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Some Variations on the Dedekind Conjecture

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Abstract. In this paper we prove a group theoretic statement about expressing certain characters of a finite solvable group as a sum of monomial characters. This is used to prove holomorphy of certain products of Artin L-functions which can be thought of as a variant of the Dedekind Conjecture. This variant is then used to improve, in the solvable case, a certain inequality due to R. Foote and K. Murty which bounds the orders of some Artin L-functions, at an arbitrary but fixed point in the complex plane, in terms of the order of a suitable quotient of Dedekind zeta functions. This improved inequality has a rather striking consequence regarding non-existence of simple zeros or simple poles in such quotients.

1. Introduction

A celebrated conjecture of Dedekind asserts that for any finite algebraic extension F of $\mathbb{Q}$, the zeta function $\zeta_F(s)$ is divisible by the Riemann zeta function $\zeta(s)$. That is, the quotient $\zeta_F(s)/\zeta(s)$ is entire. More generally, Dedekind conjectures that if K is a finite extension of F, then $\zeta_K(s)/\zeta_F(s)$ should be entire. This conjecture is still open.

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By the work of Aramata and Brauer (see for example, Section 5, Chapter I of [6]) the conjecture is known if K/F is Galois. If K is contained in a solvable extension of F, then Uchida [12] and van der Waall [13] have independently proved Dedekind's conjecture.

If Artin's conjecture about the holomorphy of his non-abelian L-series is true, then it is well-known that Dedekind's conjecture follows. It is also known that both Artin and Dedekind's conjectures follow from the "Selberg philosophy" of factorization of zeta functions (see Selberg [9] and Murty [8], for example).

In this paper, we present several variations on Dedekind's conjecture. These are recorded in Theorems 4.1, 5.1 and 5.3. For example, one of the consequences of our theorems (see Theorem 5.4 and Corollary 5.6) is the result that for any finite solvable Galois extension K/F, the (entire) function

$$\prod_{\chi:\chi(1)>1} L(s,\chi,K/F)^{\chi(1)},$$

where the product is over non-abelian irreducible characters, cannot have any simple zeros (see Corollary 5.6). The fact that the above product can be identified as the quotient of two zeta functions, namely $\zeta_K(s)$ and $\zeta_{K^{ab}}(s)$, where K^{ab} is the subfield of K fixed by the commutator subgroup of $\mathrm{Gal}(K/F)$, shows that it is entire. However, it is not at all clear that the function does not have simple zeros unless Artin's conjecture is assumed.

Stark [11] proved that if K/F is a Galois extension and $\zeta_K(s)$ has a simple zero at $s = s_0$, then it must necessarily come from a cyclic extension M of F. Moreover, $\zeta_M(s)$ has a simple zero at $s = s_0$ and any intermediate field whose corresponding Dedekind zeta function has $s = s_0$ as a zero must necessarily contain M. This seems to suggest some sort of "Galois theory" or "Galois packets" of zeros. This theme was initiated by K. Murty in [4] (see in particular Section 7 of that paper) where it is shown that for K/F solvable satisfying a mild condition, and for each zero ρ of $\zeta_K(s)$, there is an extension K_ρ such that $\mathrm{ord}_{s=\rho}\,\zeta_K(s) = \mathrm{ord}_{s=\rho}\,\zeta_{K_\rho}(s)$ and any extension M of F whose Dedekind zeta function has $s = \rho$ as a zero of the same order, must necessarily contain K_ρ. As proved in [4], the existence of K_ρ is easily established assuming Artin's conjecture and it would be desirable to do this without this hypothesis. We see this paper as one of the steps to be taken to achieve such a goal. As also indicated in [4], such results will have analytic consequences.

Our methods of proof develop earlier work of K. Murty [3] and Foote-Murty [1]. If K/F is a finite Galois extension of algebraic number fields with Galois group G, it is known from the work of Brauer that $L(s,\chi,K/F)$ can be extended to the entire complex plane as a meromorphic function of s. Fix $s_0 \in \mathbb{C}$. If $n(G,\chi)$ denotes the order of $L(s,\chi,K/F)$ at $s = s_0$, then Foote and Murty have shown

$$\sum_{\chi} n(G,\chi)^2 \leq (\operatorname*{ord}_{s=s_0} \zeta_K(s))^2.$$

An interesting variation of the results in [1] is an observation due to the first author which is recorded in Theorem 4.5 where in the above inequality one can eliminate the term corresponding to the trivial character and get a better inequality. One of the motivating factors of this paper has been to further improve on the inequality. We show in Theorem 4.6 that it is possible to eliminate any arbitrary abelian character from the left hand side and thereby obtain a sharper inequality. An even better inequality is obtained in Theorem 5.4 where all the terms in the left hand side corresponding to abelian characters can be eliminated. In proving these inequalities the above mentioned variations of the Dedekind conjecture has been crucial.

Another important feature of this paper concerns monomial representations of finite groups recorded in Lemmas 2.1 to 2.5 which are of interest in their own right. We believe that these ideas and theorems will eventually be useful in analytic number theory in much the same spirit as in Stark [11] or Heilbronn [2] in whose papers the Heilbronn character of Section 4 originates.

