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On the error term in a Parseval type formula in the theory of Ramanujan expansions

M. Ram Murty [a,*], Biswajyoti Saha [b]

[a] *Department of Mathematics, Queen's University, Kingston, Ontario, K7L 3N6, Canada*

[b] *Institute of Mathematical Sciences, C.I.T. Campus, Taramani, Chennai, 600 113, India*

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Given two arithmetical functions f, g we derive, under suitable conditions, asymptotic formulas with error term, for the convolution sums $\sum_{n\le N} f(n)g(n+h)$, building on an earlier work of Gadiyar, Murty and Padma. A key role in our method is played by the theory of Ramanujan expansions for arithmetical functions.

1. Introduction and statement of theorems

The past century of mathematics, in particular number theory, has witnessed a number of developments in many different directions, originating from different articles by Srinivas Ramanujan. One of these is the theory of Ramanujan expansions. In 1918,

* Corresponding author.
E-mail addresses: murty@mast.queensu.ca (M.R. Murty), biswajyoti@imsc.res.in (B. Saha).

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Ramanujan [10] introduced certain sums of roots of unity. To be precise, for positive integers r, n, he defined the following sum,

$$c_r(n) := \sum_{a \in (\mathbb{Z}/r\mathbb{Z})^*} \zeta_r^{an}, \tag{1}$$

where ζ_r denotes a primitive r-th root of unity. These sums are now known as Ramanujan sums. Among other significant properties of Ramanujan sums, we list a few, which can be obtained from elementary observations. For a more elaborate account on Ramanujan sums, we refer the reader to the texts [12,11] and the survey articles [7,9]. We know:

a) For any r, n, $c_r(n) \in \mathbb{Z}$. This can be seen by reading the sum in (1) as the trace of the algebraic integer ζ_r^n.
b) For fixed n, $c_r(n)$ is a multiplicative function i.e. for r_1, r_2 with $\gcd(r_1, r_2) = 1$ we have $c_{r_1 r_2}(n) = c_{r_1}(n) c_{r_2}(n)$. This is essentially due to the fact that, for r_1, r_2 with $\gcd(r_1, r_2) = 1$, the fields $\mathbb{Q}(\zeta_{r_1})$ and $\mathbb{Q}(\zeta_{r_2})$ are linearly disjoint.
c) $c_r(\cdot)$ is a periodic function with period r. In fact, $c_r(n) = c_r(\gcd(n, r))$.
d) $c_r(n)$ can be expressed in terms of the Möbius function and written as

$$c_r(n) = \sum_{d \mid \gcd(n,r)} \mu(r/d) d.$$

Ramanujan used these sums to derive point-wise convergent series expansion of various arithmetical functions of the form $\sum_r a_r c_r(n)$, which are now called Ramanujan expansions. More precisely, given an arithmetical function f, we say f admits a Ramanujan expansion, if

$$f(n) = \sum_r \hat{f}(r) c_r(n)$$

for appropriate complex numbers $\hat{f}(r)$ and the series on the right hand side converges. Existence of such expansions for a given arithmetical function and their convergence properties have been studied extensively in the past, for example in [13,4,11]. However, we do not discuss these here. In this article we focus on a different theme.

In [3], Gadiyar, Murty and Padma have studied sums of the kind $\sum_{n \le N} f(n) g(n+h)$ for two arithmetical functions f, g with absolutely convergent Ramanujan expansions. They derived asymptotic formulas which are analogous to Parseval's formula in the case of Fourier series expansions. However it seems that the study of error term for these sums has not been carried out before. Under certain additional hypotheses we extend their results and provide explicit error terms. To be precise we prove,

Theorem 1. *Suppose that f and g are two arithmetical functions with absolutely convergent Ramanujan expansion:*

$$f(n) = \sum_r \hat{f}(r) c_r(n), \qquad g(n) = \sum_s \hat{g}(s) c_s(n),$$

respectively. Further suppose that

$$|\hat{f}(r)|, |\hat{g}(r)| \ll \frac{1}{r^{1+\delta}}$$

for some $\delta > 1/2$. Then, we have,

$$\sum_{n \le N} f(n) g(n) = N \sum_r \hat{f}(r) \hat{g}(r) \phi(r) + O\left(N^{\frac{2}{1+2\delta}} (\log N)^{\frac{5+2\delta}{1+2\delta}}\right).$$

