assertions, if $t$ is even, $\psi_t(z) - \psi_t(\overline{z}) = 2i\Im(\psi_t(z))$ and for $t$ odd, $\psi_t(z) + \psi_t(\overline{z}) = 2\Re(\psi_t(z))$. $\square$

## 4 Implications of conjectures of Schneider and Gel'fond

We now explore implications of conjectures of Schneider and Gel'fond involving algebraic powers of algebraic numbers. Note that throughout we define log as the principal value of the logarithm with argument in $(-\pi, \pi]$ and define $\alpha^\beta$ as $e^{\beta \log(\alpha)}$. After the solution to Hilbert's seventh problem, solved by Gel'fond and Schneider independently, the two were led to formulate some conjectures about the algebraic independence of various algebraic powers of algebraic numbers. For more details surrounding these conjectures, we refer the reader to [11, 12] and we point out that in [11], the following conjecture is credited to Schneider alone.

**Conjecture 7** (Gel'fond–Schneider) *If $\alpha \neq 0, 1$ is algebraic and if $\beta$ is algebraic of degree $d \geq 2$, then the $d-1$ numbers*

$$\alpha^\beta, \; \ldots, \; \alpha^{\beta^{d-1}}$$

*are algebraically independent.*

In previous work, in particular [6], we referred to the previous conjecture as "Schneider's conjecture"; however, the conjecture should be credited to both Gel'fond and Schneider. In this paper, we will refer to Conjecture 7 as the "Gel'fond–Schneider conjecture".

In 1949, Gel'fond [3] proved that the Gel'fond–Schneider conjecture is true for the cases when $d = 2$ or 3. Gel'fond also conjectured that $\log \alpha$ and $\alpha^\beta$ are algebraically independent. Putting this together with Conjecture 7, we have the conjecture that the numbers

$$\log \alpha, \; \alpha^\beta, \; \ldots, \; \alpha^{\beta^{d-1}}$$

are algebraically independent. Following [11], in [6] we called this the "Gel'fond–Schneider conjecture", but here we will refer it as the "second Gel'fond–Schneider conjecture". For our purposes, $\alpha$ will be a root of unity and we use a modified version of the conjecture that

$$\pi, \; \alpha^\beta, \; \ldots, \; \alpha^{\beta^{d-1}}$$

are algebraically independent. In some cases, we need much less, and we state our special version of the second Gel'fond–Schneider conjecture here.

**Conjecture 8** *If $\alpha \neq 0, 1$ is algebraic and if $\beta$ is irrational algebraic, then*

$$\log \alpha \quad \textit{and} \quad \alpha^\beta$$

*are algebraically independent. For $\alpha$ a root of unity, this implies that $\pi$ and $\alpha^\beta$ are algebraically independent.*

We now state conditional results extending Theorems 1, 5, and 6. Note that the calculations involved in finding explicit values in the proofs of those theorems required that $\alpha$ and $\alpha + 1/2$ along with their complex conjugates satisfy a symmetry condition $\alpha = -\overline{\alpha} + n$ for some integer $n$. This symmetry is needed here as well.

**Definition** Let

$$\mathcal{S} := \left\{ \alpha \in \overline{\mathbb{Q}} : \Re(\alpha) \in \frac{1}{2}\mathbb{Z} \right\}$$

so that $\mathcal{S}$ is the collection of all algebraic numbers $\alpha$ which satisfy the symmetry $\alpha = -\overline{\alpha} + n$ for some integer $n$. In particular, this set contains all algebraic numbers which are purely imaginary.

Additionally, we recall Lemma 9 from [6] in which the derivatives of the cotangent function are studied.