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2. Some Group Theoretic Preliminaries

In this section we state and prove some lemmas regarding representations of finite groups. Some of them are known (see [3]) and some are new and are stated in a way best suited to our needs. Before we do that we introduce some notations and also mention some well known and "user friendly" theorems that will be used almost throughout this article.

Let G be a finite group. By a representation (π, V) of G we mean a group homomorphism π of G into $\mathrm{GL}(V)$ where V is a finite dimensional complex vector space. To (π, V) is associated its character χ_π defined by $\chi_\pi(g) = \mathrm{Tr}_V(\pi(g))$. The character is a class function, i.e. is constant on conjugacy classes. Note that $\chi_\pi(1) = \dim_{\mathbb{C}}(V)$ which will be called the dimension or degree of the representation/character.

The canonical inner product on the space of $\mathbb{C}$–valued functions is given by

$$\langle f_1, f_2\rangle_G = \frac{1}{|G|}\sum_{g\in G} f_1(g)\overline{f_2(g)}.$$

Schur's orthogonality (see Chapter 2 of [10]) states that if χ_1 and χ_2 are the characters of two distinct irreducible representations then $\langle\chi_1, \chi_2\rangle_G = 0$ and if χ is the character of an irreducible representation of G then $\langle\chi, \chi\rangle_G = 1$. So the character of a representation completely determines the representation. Henceforth we will ("in the tradition of finite group theorists") use the term character interchangeably for both a representation and its character. So a phrase like "let χ be an irreducible character of G" stands for let $\chi = \chi_\pi$ for some irreducible representation (π, V) of G.

There are finitely many distinct irreducible characters of G. The set of all distinct irreducible characters will be denoted by $\widehat{G}$. There are two canonical characters, one of which is the *trivial character* and shall be denoted $\mathbf{1}_G$ and the other is what is called the *regular representation/character* which will be denoted by $\mathcal{R}_G$ and in terms of the irreducible characters it is given by:

$$\mathcal{R}_G = \sum_{\chi\in\widehat{G}} \chi(1)\,\chi.$$

Now let H be a subgroup of G. Let ψ be a character of H. Then the character of G obtained by inducing ψ from H to G, denoted by $\mathrm{Ind}_H^G(\psi)$ or if there is no chance of a confusion by ψ^G, is given by

$$\mathrm{Ind}_H^G(\psi)(g) = \frac{1}{|H|}\sum_{x\in T(g,H)} \psi(xgx^{-1})$$

where $T(g, H)$ is the 'transporter' of g into H for the conjugation action of G on itself, i.e., $T(g, H) = \{x \in G : xgx^{-1} \in H\}$. A character χ of G is said to be *monomial* if $\chi = \mathrm{Ind}_H^G(\psi)$ for some one dimensional character ψ of a subgroup H of G. In terms of induced characters one has $\mathcal{R}_G = \mathrm{Ind}_{\{1\}}^G(1_{\{1\}})$. Some basic facts about induced representations are recorded below.

1. **Frobenius Reciprocity.** If χ is a character of a finite group G and ψ is a character of a subgroup H of G then

$$\langle\chi, \psi^G\rangle_G = \langle\chi|_H, \psi\rangle_H$$

2. **Mackey's Theorem.** Let ψ be a character of H and let K be a subgroup of G then

$$\mathrm{Ind}_H^G(\psi)|_K = \sum_{x\in K\backslash G/H} \mathrm{Ind}_{K\cap xHx^{-1}}^K(\psi^x)$$

where $\psi^x(g) = \psi(x^{-1}gx)$ for all $g \in K\cap xHx^{-1}$.

Lemmas 2.1, 2.2, 2.3 and 2.4 below are implicit in K. Murty's paper [3] and are exposed here for the sake of clarity, completeness and generalisation.

Lemma 2.1. *Let G be a finite group and let A be an abelian normal subgroup. Let ϵ be a one dimensional character of A. Let T_ϵ be the stabilizer of ϵ, i.e.*

$$T_\epsilon = \{g \in G : \epsilon(gag^{-1}) = \epsilon(a) \ \ \forall a \in A\}.$$

Let

$$\mathrm{Ind}_A^{T_\epsilon}(\epsilon) = \sum_{\psi \in \widehat{T_\epsilon}} m_\psi \cdot \psi.$$

Then the collection $\{\mathrm{Ind}_{T_\epsilon}^G(\psi)\}_{m_\psi \neq 0}$ are distinct irreducible characters of G.

Proof. Observe that

$$\mathrm{Ind}_A^{T_\epsilon}(\epsilon)(g) = \begin{cases} [T_\epsilon : A]\epsilon(g) & \text{if } g \in A \\ 0 & \text{if } g \notin A. \end{cases} \tag{1}$$

This follows easily from the formula for an induced character, the definition of T_ϵ and the normality of A. Therefore we have

$$\sum_{\psi \in \widehat{T_\epsilon}} m_\psi \cdot \psi|_A = [T_\epsilon : A]\epsilon. \tag{2}$$

This implies that $\psi|_A$ is a multiple of ϵ. Also

$$m_\psi = \langle \psi, \mathrm{Ind}_A^{T_\epsilon}(\epsilon)\rangle_{T_\epsilon} = \langle \psi|_A, \epsilon\rangle_A.$$

