The Zeta Mahler measure of $(z^n - -1)/(z - -1)$

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The Zeta Mahler measure of $(z^n-1)/(z-1)$

Arunabha Biswas and M. Ram Murty

Dedicated to the memory of S. Srinivasan

Abstract. We consider the k-higher Mahler measure $m_k(P)$ of a Laurent polynomial P as the integral of $\log^k|P|$ over the complex unit circle and zeta Mahler measure as the generating function of the sequence $\{m_k(P)\}$. In this paper we derive a few properties of the zeta Mahler measure of the polynomial $P_n(z) := (z^n-1)/(z-1)$ and propose a conjecture.

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1. Introduction

The Mahler measure of a polynomial $P(z)$ in $\mathbb{C}[z]$ is defined as

$$M(P) := |a|\prod_{j=1}^{d}\max\{1,|r_j|\},$$

where $P(z) = a\prod_{j=1}^{d}(z-r_j)$. The study of Mahler measures of irreducible polynomials with integer coefficients has been the focus of intense research for more than fifty years. For example, it is not hard to show that for polynomials with integer coefficients, $M(P) = 1$ if and only if $\pm P$ is a power of z times a cyclotomic polynomial. This follows from an old result of Kronecker (see for example, the excellent survey of Smyth [Sm08]). A celebrated conjecture of Lehmer predicts that there is a constant $c > 1$ such that if P is a monic irreducible polynomial with integer coefficients, then $M(P) > c$. In fact, Lehmer suggests that c can be taken as the Mahler measure of the polynomial

$$z^{10}+z^9-z^7-z^6-z^5-z^4-z^3+z+1.$$

For a non-zero rational function $P(z) \in \mathbb{C}[z]$, the k-higher Mahler measure of P is defined [KLO08] as

$$m_k(P) = \int_0^1 \log^k\left|P\left(e^{2\pi it}\right)\right|\, dt.$$

For $k = 1$ and $P(z) \in \mathbb{C}[z]$, this coincides with the classical (log) Mahler measure defined as

$$m(P) = \log\left(M(P)\right) = \log\left(|a|\prod_{j=1}^{d}\max\{1,|r_j|\}\right), \text{ for } P(z) = a\prod_{j=1}^{d}(z-r_j),$$

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that is, $m_1(P) = m(P)$, by Jensen's formula (see [EvWa09], p. 7) which is given as,

$$\int_0^1 \log \left|\alpha - e^{2\pi i\theta}\right| \, d\theta = \log\left(\max\{1, |\alpha|\}\right) \quad \text{for some} \quad \alpha \in \mathbb{C}.$$

The k-higher Mahler measure was introduced by Kurokawa, Lalín, and Ochiai [KLO08]. In [Aka09], Akatsuka defined what has been now called as Akatsuka's zeta Mahler measure as:

$$Z(s, P) := \int_0^1 \left|P\left(e^{2\pi it}\right)\right|^s \, dt, \quad s \in \mathbb{C}. \tag{1.1}$$

The terms, $m_k(P)$'s, are determined by the coefficients of the Taylor series expansion of Akatsuka's zeta Mahler measure, that is,

$$Z(s, P) = \sum_{k=0}^{\infty} \frac{m_k(P)}{k!} \, s^k.$$

Thus, $Z(s, P)$ is the generating function of the $m_k(P)$'s. After that, several authors studied higher Mahler measure; for example, Sasaki [Sa10], Borwein and Straub [BoSt12], Borwein, Borwein, Straub and Wan [BBSW12], Biswas [Bis14], Biswas and Monico [BisMo14], Lalín and Lechasseur [LaLe16], and Lalín [La16].

Finding exact formulas for k-higher Mahler measure is usually very difficult. Given a polynomial, deriving a functional expression of $Z(s, P)$ and taking its Taylor expansion seems a much better and practical approach for the calculation of $m_k(P)$ than using its definition.

In [KLO08] and [Bis14], the authors used two different (but equivalent) functional expressions of $Z(s, P)$ to calculate k-higher Mahler measure $m_k(P)$ for the polynomial $P(z) = z - r$ with $|r| = 1$ and unfortunately this is the only single variable polynomial whose explicit k-higher Mahler measure is known.