Theorem 2. *Let f and g be two arithmetical functions with the same hypotheses as in Theorem 1 and h be a positive integer. Then we have,*

$$\sum_{n \le N} f(n) g(n+h) = N \sum_r \hat{f}(r) \hat{g}(r) c_r(h) + O\left(N^{\frac{2}{1+2\delta}} (\log N)^{\frac{5+2\delta}{1+2\delta}}\right).$$

In his article [10], Ramanujan showed that, for real variable $s > 0$,

$$\frac{\sigma_s(n)}{n^s} = \zeta(s+1) \sum_r \frac{c_r(n)}{r^{s+1}}, \tag{2}$$

where $\sigma_s(n) = \sum_{d|n} d^s$. Hence as an immediate corollary to Theorem 2 we get,

Corollary 1. *For $s, t > 1/2$ and any non-negative integer h, we have,*

$$\sum_{n \le N} \frac{\sigma_s(n)}{n^s} \frac{\sigma_t(n+h)}{(n+h)^t} = N \frac{\zeta(s+1)\zeta(t+1)}{\zeta(s+t+2)} \sigma_{-(s+t+1)}(h) + O\left(N^{\frac{2}{1+2\delta}} (\log N)^{\frac{5+2\delta}{1+2\delta}}\right),$$

where $\delta = \min\{s, t\}$.

This result, without the explicit error term was mentioned by Ingham in his article [5]. A proof of Ingham's result was obtained in [3].

Ramanujan [10] also showed that, for $s > 0$

$$\frac{\phi_s(n)}{n^s} \zeta(s+1) = \sum_r \frac{\mu(r)}{\phi_{s+1}(r)} c_r(n), \tag{3}$$

where $\phi_s(n) := n^s \prod_{\substack{p|n \\ p\ prime}} (1 - p^{-s})$. Note that, $\frac{1}{\phi_s(r)} \ll \frac{1}{r^s}$. Hence as a corollary we have,

Corollary 2. *If $s, t > 1/2$ and h is a non-negative integer, then*

$$\sum_{n\leq N} \frac{\phi_s(n)}{n^s}\frac{\phi_t(n+h)}{(n+h)^t} = N\Delta(h) + O\left(N^{\frac{2}{1+2\delta}}(\log N)^{\frac{5+2\delta}{1+2\delta}}\right),$$

where

$$\Delta(h) \\ = \prod_{p|h}\left[\left(1-\frac{1}{p^{s+1}}\right)\left(1-\frac{1}{p^{t+1}}\right)+\frac{p-1}{p^{s+t+2}}\right]\prod_{p\nmid h}\left[\left(1-\frac{1}{p^{s+1}}\right)\left(1-\frac{1}{p^{s+1}}\right)-\frac{1}{p^{s+t+2}}\right]$$

and $\delta = \min\{s, t\}$.

2. Preliminaries

For the sake of completeness we collect a few basic results about certain arithmetical functions. They can be found easily in the texts like [1,8,11]. We first list the results about the average order of Euler's phi function $\phi(\cdot)$ and the Möbius function $\mu(\cdot)$.

Proposition 1. *For any real number $x \geq 1$,*

$$\sum_{k\leq x}\phi(k) = \frac{3}{\pi^2}x^2 + O(x\log x).$$

Definition 1. The Mertens function $M(\cdot)$ is defined for all positive integers n as

$$M(n) := \sum_{k\leq n}\mu(k)$$

where $\mu(\cdot)$ is the Möbius function. The above definition can be extended to any real number $x \geq 1$ by defining,

$$M(x) := \sum_{k\leq x}\mu(k).$$

Essentially, the error term in the prime number theorem, due to de la Vallée-Poussin [2] gives us,

Proposition 2. *For any real number $x \geq 1$,*

$$M(x) = \sum_{k\leq x}\mu(k) = O\left(xe^{-c\sqrt{\log x}}\right),$$

where c is some positive constant.

Remark 1. This error term has been improved independently by Korobov [6] and Vinogradov [14] in 1958 to $O\left(xe^{-c\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}}\right)$ with $c > 0$, the best known to date. An equivalent statement of the Riemann hypothesis is $M(x) = O\left(x^{\frac{1}{2}+\epsilon}\right)$ for any $\epsilon > 0$.

Next we note the following very useful theorem known as *"partial summation"*.

Theorem 3. *Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers and $f(t)$ is a continuously differentiable function on $[1, x]$. Let $A(t) := \sum_{n\le t} a_n$. Then,*

$$\sum_{n\le x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)\, dt.$$

Let $d_k(n)$ be the number of ways writing n as a product of k numbers. Sometimes we write $d(n)$ to denote $d_2(n)$. Using Theorem 3 we can derive the following result about the average order of the arithmetical function $d_k(\cdot)$.