Hence

$$\psi|_A = m_\psi \epsilon. \tag{3}$$

From Equations (2) and (3) we get

$$\sum_\psi m_\psi^2 = [T_\epsilon : A]. \tag{4}$$

We now claim that Equation (4) implies

$$\langle \mathrm{Ind}_A^G(\epsilon), \mathrm{Ind}_A^G(\epsilon)\rangle_G = \langle \mathrm{Ind}_A^{T_\epsilon}(\epsilon), \mathrm{Ind}_A^{T_\epsilon}(\epsilon)\rangle_{T_\epsilon}. \tag{5}$$

The basic point is that if we induce ϵ from A to G then all the "splitting" occurs at T_ϵ itself and the lemma follows easily from this. So it now suffices to prove Equation (5). To this end observe that

$$[T_\epsilon : A] = \sum m_\psi^2 = \langle \mathrm{Ind}_A^{T_\epsilon}(\epsilon), \mathrm{Ind}_A^{T_\epsilon}(\epsilon)\rangle_{T_\epsilon} \leq \langle \mathrm{Ind}_A^G(\epsilon), \mathrm{Ind}_A^G(\epsilon)\rangle_G$$

and using Mackey's theorem and the definition of T_ϵ we get that

$$\langle \mathrm{Ind}_A^G(\epsilon), \mathrm{Ind}_A^G(\epsilon)\rangle_G = [T_\epsilon : A].$$

This proves Equation (5). □

Lemma 2.2. *Let G be a finite group. Let $G = H \cdot A$ where A is an abelian normal subgroup and H is a subgroup such that $H \cap A = \{1\}$. Let ϕ be an irreducible character of H. Then*

$$\mathrm{Ind}_H^G(\phi) = \sum \mathrm{Ind}_{H_i}^G(\phi_i)$$

where the ϕ_i's are one dimensional characters of some subgroups H_i's of G. Further for all i one has $A \subset H_i$ and that the collection $\{\mathrm{Ind}_{H_i}^G(\phi_i)\}_i$ are all distinct irreducible characters of G.

Proof. Let

$$\mathrm{Ind}_H^G(\phi) = \sum_{\chi \in \widehat{G}} m_\chi \cdot \chi. \tag{6}$$

Then restricting Equation (6) to A and using Mackey's theorem and the hypothesis on H and A we get

$$\sum_{\chi \in \widehat{G}} m_\chi \cdot \chi|_A = \sum_{\epsilon \in \widehat{A}} \epsilon. \tag{7}$$

Hence for each $\chi \in \widehat{G}$ we have $m_\chi = 0$ or 1 and also $\langle \chi|_A, \epsilon\rangle_A = 0$ or 1. The lemma now follows from the method of "little groups" of Mackey and Wigner (see Section 8.2 of [10]). □

Lemma 2.3. *Let G be a finite group. Let A be an abelian normal subgroup and H a subgroup of G such that $G = H \cdot A$. Then*

$$\mathrm{Ind}_H^G(1_H) = \sum \mathrm{Ind}_{H_i}^G(\psi_i)$$

where ψ_i's are one dimensional characters of some subgroups H_i's. Further for all i one has $A \subset H_i$ and that the collection $\{\mathrm{Ind}_{H_i}^G(\psi_i)\}_i$ consists of distinct irreducible characters of G.

Proof. This lemma is almost exactly as Lemma 2.2 except that H may intersect A but then we decompose only that character of G which is the induction of the trivial character of H. Let

$$\mathrm{Ind}_H^G(\mathbf{1}_H) = \sum_{\chi \in \widehat{G}} m_\chi \cdot \chi. \tag{8}$$

As in Lemma 2.2, restricting to A we get

$$\sum_{\chi \in \widehat{G}} m_\chi \cdot \chi|_A = \sum_{\epsilon \in \widehat{A/H \cap A}} \epsilon. \tag{9}$$

We therefore get that $\mathrm{Ind}_H^G(\mathbf{1}_H)$ is multiplicity free and that if χ occurs in it then $\chi|_A = \epsilon$ for some one dimensional character ϵ of A which is trivial on $H \cap A$. The lemma now follows using Frobenius reciprocity and Lemma 2.1. □

Lemma 2.4. *Let G be a finite solvable group. Let H be a subgroup. Then*

$$\mathrm{Ind}_H^G(\mathbf{1}_H) = \mathbf{1}_G + \sum \mathrm{Ind}_{H_i}^G(\psi_i)$$

where ψ_i's are 1-dimensional, non-trivial characters of some subgroups H_i's of G.

Proof. We prove this lemma by a double induction on the cardinality $|G|$ of G and the index $[G : H]$ of H in G. We consider two cases : Suppose H contains a non-trivial abelian normal subgroup A of G then since $|G/A| < |G|$ by induction we have

$$\mathrm{Ind}_{H/A}^{G/A}(\mathbf{1}_{H/A}) = \mathbf{1}_{G/A} + \sum \mathrm{Ind}_{H_i/A}^{G/A}(\psi_i).$$

Inflating everything to G via the canonical map from G to G/A we get the lemma.