In [Aka09], Akatsuka found out closed form functional expressions of $Z(s, P)$'s for the polynomials $P_{A_1}(z) := z - c$ for any $c \in \mathbb{C}$ and $P_{A_2}(z) := z + z^{-1} + r$ for any $r \in \mathbb{R}$. Unfortunately these P_{A_1} and P_{A_2} are the only examples of single variable polynomials whose closed form functional expressions of zeta Mahler measure are known.

Unless the higher Mahler measure could be given an interpretation as some sort of regulator, it does not seem likely that it will be possible to compute many explicit values. But we need to get going with a few explicit examples before getting an idea of a more general pattern.

In [LaSi11] Lalin and Sinha used polynomials of type $P_n(z) := (z^n - 1)/(z - 1)$ with $n \in \mathbb{N}$, $n > 1$, while answering analogous Lehmer's question [EvWa09] for odd indexed higher Mahler measure. Although the main purpose of their paper was not formulating a practically useful form of $m_{2k+1}(P_n)$, the expression that the authors achieved is significantly complicated. They used a direct calculation using the definition here.

2. Akatsuka's zeta Mahler measure for $P_n(z) = (z^n - 1)/(z - 1)$

In this section, we explore a few properties of $Z(s, P_n)$.

Using the familiar identities $e^{i\theta} = \cos\theta + i\sin\theta$ and $\cos(2\theta) = 1 - 2\sin^2\theta$, we see that,

$$\left|P_n\left(e^{2\pi it}\right)\right| = \left|\frac{e^{2\pi nit} - 1}{e^{2\pi it} - 1}\right| = \left|\frac{\sin(n\pi t)}{\sin(\pi t)}\right|. \tag{2.1}$$

Then using the definition (1.1) of Akatsuka's zeta Mahler measure and (2.1) we see that,

$$Z(s, P_n) \quad = \quad \int_0^1 \left|\frac{\sin(n\pi t)}{\sin(\pi t)}\right|^s \, dt. \tag{2.2}$$

We will prove:

Theorem 2.3. *$Z(2s, P_n) \in \mathbb{N}$ for all $s \in \mathbb{N}$.*

Proof. For the convenience of calculation we shall show $Z(2s, P_{n+1}) \in \mathbb{N}$ for all $s \in \mathbb{N}$. Using the familiar identity

$$\frac{x^{n+1}-y^{n+1}}{x-y} = x^n + x^{n-1}y + \cdots + xy^{n-1} + y^n$$

we see that

$$\frac{\sin((n+1)t)}{\sin(t)} = \frac{e^{i(n+1)t}-e^{-i(n+1)t}}{e^{it}-e^{-it}} = e^{int} + e^{i(n-2)t} + \cdots + e^{-i(n-2)t} + e^{-int}.$$

Now applying the multinomial theorem, we see,

$$\left[\frac{\sin((n+1)t)}{\sin(t)}\right]^k = \sum_{j_0,\cdots,j_n} \binom{k}{j_0,\cdots,j_n} \exp\left(i\sum_{r=0}^{n}(n-2r)j_r t\right),$$

where

$$\binom{k}{j_0,\cdots,j_n} := \frac{k!}{j_0!,\cdots,j_n!} \quad \text{with} \quad k = j_0 + \cdots + j_n$$

are the usual multinomial coefficients. Now if k is even (say $k = 2s$) then after a suitable change of variables,

$$Z(2s, P_{n+1}) = \frac{1}{\pi}\int_0^{\pi}\left[\frac{\sin((n+1)t)}{\sin(t)}\right]^{2s} \mathrm{d}t = \sum_{j_1+2j_2+\cdots+nj_n=ns} \binom{2s}{j_0,\cdots,j_n}.$$

Since $Z(2s, P_{n+1})$ is expressed as sum of multinomial coefficients, it must be a positive integer.

Theorem 2.4. $\dfrac{Z(m(s+1), P_n)}{Z(ms, P_n)} \to n^m$ *as* $s \to \infty$.

To prove this theorem we use Laplace's saddle-point method (see [Wo89], p. 57) as described in the following proposition 2.5 which we recall for ease of exposition.

Proposition 2.5. *Suppose that h is a real-valued C^2-function defined on the interval $(a, b) \subset \mathbb{R}$. If we further suppose that h has a unique maximum at $\xi = c$ with $a < c < b$ so that $h'(c) = 0$ and $h''(0) < 0$, then, we have*

$$\int_a^b e^{\lambda h(t)}\,\mathrm{d}t \sim e^{\lambda h(c)}\left(\frac{-2\pi}{\lambda h''(c)}\right)^{\frac{1}{2}}$$

as $\lambda \to \infty$.