**Lemma** (Lemma 9 of [6]) *For* $k \geq 2$,

$$\frac{d^{k-1}}{dz^{k-1}}(\pi \cot(\pi z)) = (2\pi i)^k \left( \frac{A_{k,1}}{e^{2\pi i z} - 1} + \cdots + \frac{A_{k,k}}{(e^{2\pi i z} - 1)^k} \right),$$

*where each* $A_{i,j} \in \mathbb{Z}$ *with* $A_{k,1}, A_{k,k} \neq 0$.

A similar lemma can be shown for the tangent function and we leave the details to the reader.

**Theorem 9** *Let* $\alpha \in \mathcal{S} \setminus \mathbb{Z}$. *If Conjecture 8 is true, then* $\Gamma(\alpha)$, $\Im(\psi_t(\alpha))$ for $t$ even, *and* $\Re(\psi_t(\alpha))$ for $t$ odd, *are transcendental.*

*Proof* Using the symmetry of $\alpha$ with $\overline{\alpha}$, the explicit values for $|\Gamma(\alpha)|^2$ and each case of $\psi_t(\alpha)$ can be found in Theorems 1, 5, and 6. For $|\Gamma(\alpha)|^2$ and $\psi(\alpha)$ the result is clear by rewriting each trigonometric function in terms of exponentials and noting that Conjecture 8 implies that $\pi$ and $(e^{\pi i})^{\alpha}$ are algebraically independent. Since $|\Gamma(\alpha)|^2$ is transcendental, $\Gamma(\alpha)$ must be transcendental as well. For $t \geq 1$, Lemma 9 of [6] stated above shows the result for $\psi_t(\alpha)$. □

Analogous to Theorem 1, Theorem 9 gives an infinite set of (conditionally) transcendental numbers. In particular, if $\tau \in \overline{\mathbb{Q}} \cap \mathbb{R} \setminus \{0\}$, then Conjecture 8 implies that $\Gamma(i\tau)$ is transcendental. As we saw earlier, for example, $\tau \in \mathbb{Q}$, the result is unconditional in some cases. Additionally, for any irrational $\alpha \in \mathcal{S}$, we have the following unconditional result. We need only recall the theorem of Gel'fond in which $\alpha^{\beta}$ is transcendental for $\alpha \neq 0, 1$ and $\beta$ is irrational algebraic.

**Corollary 10** *If* $\alpha \in \mathcal{S} \setminus \mathbb{Q}$, *then* $|\Gamma(\alpha)|^2/\pi$ *is transcendental and therefore* $\Gamma(\alpha)/\sqrt{\pi}$ *is transcendental.*

By examining values $\Gamma(\alpha)$ (or more precisely $|\Gamma(\alpha)|^2$) individually, we need a special case of the second Gel'fond–Schneider conjecture in order to conclude transcendence. If we assume the weaker Gel'fond–Schneider conjecture, we get *almost* the same result, up to integer translations and complex conjugation.

**Theorem 11** *For* $\alpha_1, \alpha_2 \in \mathcal{S} \setminus \mathbb{Z}$ *so that* $\alpha_1 \not\equiv \pm\alpha_2 \pmod{\mathbb{Z}}$, *if the Gel'fond–Schneider conjecture is true, then*

$$\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_2)} \notin \overline{\mathbb{Q}}.$$

*Moreover, the set* $\{|\Gamma(\alpha)|^2 : \alpha \in (\mathcal{S} \setminus \mathbb{Z})/\mathbb{Z}\}$ *contains at most one algebraic number.*

*Proof* Since $\mathcal{S} \cap \mathbb{Q} \setminus \mathbb{Z} = \{k + \frac{1}{2} : k \in \mathbb{Z}\}$, we deduce that at least one of $\alpha_1, \alpha_2$ must be irrational. Without loss of generality, $\alpha_1 \notin \mathbb{Q}$. Suppose $|\Gamma(\alpha_1)|^2/|\Gamma(\alpha_2)|^2 = \theta_1$ is algebraic. Applying the method described in the proof of Theorem 1, we replace our quotient with values of the sine function and have