Now suppose H does not contain any such subgroup. Since G is solvable it definitely has an abelian normal subgroup A. Let $G_1 = HA$. So G_1 strictly contains H. If $G_1 = G$ then we are done by Lemma 2.3 and Frobenius reciprocity. If $G_1 \neq G$ then by induction, since $|G_1| < |G|$, we have

$$\mathrm{Ind}_H^{G_1}(\mathbf{1}_H) = \mathbf{1}_{G_1} + \sum \mathrm{Ind}_{H_i}^{G_1}(\psi_i).$$

Inducing now to G we get

$$\mathrm{Ind}_H^G(\mathbf{1}_H) = \mathrm{Ind}_{G_1}^G(\mathbf{1}_{G_1}) + \sum \mathrm{Ind}_{H_i}^G(\psi_i).$$

But now $[G : G_1] < [G : H]$. So applying the lemma for the subgroup G_1 of G in the above equation we are done. □

Let G be a finite solvable group. Let $G^0 := G$ and for all $i \geq 1$ define G^i to be $[G^{i-1}, G^{i-1}]$. This series is called the *derived series* of the group G. Since G is solvable, by definition this series is eventually trivial. Using this series one may define the *level of an irreducible character* χ, denoted $\ell(\chi)$, as the least non-negative integer m such that χ is trivial on G^m. For instance the level one characters are exactly the abelian characters which are non-trivial. We remark here that we use this notion of level only in Remarks 5.2, 5.5 and Theorem 5.3.

Lemma 2.5. *Let G be a finite solvable group. Let H be a subgroup of G. Let $\{G^i\}_{i\geq 0}$ denote the derived series of G. Let k be the least non-negative integer such that $G^{k+1} = \{1\}$. (Assume that G has more than one element.) Then for all $i \geq 1$*

$$\mathrm{Ind}_H^G(\mathbf{1}_H) = \mathrm{Ind}_{H \cdot G^i}^G(\mathbf{1}_{H \cdot G^i}) + \sum \mathrm{Ind}_{H_j}^G(\psi_j)$$

where ψ_j's are one dimensional characters of some subgroups H_j of G and j runs over some indexing set which might possibly be empty and all of these depending on H and i.

Proof. The proof is by a double induction on $|G|$ the cardinality of G and $[G : H]$ the index of H in G. Let H_1 denote the subgroup $H \cdot G^k$ of G. We take up various cases.

1. If $H = H_1$ then by induction argument, the lemma is true for the group G/G^k and the subgroup H/G^k and then inflate the result to G by the canonical map from G to G/G^k.

2. If $H \subset H_1 = G$ then the lemma is true which can be seen by using Lemma 2.3.

3. If $H \subset H_1 \subset G$ then by Lemma 2.3 we have

$$\mathrm{Ind}_H^{H_1}(1_H) = 1_{H_1} + (*)$$

where $(*)$ is a sum of monomial characters of H_1. Now inducing the above equation from H_1 to G we get

$$\mathrm{Ind}_H^G(1_H) = \mathrm{Ind}_{H_1}^G(1_{H_1}) + (**)$$

where $(**)$ is sum of monomial characters of G. By induction argument, since the index of H_1 in G is less than the index of H in G we have

$$\mathrm{Ind}_{H_1}^G(1_{H_1}) = \mathrm{Ind}_{H_1 \cdot G^i}^G(1_{H_1 \cdot G^i}) + (***)$$

where $(***)$ is a sum of monomial characters of G. The lemma follows in this case by using the fact that $H_1 \cdot G^i = H \cdot G^k \cdot G^i = H \cdot G^i$. □

3. Preliminaries on Artin L–functions

In this section we recall the definition of Artin L–functions and also mention some basic properties of the same. All of this is well known and no proofs are given but for pointing out some references.

Let $\mathbb{Q}$ denote the field of rational numbers. By a number field we mean a finite extension K over $\mathbb{Q}$. Let K/F be a Galois extension of number fields. Let $G = \mathrm{Gal}(K/F)$ be its Galois group. Let (ϕ, V) be a finite dimensional representation of G. An Artin L–function is a meromorphic function $L(s, \phi, K/F)$ attached to this data.

For each prime ideal $\mathfrak{P}$ of K let

$$D_{\mathfrak{P}} = \{\sigma \in G : \mathfrak{P}^\sigma = \mathfrak{P}\}$$

be its decomposition group and let

$$I_{\mathfrak{P}} = \{\sigma \in G : \sigma(x) \equiv x \pmod{\mathfrak{P}}\}$$

be its inertia group. The inertia group is a normal subgroup of the decomposition group and the quotient $D_{\mathfrak{P}}/I_{\mathfrak{P}}$ is a cyclic group generated by the Frobenius automorphism $\sigma_{\mathfrak{P}}$. This automorphism has the property that

$$\sigma_{\mathfrak{P}}(x) \equiv x^{N(\mathfrak{p})} \pmod{\mathfrak{P}}$$

where $\mathfrak{p} = \mathfrak{P} \cap F$ and N is the absolute norm from F to $\mathbb{Q}$. For any $\mathfrak{P}|\mathfrak{p}$, the Frobenius elements $\sigma_{\mathfrak{P}}$ are well defined modulo $I_{\mathfrak{P}}$. When the inertia is trivial, i.e. when $\mathfrak{P}$ is unramified in K/F, the conjugacy class of $\sigma_{\mathfrak{P}}$ is called the Artin symbol of $\mathfrak{p}$. The Artin L–function is defined by:

$$L(s, \phi, K/F) := \prod_{\mathfrak{p}} \det(1 - \phi(\sigma_{\mathfrak{p}})N(\mathfrak{p})^{-s}|_{V^{I_{\mathfrak{P}}}})^{-1}.$$

From the definition it is clear that the Artin L–function $L(s, \phi, K/F)$ depends only on the character χ_ϕ of ϕ and so we may talk of the Artin L–function $L(s, \chi, K/F)$ attached to a character χ of G.