Proof of Theorem 2.4. We want to apply Proposition 2.5 here with

$$g(t) := e^{h(t)} := \left(\frac{\sin(n\pi t)}{\sin(\pi t)}\right)^2 \quad \text{and} \quad \lambda := \frac{s}{2}.$$

We recall two equivalent expressions for the Fejér kernel $F_n(x)$ as

$$F_n(x) = \frac{1}{n}\left(\frac{\sin(nx/2)}{\sin(x/2)}\right)^2 = \sum_{|j|\leqslant n}\left(1 - \frac{|j|}{n}\right)e^{ijx}.$$

Therefore,

$$g(t) = e^{h(t)} = \left(\frac{\sin(n\pi t)}{\sin(\pi t)}\right)^2 = \sum_{|j|\leqslant n}(n-|j|)e^{i2\pi jt} = n + 2\sum_{j=1}^{n}(n-j)\cos(2\pi jt).$$

We see that $g(t)$ (and hence $h(t)$) attains its maximum at $t = 0$ and the maximum is

$$g(0) = n + 2\sum_{j=1}^{n}(n-j) = n^2.$$

Now we have, on the one hand

$$g'(t) = h'(t)e^{h(t)} \quad \text{and} \quad g''(t) = h''(t)e^{h(t)} + (h'(t))^2 e^{h(t)},$$

and on the other

$$g'(t) = -4\pi\sum_{j=1}^{n}(nj - j^2)\sin(2\pi jt) \quad \text{and} \quad g''(t) = -8\pi^2\sum_{j=1}^{n}(nj^2 - j^3)\cos(2\pi jt).$$

Therefore, from the first expression of $g''(t)$ we get

$$g''(0) = h''(0)n^2, \tag{2.6}$$

and from the next expression we get

$$g''(0) = -8\pi^2\sum_{j=1}^{n}(nj^2 - j^3) = -(2/3)\pi^2(n^4 - n^2), \tag{2.7}$$

using the identities

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{j=1}^{n} j^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Now comparing values of $g''(0)$ from (2.6) and (2.7) we see that

$$h''(0) = -(2/3)\pi^2(n^2 - 1),$$

and this implies, $h''(0) = -(2/3)\pi^2(n^2 - 1) < 0$.

Now applying Theorem 2.5, as $\lambda = s/2 \to \infty$ (and hence $s \to \infty$) we get

$$\int_0^1 \left|\frac{\sin(n\pi t)}{\sin(\pi t)}\right|^s \mathrm{d}t = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{\sin(n\pi t)}{\sin(\pi t)}\right)^{2\lambda} \mathrm{d}t \sim g(0)^{\lambda}\left(\frac{-2\pi}{\lambda h''(0)}\right)^{\frac{1}{2}} = n^s\left(\frac{6}{s\pi(n^2-1)}\right)^{\frac{1}{2}}.$$

Since the limit of $(\sin t)/t \to 1$ as $t \to 0\pi$, the second integrand above is well-defined.

That means,

$$Z(s, P_n) \sim n^s\left[\frac{6}{s\pi(n^2-1)}\right]^{1/2} \tag{2.8}$$

as $s \to \infty$. Therefore, as $s \to \infty$ we have,

$$\frac{Z(m(s+1), P_n)}{Z(ms, P_n)} \sim \frac{n^{m(s+1)}\left[6/(m(s+1)\pi(n^2-1))\right]^{1/2}}{n^{ms}\left[6/(ms\pi(n^2-1))\right]^{1/2}}.$$

Now taking limit $s \to \infty$ both sides we get our desired result.

3. A conjecture

In both [KLO08] (in p. 284) and in [Aka09] (in p. 2724) we see the following formula:

$$Z(s, P_2) = \frac{2^s}{\pi} B\left(\frac{s+1}{2}, \frac{1}{2}\right).$$

Keeping this in mind, we propose the following conjecture:

Conjecture 3.1. *Let $Z(s, P_n)$ be as described before. Then*

$$Z(s, P_n) = g(s, n)\ \frac{n^s}{\pi} B\left(\frac{s+1}{2}, \frac{1}{2}\right) \tag{3.2}$$

where $B(x, y)$ is the usual beta function and $g(s, n)$ lies in $(0, 1)$ and can be written in terms of well-known functions. Additionally,

$$\lim_{s\to\infty} g(s, n) = \sqrt{\frac{3}{n^2 - 1}} \tag{3.3}$$

for any fixed $n \geq 2$ and

$$\lim_{n\to\infty} g(s, n) = 0 \tag{3.4}$$

for fixed $s \geq 0$.