$$\theta = \frac{e^{\pi i \alpha_2} - e^{-\pi i \alpha_2}}{e^{\pi i \alpha_1} - e^{-\pi i \alpha_1}}$$

for some $\theta \in \overline{\mathbb{Q}}$. By the primitive element theorem, there is an algebraic $\beta$ of degree $d \geq 2$ such that $\mathbb{Q}(\beta) = \mathbb{Q}(\alpha_1, \alpha_2)$ and we can write each

$$\alpha_j = \frac{1}{M} \sum_{a=0}^{d-1} n_{a,j} \beta^a$$

for some integers $M, n_{a,j}$. Let $\alpha = e^{\pi i/M}$ and define $x_a = \alpha^{\beta^a} = e^{\pi i \beta^a/M}$ for $a = 1, \ldots, d-1$ so that

$$e^{\pi i \alpha_j} = \prod_{a=0}^{d-1} e^{\pi i n_{a,j} \beta^a/M} = \gamma_j x_1^{n_{1,j}} \cdots x_{d-1}^{n_{d-1,j}},$$

where $\gamma_j = e^{\pi i n_{0,j}/M}$ is a root of unity. If the Gel'fond–Schneider conjecture is true, then

$$x_1, \ \ldots, \ x_{d-1}$$

are algebraically independent and the equation

$$\theta = \frac{\gamma_2 x_1^{n_{1,2}} \cdots x_{d-1}^{n_{d-1,2}} - \gamma_2^{-1} x_1^{-n_{1,2}} \cdots x_{d-1}^{-n_{d-1,2}}}{\gamma_1 x_1^{n_{1,1}} \cdots x_{d-1}^{n_{d-1,1}} - \gamma_1^{-1} x_1^{-n_{1,1}} \cdots x_{d-1}^{-n_{d-1,1}}}$$

implies that the function

$$F(X_1, \ldots, X_{d-1}) = \frac{\gamma_2 X_1^{n_{1,2}} \cdots X_{d-1}^{n_{d-1,2}} - \gamma_2^{-1} X_1^{-n_{1,2}} \cdots X_{d-1}^{-n_{d-1,2}}}{\gamma_1 X_1^{n_{1,1}} \cdots X_{d-1}^{n_{d-1,1}} - \gamma_1^{-1} X_1^{-n_{1,1}} \cdots X_{d-1}^{-n_{d-1,1}}}$$

is constant. We show that $F$ is not constant by examining $F$ at some special points. Let $y$ be a new indeterminate and for some $\overline{e} = (e_1, \ldots, e_{d-1}) \in \mathbb{Z}^{d-1}$ to be specified later, let $X_i = y^{e_i}$. Writing $\overline{n_j} = (n_{1,j}, \ldots, n_{d-1,j})$ we have that

$$G(y) := F\left(y^{e_1}, \ldots, y^{e_{d-1}}\right) = \frac{\gamma_2 y^{\overline{n_2}\cdot\overline{e}} - \gamma_2^{-1} y^{-\overline{n_2}\cdot\overline{e}}}{\gamma_1 y^{\overline{n_1}\cdot\overline{e}} - \gamma_1^{-1} y^{-\overline{n_1}\cdot\overline{e}}}.$$

To ensure that $G(y)$ is not constant, and that we are not dividing by 0, we require that

$$(\overline{n_1} \pm \overline{n_2}) \cdot \overline{e} \neq 0,$$
$$\overline{n_1} \cdot \overline{e} \neq 0.$$

To prove the existence of such an $\overline{e}$, examine the box $B_D = (0, D]^{d-1}$, for positive integer $D$, which contains a total of $D^{d-1}$ lattice points. We wish to avoid points which satisfy the equations

$$(\overline{n_1} \pm \overline{n_2}) \cdot \overline{e} = 0,$$
$$\overline{n_1} \cdot \overline{e} = 0.$$