We now recall the basic facts of Artin L–functions that we need. For a proof of these and other basic material on Artin L-functions the reader may refer to [6] and also the references therein.

Theorem 3.1 (E. Artin). *Let G be the Galois group of a Galois extension K/F of number fields. Let H be a subgroup of G. Let χ, χ_1 and χ_2 be characters of G. Let ψ be a character of H.*

(1) $L(s, \chi_1 \oplus \chi_2, K/F) = L(s, \chi_1, K/F)L(s, \chi_2, K/F)$.

(2) $L(s, \psi, K/K^H) = L(s, \mathrm{Ind}_H^G(\psi), K/F)$.

(3) *If χ is non-trivial and one dimensional then $L(s, \chi, K/F)$ extends to an entire function of s.*

(4) *If χ is a monomial character not containing the trivial character then $L(s, \chi, K/F)$ extends to an entire function of s.*

In the above theorem Statements (1) and (2) are easy to prove. Statement (3) is called the abelian reciprocity law. It is part of class field theory and is proved by showing that this L–function is also the L–function attached to a certain a Hecke character and this latter function is known to be entire by the work of Hecke (see for example, [5]). Statement (4) follows trivially from (2) and (3). The famous Artin conjecture predicts that if χ is irreducible, and unequal to the trivial character, then $L(s, \chi, K/F)$ extends to an entire function. The next theorem describes the relation of Artin L–functions with zeta functions of number fields.

Theorem 3.2. *Let G be the Galois group of a Galois extension K/F of number fields. Then*

(1) $L(s, \mathbf{1}_G, K/F) = \zeta_F(s)$.

(2) $L(s, \mathcal{R}_G, K/F) = \zeta_K(s)$.

(3) **Artin-Takagi factorization:**

$$\zeta_K(s) = \prod_{\chi \in \widehat{G}} L(s, \chi, K/F)^{\chi(1)}.$$

4. Variations of Dedekind Conjecture I

A generalization of the Uchida-van der Waall's Theorem

Theorem 4.1. *Let K/F be a solvable extension of number fields, i.e. K/F is a Galois extension of number fields and $G = \mathrm{Gal}(K/F)$ is solvable. Let χ be a one dimensional character of G. Then for every subgroup H of G and for every one dimensional character ψ of H, if $m(\chi, \psi) = \langle \chi, \mathrm{Ind}_H^G \psi \rangle_G$ then*

$$\frac{L(s, \mathrm{Ind}_H^G(\psi), K/F)}{L(s, \chi, K/F)^{m(\chi,\psi)}}$$

is regular at $s = s_0 \neq 1$.

Remark 4.2. Note that the theorem generalizes the Uchida-van der Waall's theorem. This states that if E/F is an extension of number fields and if K/F be the normal closure of E/F such that K/F is solvable then $\frac{\zeta_E(s)}{\zeta_F(s)}$ is an entire function of s. To see this apply the theorem to the case when $H = \mathrm{Gal}(K/E)$ and both χ and ψ are the trivial characters. The behaviour of the quotient at $s = 1$ is easily determined. For instance, if $\chi = \mathbf{1}_G$, then, $m(\chi, \psi) \neq 0$ only if $\psi = \mathbf{1}_H$ in which case the quotient is regular at $s = 1$. If $\psi = \mathbf{1}_H$ and $\chi \neq \mathbf{1}_G$, the quotient has a simple pole at $s = 1$.

Proof. Observe that if χ does not occur in $\mathrm{Ind}_H^G(\psi)$, i.e. if $m(\chi, \psi) = 0$ then the theorem follows from Statement (4) of Theorem 3.1 if ψ is non-trivial and if ψ is trivial this follows from holomorphy of Dedekind zeta functions outside $s = 1$. If χ does occur in $\mathrm{Ind}_H^G(\psi)$ then since both are one dimensional characters we have that $\chi|_H = \psi$ or in other words $m(\chi, \psi) = 1$. Now by Lemma 2.4 we have

$$\mathrm{Ind}_H^G(\mathbf{1}_H) = \mathbf{1}_G + \sum \mathrm{Ind}_{H_i}^G(\psi_i)$$

with the ψ_i's being one dimensional. Twisting the above equation by χ by which we mean tensoring by the representation χ or in terms of characters multiplying by χ we get

$$\chi \cdot \mathrm{Ind}_H^G(\mathbf{1}_H) = \chi + \sum \chi \cdot \mathrm{Ind}_{H_i}^G(\psi_i).$$