We want to recall a well known application of Stirling's approximation that says

$$B(x, y) \sim \Gamma(y)x^{-y}$$

for large x and fixed y. Using it we see that

$$B\left(\frac{s+1}{2}, \frac{1}{2}\right) \sim \Gamma\left(\frac{1}{2}\right)\left(\frac{s+1}{2}\right)^{-1/2} = \left(\frac{2\pi}{s+1}\right)^{1/2} \sim \left(\frac{2\pi}{s}\right)^{1/2}.$$

Combining this with Conjecture 3.1 we see that

$$Z(s, P_n) \sim g(s, n)\ n^s \left(\frac{2}{s\pi}\right)^{1/2} \tag{3.5}$$

Clearly we see a remarkable similarity between (3.5) and (2.8); and this gives us hope about the truthfulness of the conjecture 3.1. Numerical evidence shows that $g(s, n)$ is a decreasing function of s for fixed n and of n for fixed s.

It is easy to confirm our claim (3.3) when we determine $g(s, n)$ from (3.2) by the use of (2.8).

We also confirm our claim (3.4) by invoking the following lemma. Hirotaka Akatsuka provided us this lemma including its proof in a private communication.

Lemma 3.6. *Let $Z(s, P_n)$ be as described before. Then:*

1. $\displaystyle\lim_{n\to\infty} Z(s, P_n) = \frac{2}{\pi s} \tan\frac{\pi s}{2}$ *when* $0 < s < 1$,

2. $\displaystyle\lim_{n\to\infty} \frac{Z(s, P_n)}{\log n} = \frac{4}{\pi^2}$ *when* $s = 1$,

3. $\displaystyle\lim_{n\to\infty}\frac{Z(s,P_n)}{n^{s-1}}=\frac{2}{\pi}\int_0^\infty\left|\frac{\sin u}{u}\right|^s \mathrm{d}u$ *when* $s>1$.

Proof of Lemma 3.6. We can look at exercise problem 1.1 in page 71 of [Kat04] for a similar statement of the second claim and Lemma 2.1 in page 1383 of [AAJRS07] for that of the third claim of this lemma. So, let us just show the proof the first claim. To simply the calculations, we will show that

$$\lim_{N\to\infty} Z(s,P_{2N})=\frac{2}{\pi s}\tan\frac{\pi s}{2} \tag{3.7}$$

when $0<s<1$. Because of the symmetry, we have,

$$\int_0^1\left|\frac{\sin(2N\pi t)}{\sin(\pi t)}\right|^s \mathrm{d}t=\frac{1}{2}\int_0^{1/2}\left|\frac{\sin(2N\pi t)}{\sin(\pi t)}\right|^s \mathrm{d}t. \tag{3.8}$$

We now divide the integral into

$$\int_0^{1/2}\left|\frac{\sin(2N\pi t)}{\sin(\pi t)}\right|^s \mathrm{d}t=\sum_{k=0}^{N-1}\int_{\frac{k}{2N}}^{\frac{k+1}{2N}}\left|\frac{\sin(2N\pi t)}{\sin(\pi t)}\right|^s \mathrm{d}t \tag{3.9}$$

After the change of variables $u=t-k/(2N)$, the integrals on the right hand side become

$$\begin{aligned}\int_{\frac{k}{2N}}^{\frac{k+1}{2N}}\left|\frac{\sin(2N\pi t)}{\sin(\pi t)}\right|^s \mathrm{d}t &= \int_0^{\frac{1}{2N}}\left|\frac{\sin(2N\pi u)}{\sin\left(\pi\left(u+\frac{k}{2N}\right)\right)}\right|^s \mathrm{d}u\\ &= \int_0^{\frac{1}{2N}}\left(\frac{\sin(2N\pi u)}{\sin\left(\pi\left(u+\frac{k}{2N}\right)\right)}\right)^s \mathrm{d}u.\end{aligned}$$