Note that these equations are not satisfied trivially because $\alpha_1 \neq \pm\alpha_2 \pmod{\mathbb{Z}}$ and $\alpha_1 \notin \mathbb{Q}$. Since there are at most $D^{d-2}$ lattice points in $B_D$ which satisfy each equation, for $D$ large enough we choose $\overline{e}$ from at least

$$D^{d-1} - 3D^{d-2} > 1$$

remaining lattice points. With such an $\overline{e}$ chosen, it is clear that $G(y)$ is not constant which implies that the quotient $|\Gamma(\alpha_1)|^2/|\Gamma(\alpha_2)|^2$ is not algebraic. The final assertion follows immediately. $\square$

The Gel'fond–Schneider conjecture is known to be true for $d = 2$ or 3, and we immediately have the following unconditional result.

**Corollary 12** *For $\alpha_1, \alpha_2 \in \mathcal{S} \setminus \mathbb{Z}$ with $\alpha_1 \not\equiv \pm\alpha_2 \pmod{\mathbb{Z}}$ and $[\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$ is 2 or 3,*

$$\frac{\Gamma(\alpha_1)}{\Gamma(\alpha_2)} \notin \overline{\mathbb{Q}}$$

*so that at most one of $\Gamma(\alpha_1), \Gamma(\alpha_2)$ is algebraic.*

Similar to Theorem 9, if we assume the second Gel'fond–Schneider conjecture to be true, then we can extend Theorems 2, 3, and 4.

**Theorem 13** *Let $A(z) \in \overline{\mathbb{Q}}[z]$ be monic with roots $\alpha_1, \ldots, \alpha_t \in \mathcal{S} \setminus \mathbb{N}$ with $\Re(s) = \Re(\sum_{j=1}^{t} \alpha_j) = 0$. If the second Gel'fond–Schneider conjecture is true, then the products*

$$|P|^2 = \prod_{n=1}^{\infty} \frac{A(n)\overline{A(n)}}{n^{2t}} \quad \textit{and therefore} \quad P = \prod_{n=1}^{\infty} \frac{A(n)}{n^t} e^{s/n}$$

*are transcendental.*

*Proof* From (10) we have,

$$|P|^2 = \frac{|\alpha_1|^{-2} \cdots |\alpha_t|^{-2}}{|\Gamma(-\alpha_1)|^2 \cdots |\Gamma(-\alpha_t)|^2}.$$

Suppose that $A(z)$ has no irrational roots. If all roots are in $\mathbb{Q} \setminus \mathbb{Z}$, then since $\mathbb{Q} \cap \mathcal{S} \setminus \mathbb{Z}$ contains only half-integers, the product is unconditionally transcendental due to the transcendence of $\pi$. If there are any (negative) integer roots which contribute integer factorial values to the product, then there must also be positive roots from $\mathbb{Q} \setminus \mathbb{Z}$ since $\Re(\sum_{j=1}^{t} \alpha_j) = 0$. Again the product is unconditionally transcendental in this case.

Now suppose there is at least one irrational root. Due to the symmetry between $\alpha_j$ and $\overline{\alpha_j}$, and following the method demonstrated in the proof of Theorem 1, each value $|\Gamma(\alpha_j)|^2$ can be replaced with an algebraic multiple of $\pi / \sin(\pi\alpha_j)$. Thus, for some non-zero $\xi \in \overline{\mathbb{Q}}$, the product is equal to

$$\xi \pi^{-t} \prod_{j=1}^{t} \bigl(e^{\pi i \alpha_j} - e^{-\pi i \alpha_j}\bigr). \tag{19}$$