Since tensoring "commutes" with induction we have

$$\mathrm{Ind}_H^G(\psi) = \chi + \sum \mathrm{Ind}_{H_i}^G(\psi_i')$$

where $\psi_i' = \chi|_{H_i} \cdot \psi_i$. We hence get that

$$\frac{L(s, \mathrm{Ind}_H^G(\psi), K/F)}{L(s, \chi, K/F)^{m(\chi,\psi)}} = \prod_i L(s, \mathrm{Ind}_{H_i}^G(\psi_i'), K/F).$$

The right hand side is entire outside $s = 1$ which follows from Statement (4) of Theorem 3.1 and holomorphy of Dedekind zeta functions in this region. □

Applications to the Heilbronn Character I

We begin in this paragraph by recalling the Heilbronn character which studies the behaviour of Artin L–functions at an arbitrary but fixed point $s_0 \in \mathbb{C}$.

Let G be the Galois group of a Galois extension K/F of number fields. Let χ be an irreducible character of G. Let the order of the Artin L–function $L(s, \chi, K/F)$ at $s = s_0$ be denoted by $n(G, \chi)$, i.e.

$$n(G, \chi) = \operatorname*{ord}_{s=s_0} L(s, \chi, K/F). \tag{10}$$

The **Heilbronn character** is not really a character but a class function defined on G given by the formula:

$$\Theta_G(g) = \sum_{\chi \in \widehat{G}} n(G, \chi)\chi(g). \tag{11}$$

If Artin's conjecture is true and $s_0 \neq 1$, then Θ_G is a character. A basic observation due to Heilbronn (see [2]) is the following proposition.

Proposition 4.3 (Heilbronn). *Let G be the Galois group of a Galois extension K/F of number fields. Let H be a subgroup of G. Then*

$$\Theta_G|_H = \Theta_H.$$

Here Θ_H is the Heilbronn character for the extension K/K^H.

We now quote two theorems, the first is due to R. Foote and V. K. Murty (see [1]) and the second theorem is due to the first author (see [7]).

Theorem 4.4 (R. Foote-V. K. Murty). *Let G be the Galois group of a Galois extension K/F of number fields. Then*

$$\sum_{\chi\in\widehat{G}} n(G,\chi)^2 \leq (\operatorname*{ord}_{s=s_0} \zeta_K(s))^2.$$

Observe that the above theorem implies that $\zeta_K(s)/L(s,\chi,K/F)$ is regular at $s = s_0 \neq 1$.

Theorem 4.5 (M. Ram Murty). *Let G be the Galois group of a solvable Galois extension K/F of number fields. Then*

$$\sum_{\substack{\chi\in\widehat{G}\\ \chi\neq 1_G}} n(G,\chi)^2 \leq \left(\operatorname*{ord}_{s=s_0}\left(\frac{\zeta_K(s)}{\zeta_F(s)}\right)\right)^2.$$

We now give a generalization of Theorem 4.5. The following theorem essentially states that instead of cutting out the trivial character of G we can cut out any one dimensional character of G.

Theorem 4.6. *Let G be the Galois group of a solvable Galois extension K/F of number fields. Let χ_0 be any one dimensional character of G. Then*

$$\sum_{\substack{\chi\in\widehat{G}\\ \chi\neq\chi_0}} n(G,\chi)^2 \leq \left(\operatorname*{ord}_{s=s_0}\left(\frac{\zeta_K(s)}{L(s,\chi_0,K/F)}\right)\right)^2.$$

Remark 4.7. This clearly generalizes Theorem 4.5 by taking χ_0 to be the trivial character.

Proof. Define a "truncated" Heilbronn character with respect to χ_0 as

$$\Theta_G^{\chi_0}(g) := \sum_{\substack{\chi\in\widehat{G}\\ \chi\neq\chi_0}} n(G,\chi)\chi(g). \tag{12}$$

Taking norms on both sides, and using orthogonality of irreducible characters we get

$$|\Theta_G^{\chi_0}|^2 = \frac{1}{|G|}\sum_{g\in G}|\Theta_G^{\chi_0}(g)|^2 = \sum_{\chi\neq\chi_0} n(G,\chi)^2. \tag{13}$$

We now analyze $|\Theta_G^{\chi_0}(g)|$ a bit carefully. In what follows, the cyclic subgroup of G generated by an element $g \in G$ will be denoted $\langle g\rangle$. Using Proposition 4.3 we have

$$\begin{aligned}\Theta_G^{\chi_0}(g) &= \Theta_G(g) - n(G,\chi_0)\chi_0(g)\\ &= \Theta_{\langle g\rangle}(g) - n(G,\chi_0)\chi_0(g)\\ &= \sum_{\psi\in\widehat{\langle g\rangle}} \left(n(\langle g\rangle,\psi) - n(G,\chi_0)\langle\chi_0,\psi^G\rangle_G\right)\psi(g).\end{aligned}$$