Estimating the denominator trivially, we get [1]

$$\begin{aligned}\frac{1}{\left(\sin\left(\frac{\pi(k+1)}{2N}\right)\right)^s}\int_0^{\frac{1}{2N}}(\sin(2N\pi u))^s\,\mathrm{d}u &\le \int_{\frac{k}{2N}}^{\frac{k+1}{2N}}\left|\frac{\sin(2N\pi t)}{\sin(\pi t)}\right|^s \mathrm{d}t\\ &\le \frac{1}{\left(\sin\left(\frac{\pi k}{2N}\right)\right)^s}\int_0^{\frac{1}{2N}}(\sin(2N\pi u))^s\,\mathrm{d}u.\end{aligned}$$

Now using the fact

$$\int_0^{\frac{1}{2N}}(\sin(2N\pi u))^s\,\mathrm{d}u=\frac{1}{2\pi N}B\left(\frac{s+1}{2},\frac{1}{2}\right),$$

we see

$$\frac{B\left(\frac{s+1}{2},\frac{1}{2}\right)}{2N\pi\left(\sin\left(\frac{\pi(k+1)}{2N}\right)\right)^s}\le\int_{\frac{k}{2N}}^{\frac{k+1}{2N}}\left|\frac{\sin(2N\pi t)}{\sin(\pi t)}\right|^s \mathrm{d}t\le\frac{B\left(\frac{s+1}{2},\frac{1}{2}\right)}{2N\pi\left(\sin\left(\frac{\pi k}{2N}\right)\right)^s}.$$

Plugging this in "(3.8)" (and also using the footnote), we obtain

$$\begin{aligned}0 &\le \int_0^{1/2}\left|\frac{\sin(2N\pi t)}{\sin(\pi t)}\right|^s \mathrm{d}t-\frac{B\left(\frac{s+1}{2},\frac{1}{2}\right)}{2N\pi}\sum_{k=1}^{N}\frac{1}{\left(\sin\left(\frac{\pi k}{2N}\right)\right)^s}\\ &\le \frac{\pi^s N^{-(1-s)}}{2}+\frac{B\left(\frac{s+1}{2},\frac{1}{2}\right)}{2N\pi\left(\sin\left(\frac{\pi}{2N}\right)\right)^s}-\frac{B\left(\frac{s+1}{2},\frac{1}{2}\right)}{2N\pi\left(\sin\left(\frac{\pi}{2}\right)\right)^s}.\end{aligned} \tag{3.10}$$

[1] When $k=0$, we need to modify the upper bound. By using the facts $\sin(\pi\theta)\ge 2\theta$ for $\theta\in[0,1/2]$ and $\sin(\pi\theta)\le\pi\theta$ for $\theta\in[0,\infty)$, the middle integral is bounded above by $\pi^s N^{-(1-s)}/2$.

By the definition of the Riemann integral we expect

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}\frac{1}{\left(\sin\left(\frac{\pi k}{2N}\right)\right)^s}=\int_0^{1/2}\frac{\mathrm{d}v}{(\sin\left(\frac{\pi v}{2}\right))^s}=\frac{1}{\pi}B\left(\frac{1-s}{2},\frac{1}{2}\right). \tag{3.11}$$

Since the middle integral is improper, this is not a direct consequence of the definition of the Riemann integral. But we can justify this by

$$\int_{\frac{k}{N}}^{\frac{k+1}{N}}\frac{\mathrm{d}v}{(\sin\left(\frac{\pi v}{2}\right))^s}\le\frac{1}{N\left(\sin\left(\frac{\pi k}{2N}\right)\right)^s}\le\int_{\frac{k-1}{N}}^{\frac{k}{N}}\frac{\mathrm{d}v}{(\sin\left(\frac{\pi v}{2}\right))^s}$$

Now combining the inequality (3.10) and the equality (3.11), we obtain

$$\lim_{N\to\infty}\int_0^{1/2}\left|\frac{\sin(2N\pi t)}{\sin(\pi t)}\right|^s\mathrm{d}t=\frac{1}{2\pi^2}B\left(\frac{s+1}{2},\frac{1}{2}\right)B\left(\frac{1-s}{2},\frac{1}{2}\right)$$

Using the facts $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$ and the reciprocal formula of the gamma function $\Gamma(z)$, we obtain

$$B\left(\frac{s+1}{2},\frac{1}{2}\right)B\left(\frac{1-s}{2},\frac{1}{2}\right)=\frac{2\pi}{s}\tan\left(\frac{\pi s}{2}\right),$$

and this completes the proof of (3.7).

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