By the primitive element theorem, there is an algebraic $\beta$ of degree $d \geq 2$ such that $\mathbb{Q}(\beta) = \mathbb{Q}(\alpha_1, \ldots, \alpha_t)$. Thus, we can write each

$$\alpha_j = \frac{1}{M} \sum_{a=0}^{d-1} n_{a,j} \beta^a$$

for some integers $M, n_{a,j}$. Let $\alpha = e^{\pi i/M}$ and define $x_a = \alpha^{\beta^a} = e^{\pi i \beta^a / M}$ for $a = 1, \ldots, d-1$ so that

$$e^{\pi i \alpha_j} = \prod_{a=0}^{d-1} e^{\pi i n_{a,j} \beta^a / M} = \gamma_j x_1^{n_{1,j}} \cdots x_{d-1}^{n_{d-1,j}},$$

where $\gamma_j = e^{\pi i n_{0,j}/M}$ is a root of unity. The second Gel'fond–Schneider conjecture implies that

$$\pi,\ x_1,\ \ldots,\ x_{d-1}$$

are algebraically independent and our non-zero product rewritten in terms of the $x_a$'s

$$|P|^2 = \xi \pi^{-t} \prod_{j=1}^{t} \bigl(\gamma_j x_1^{n_{1,j}} \cdots x_{d-1}^{n_{d-1,j}} - \gamma_j^{-1} x_1^{-n_{1,j}} \cdots x_{d-1}^{-n_{d-1,j}}\bigr)$$

is transcendental, thus $P$ is transcendental as well. $\square$

We now extend Theorem 3 and note that the explicit form calculated there is true for any $A(z) \in \mathbb{C}[z]$ with non-integer roots.

**Theorem 14** *Let $A(z) \in \overline{\mathbb{Q}}[z]$ be monic with roots $\alpha_1, \ldots, \alpha_t \in \overline{\mathbb{Q}} \setminus \mathbb{Z}$. The product*

$$\prod_{n=1}^{\infty} \frac{A(n)A(-n)}{n^{2t}} = \frac{\prod_{j=1}^{t} \sin(\pi\alpha_j)}{(-\pi)^t A(0)}$$

*and if the second Gel'fond–Schneider conjecture is true, the product is transcendental and therefore, at least one of*

$$\prod_{n=1}^{\infty} \frac{A(n)}{n^t} e^{s/n} \quad \text{or} \quad \prod_{n=1}^{\infty} \frac{A(-n)}{n^t} e^{-s/n}$$

*is transcendental.*

*Proof* With the explicit value given in terms of the sine function, we note that if all roots are rational, $\pi$ forces the product to be unconditionally transcendental. If there is at least one irrational root, then the proof is exactly the same as that of Theorem 13 beginning from Eq. (19). □

**Corollary 15** *Let $A(z) \in \overline{\mathbb{Q}}[z]$ be monic and even with roots $\pm\alpha_1, \ldots, \pm\alpha_t \in \overline{\mathbb{Q}} \setminus \mathbb{Z}$. The product*

$$\prod_{n=1}^{\infty} \frac{A(n)}{n^{2t}} = \pi^{-t} \prod_{j=1}^{t} \frac{\sin(\pi\alpha_j)}{\alpha_j}$$

*and is transcendental if the second Gel'fond–Schneider conjecture is true.*

Note that in Theorems 13, 14 and Corollary 15, we can multiply each product by an appropriate power of $\pi$ and be left with a finite expression containing no $\pi$. In this setting, if we require that $A(z)$ has at least one irrational root, then we can still conclude transcendence by assuming the weaker Gel'fond–Schneider conjecture. We leave the details to the reader.