Now we use Theorem 4.1 with $H = \langle g\rangle$ and any $\psi \in \widehat{\langle g\rangle}$ to get

$$n(\langle g\rangle,\psi) - n(G,\chi_0)\langle\chi_0,\psi^G\rangle_G \geq 0.$$

Using Theorem 3.2 and transitivity of induction we have

$$\begin{aligned}|\Theta_G^{\chi_0}(g)| &\leq \sum_{\psi\in\widehat{\langle g\rangle}} \left(n(\langle g\rangle,\psi) - n(G,\chi_0)\langle\chi_0,\psi^G\rangle_G\right)\\ &= \operatorname*{ord}_{s=s_0}\prod_{\psi\in\widehat{\langle g\rangle}}\left(\frac{L(s,\psi,K/K^{\langle g\rangle})}{L(s,\chi_0,K/F)^{\langle\chi_0,\psi^G\rangle_G}}\right)\\ &= \operatorname*{ord}_{s=s_0}\left(\frac{L(s,\mathcal{R}_{\langle g\rangle},K/K^{\langle g\rangle})}{L(s,\chi_0,K/F)^{\langle\chi_0,\mathcal{R}^G_{\langle g\rangle}\rangle_G}}\right)\\ &= \operatorname*{ord}_{s=s_0}\left(\frac{\zeta_K(s)}{L(s,\chi_0,K/F)^{\langle\chi_0,\mathcal{R}_G\rangle_G}}\right).\end{aligned}$$

Notice that since χ_0 is a one dimensional character we have $\langle \chi_0, \mathcal{R}_G \rangle_G = \chi_0(1) = 1$. Therefore for all $g \in G$ we have

$$|\Theta_G^{\chi_0}(g)| \leq \underset{s=s_0}{\mathrm{ord}} \left(\frac{\zeta_K(s)}{L(s, \chi_0, K/F)} \right).$$

Now the theorem follows by using Equation (13). □

5. Variations of Dedekind Conjecture II

A further Generalization of the Uchida-van der Waall's Theorem

Theorem 5.1. *Let G be the Galois group of a solvable Galois extension K/F of number fields. Let H be a subgroup of G and let ψ be any one dimensional character of H. Let S_ψ be the set of distinct one dimensional characters of G which on restricting to H are equal to ψ. Then*

$$\frac{L(s, \mathrm{Ind}_H^G(\psi), K/F)}{\prod_{\chi \in S_\psi} L(s, \chi, K/F)}$$

is an entire function of s.

Proof. Note that if S_ψ is empty then the theorem follows from statement (4) of Theorem 3.1. We may therefore assume that S_ψ is non-empty. Let $\chi_0 \in S_\psi$. By Lemma 2.5 for the case when $i = 1$ we have

$$\mathrm{Ind}_H^G(\mathbf{1}_H) = \mathrm{Ind}_{H\cdot[G,G]}^G(\mathbf{1}_{H\cdot[G,G]}) + \sum \mathrm{Ind}_{H_j}^G(\psi_j).$$

Multiplying this equation by χ_0 we get, using the fact that tensoring 'commutes' with induction, that

$$\mathrm{Ind}_H^G(\psi) = \mathrm{Ind}_{H\cdot[G,G]}^G(\chi_0|_{H\cdot[G,G]}) + (*)$$

where $(*)$ is a sum of monomial characters of G. Now note that $\mathrm{Ind}_{H\cdot[G,G]}^G(\chi_0|_{H\cdot[G,G]})$ is exactly $\sum_{\chi \in S_\psi} \chi$ from which the theorem follows. □

Remark 5.2. Note that in the proof of the above theorem, the main point is of a group theoretic nature and that is Lemma 2.5. But we have used that lemma to cut out only the abelian piece. The lemma itself suggests that we should be able to divide $L(s, \mathrm{Ind}_H^G(\psi), K/F)$ by all the $L-$functions corresponding to characters whose levels are not greater than i and prove that this quotient is entire. We are not able to prove this theorem at this point of time. However we can prove a weaker version which is the next theorem.

Theorem 5.3. *Let G be the Galois group of a solvable Galois extension K/F of number fields. Let H be a subgroup of G and let ψ be any one dimensional character of H. Let S_ψ be the set of distinct one dimensional characters of G which on restricting to H are equal to ψ. Let ψ' be the character of $H \cdot G^i$ which is obtained from ψ by trivially extending it on G^i. If S_ψ is non-empty then*

$$\frac{L(s, \mathrm{Ind}_H^G(\psi), K/F)}{\prod_{\ell(\chi) \leq i} L(s, \chi, K/F)^{\langle \chi, \psi'^G \rangle_G}}$$

is entire as a function of s.

Proof. By Lemma 2.5 we have

$$\mathrm{Ind}_H^G(\mathbf{1}_H) = \mathrm{Ind}_{H\cdot G^i}^G(\mathbf{1}_{H\cdot G^i}) + \sum \mathrm{Ind}_{H_j}^G(\psi_j).$$

Since S_ψ is non-empty, let χ_0 be an abelian character of G which on restricting to H gives ψ. Twist the above equation by χ_0 to get

$$\mathrm{Ind}_H^G(\psi) = \mathrm{Ind}_{H\cdot G^i}^G(\psi') + (*)$$

where $(*)$ is a sum of monomial characters. Now $\mathrm{Ind}_{H\cdot G^i}^G(\psi')$ is exactly the sum of characters of G occurring in $\mathrm{Ind}_H^G(\psi)$ which are trivial on G^i, i.e. have level less than or equal to i. The rest of the proof is as in Theorem 5.1. □