## 5 Concluding remarks

In some of the above results on infinite products (1), we rely on assuming that $A(z)$ has no roots in $\mathbb{N}$, however, these cases can be handled with only a slight variation. For simplicity, assume that $A(z)$ is depressed with degree $t$, having roots $n_1, \ldots, n_l \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_{t-l} \in \mathbb{C} \setminus \mathbb{N}$. We have

$$\prod_{n=1}^{\infty}{}' \frac{A(n)}{n^t} = \prod_{n=1}^{\infty}{}' \left[ \prod_{i=1}^{l} \left(1 - \frac{n_i}{n}\right) \prod_{j=1}^{t-l} \left(1 - \frac{\alpha_j}{n}\right) \right],$$

where $\prod'$ represent the products avoiding any roots of $A(z)$. Similar to (8), we insert exponential factors,

$$\prod_{n=1}^{\infty}{}' \left[\prod_{i=1}^{l}\left(1-\frac{n_i}{n}\right)e^{n_i/n}\prod_{j=1}^{t-l}\left(1-\frac{\alpha_j}{n}\right)e^{\alpha_j/n}\right]$$

and interchange the order of multiplication to obtain

$$\left[\prod_{i=1}^{l}\prod_{n=1}^{\infty}{}'\left(1-\frac{n_i}{n}\right)e^{n_i/n}\right]\left[\prod_{j=1}^{t-l}\prod_{n=1}^{\infty}{}'\left(1-\frac{\alpha_j}{n}\right)e^{\alpha_j/n}\right].$$

By inserting the obvious missing factors from each infinite product, we can write the product as

$$e^l\left[\prod_{i=1}^{l}\xi_i\prod_{n\neq n_i}\left(1-\frac{n_i}{n}\right)e^{n_i/n}\right]\left[\prod_{j=1}^{t-l}\lambda_j\prod_{n=1}^{\infty}\left(1-\frac{\alpha_j}{n}\right)e^{\alpha_j/n}\right],$$

where

$$\xi_i=\prod_{j=1,j\neq i}^{l}\left(1-\frac{n_i}{n_j}\right)^{-1}\quad\text{and}\quad\lambda_j=\prod_{i}^{l}\left(1-\frac{\alpha_j}{n_i}\right)^{-1}.$$

Since $A(z)$ is depressed, we multiply by one final exponential factor equal to 1 to obtain

$$e^l e^{\gamma(-n_1-\cdots-\alpha_{t-l})}\left[\prod_{i=1}^{l}\xi_i\prod_{n\neq n_i}\left(1-\frac{n_i}{n}\right)e^{n_i/n}\right]\left[\prod_{j=1}^{t-l}\lambda_j\prod_{n=1}^{\infty}\left(1-\frac{\alpha_j}{n}\right)e^{\alpha_j/n}\right].$$

Note that $\Gamma(z)$ has a simple pole at each negative integer $-n$ with residue $(-1)^n/n!$ and we have

$$\lim_{z\to -n_i} e^{\gamma z}\prod_{n\neq n_i}\left(1+\frac{z}{n}\right)e^{-z/n}=\lim_{z\to -n_i}\frac{n_i e^{z/n_i}}{z(z+n_i)\Gamma(z)}=e^{-1}(-1)^{n_i+1}n_i!$$

so that

$$\prod_{n=1}^{\infty}{}'\frac{A(n)}{n^t}=\frac{(-1)^t[\prod_{i=1}^{l}(-1)^{n_i}\xi_i n_i!][\prod_{j=1}^{t-l}\lambda_j\alpha_j^{-1}]}{\Gamma(-\alpha_1)\cdots\Gamma(-\alpha_{t-l})}.$$

With the product evaluated explicitly, we see that the numerator is algebraic and so the characterization of these products relies on understanding the nature of the Gamma function at the non-integral roots of $A(z)$. As a concluding example illustrating the above technique, we evaluate a product similar to (2). One can check that the technique described above gives,

$$\prod_{n=1}^{\infty}{}'\left(1-\frac{1}{n^3}\right)=\prod_{n=2}^{\infty}\left(1-\frac{1}{n^3}\right)=\frac{1}{3\Gamma(-\rho)\Gamma(-\rho^2)}=\frac{\cosh(\pi\sqrt{3}/2)}{3\pi}$$

for $\rho = e^{2\pi i/3} = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$, and the product is transcendental by Nesterenko's theorem.

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