Applications to the Heilbronn Character II

Theorem 5.4. *Let G be the Galois group of a solvable Galois extension K/F of number fields. Let $[G, G]$ denote the commutator subgroup of*

G. Let K^{ab} denote the subfield of K/F fixed by $[G,G]$. Let S denote the set of distinct one dimensional characters of G. Then

$$\sum_{\chi\notin S} n(G,\chi)^2 \le \left(\operatorname*{ord}_{s=s_0}\left(\frac{\zeta_K(s)}{\zeta_{K^{\mathrm{ab}}}(s)}\right)\right)^2.$$

Proof. The proof of this theorem is exactly as the proof of Theorem 4.6. There we considered a truncated Heilbronn character by subtracting the term corresponding to one abelian character χ_0 and here we subtract all terms corresponding to abelian characters. So let

$$\Theta_G^1 := \Theta_G - \sum_{\chi\in S} n(G,\chi)\chi.$$

Taking norms on both sides we get

$$\sum_{\chi\notin S} n(G,\chi)^2 = \frac{1}{|G|}\sum_{g\in G}|\Theta_G^1(g)|^2.$$

Using Proposition 4.3 we get

$$\begin{aligned}\Theta_G^1(g) &= \Theta_G(g) - \sum_{\chi\in S} n(G,\chi)\chi(g)\\ &= \Theta_{\langle g\rangle}(g) - \sum_{\chi\in S} n(G,\chi)\chi(g)\\ &= \sum_{\psi\in\widehat{\langle g\rangle}}\left(n(\langle g\rangle,\psi) - \sum_{\chi\in S} n(G,\chi)\langle\chi,\psi^G\rangle_G\right)\psi(g).\end{aligned}$$

Theorem 5.1 states that for all $g\in G$ and for all $\psi\in\widehat{\langle g\rangle}$

$$n(\langle g\rangle,\psi) - \sum_{\chi\in S} n(G,\chi)\langle\chi,\psi^G\rangle_G \ge 0.$$

Using the fact that $\sum_{\chi\in S}\chi = \mathrm{Ind}_{[G,G]}^G(\mathbf{1}_{[G,G]})$ we have

$$\begin{aligned}|\Theta_G^1(g)| &\le \sum_{\psi\in\widehat{\langle g\rangle}}\left(n(\langle g\rangle,\psi) - \sum_{\chi\in S} n(G,\chi)\langle\chi,\psi^G\rangle_G\right)\\ &= \operatorname*{ord}_{s=s_0}\left(\frac{\prod_{\psi\in\widehat{\langle g\rangle}} L(s,\psi,K/K^{\langle g\rangle})}{\prod_{\psi\in\widehat{\langle g\rangle}}\prod_{\chi\in S} L(s,\chi,K/F)^{\langle\chi,\psi^G\rangle_G}}\right)\\ &= \operatorname*{ord}_{s=s_0}\left(\frac{\zeta_K(s)}{\prod_{\chi\in S} L(s,\chi,K/F)}\right)\\ &= \operatorname*{ord}_{s=s_0}\left(\frac{\zeta_K(s)}{L(s,\mathrm{Ind}_{[G,G]}^G(\mathbf{1}_{[G,G]}),K/F)}\right)\\ &= \operatorname*{ord}_{s=s_0}\left(\frac{\zeta_K(s)}{\zeta_{K^{\mathrm{ab}}}(s)}\right).\end{aligned}$$

The theorem follows. □

Remark 5.5. Just like Remark 5.2, a similar remark can be made here. Suppose K^i is the subfield of K/F left invariant by G^i the ith term in the derived series. Then Lemma 2.5 suggests that we should be able to prove that $\sum n(G,\chi)^2$ where the summation runs over all characters of G which are non-trivial on G^i, i.e. whose levels are greater than i, is bounded above by $\left(\mathrm{ord}_{s=s_0}\left(\frac{\zeta_K(s)}{\zeta_{K^i}(s)}\right)\right)^2$. We hope to return to this at a later occasion.

Corollary 5.6. *Let G be the Galois group of a solvable Galois extension K/F of number fields. Let K^{ab} be the subfield of K/F fixed by $[G,G]$. Then $\zeta_K(s)/\zeta_{K^{\mathrm{ab}}}(s)$ cannot have any simple zeros or poles.*

Proof. This follows from $\mathcal{R}_G = \mathrm{Ind}_{[G,G]}^G(\mathbf{1}) + \sum_{\chi\notin S}\chi(1)\chi$ and Theorems 3.1, 3.2 and 5.4. If $\zeta_K(s)/\zeta_{K^{\mathrm{ab}}}(s)$ had simple zeros or poles at $s=s_0$ (say), then, the right hand side of the inequality in Theorem 5.4 would be 1. But then, at most one term on the left hand side of the inequality can be non-zero. Since

$$\zeta_K(s)/\zeta_{K^{\mathrm{ab}}}(s) = \prod_{\chi\notin S} L(s,\chi)^{\chi(1)}$$

we see that at least one term in the product must contribute a zero or pole. Hence, exactly one term in the product has a zero or pole. It can't be a pole since the left hand side is entire. Thus, the quotient has a zero which must be multiple since $\chi(1) > 1$ for $\chi \notin S$. □

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