Proposition 3.

$$\sum_{n\le x} d_k(n) = \frac{x(\log x)^{k-1}}{(k-1)!} + O\left(x(\log x)^{k-2}\right).$$

We will make use of all these results from basic analytic number theory to extend the following theorems of [3].

Theorem 4 *(Gadiyar–Murty–Padma). Suppose that f and g are two arithmetical functions with absolutely convergent Ramanujan expansion:*

$$f(n) = \sum_r \hat{f}(r)c_r(n), \qquad g(n) = \sum_s \hat{g}(s)c_s(n),$$

respectively. Suppose that

$$\sum_{r,s} \left|\hat{f}(r)\hat{g}(s)\right| \gcd(r,s)d(r)d(s) < \infty.$$

Then, as N tends to infinity,

$$\sum_{n\le N} f(n)g(n) \sim N\sum_r \hat{f}(r)\hat{g}(r)\phi(r).$$

Theorem 5 *(Gadiyar–Murty–Padma). Suppose that f and g are two arithmetical functions as in Theorem 4 and h is a positive integer. Suppose further that*

$$\sum_{r,s} \left|\hat{f}(r)\hat{g}(s)\right|(rs)^{1/2}d(r)d(s) < \infty.$$

Then, as N tends to infinity,

$$\sum_{n\le N} f(n)g(n+h) \sim N\sum_{r} \hat{f}(r)\hat{g}(r)c_r(h).$$

Remark 2. Our hypotheses about $\hat{f}$, $\hat{g}$,

$$\left|\hat{f}(r)\right|, \left|\hat{g}(r)\right| \ll \frac{1}{r^{1+\delta}} \quad \text{for} \quad \delta > 1/2,$$

are extensions of the condition

$$\sum_{r,s} \left|\hat{f}(r)\hat{g}(s)\right|(rs)^{1/2}d(r)d(s) < \infty,$$

and include it as a consequence. Also note that, if we want to extend the hypothesis of Theorems 4, 5 in the form that we have in Theorems 1, 2 then $\delta > 1/2$ is the optimal choice.

To prove these theorems they prove certain lemmas about the sums of the from

$$\sum_{n\le N} c_r(n)c_s(n+h).$$

We will also make use the following.

Lemma 1.

$$\sum_{n\le N} c_r(n)c_s(n+h) = \delta_{r,s}Nc_r(h) + O(rs\log rs),$$

where $\delta_{\cdot,\cdot}$ denotes the Kronecker delta function.

Lemma 2.

$$\left|\sum_{n\le N} c_r(n)c_s(n+h)\right| \le d(r)d(s)\sqrt{rsN(N+h)}.$$

There are other related results in [3] which are of independent interest but not relevant to our work here.

3. Proofs of the theorems

3.1. Proof of Theorem 1

We start as it is done in [3]. Let U be a parameter tending to infinity which is to be chosen later. We have by absolute convergence of the series,

$$\sum_{n\le N} f(n)g(n) = \sum_{n\le N}\sum_{r,s} \hat{f}(r)\hat{g}(s)c_r(n)c_s(n)$$
$$= A + B, \quad \text{where}$$
$$A = \sum_{n\le N}\sum_{\substack{r,s\\ rs\le U}} \hat{f}(r)\hat{g}(s)c_r(n)c_s(n) \quad \text{and} \quad B = \sum_{n\le N}\sum_{\substack{r,s\\ rs> U}} \hat{f}(r)\hat{g}(s)c_r(n)c_s(n).$$

Interchanging summations and applying Lemma 1 (for $h = 0$) we get,

$$A = N\sum_{r^2\le U} \hat{f}(r)\hat{g}(r)\phi(r) + O(U\log U)$$
$$= C + D + O(U\log U), \quad \text{where}$$
$$C = N\sum_{r} \hat{f}(r)\hat{g}(r)\phi(r) \quad \text{and} \quad D = -N\sum_{r^2>U} \hat{f}(r)\hat{g}(r)\phi(r).$$

Note that C is the main term according to our theorem. Using the hypothesis we get,

$$D = O\left(N\sum_{r>\sqrt{U}} \frac{\phi(r)}{r^{2+2\delta}}\right).$$

Using Theorem 3 and Proposition 1 we get,

$$O\left(\sum_{r>\sqrt{U}} \frac{\phi(r)}{r^{2+2\delta}}\right) = O\left(\frac{1}{U^\delta} + \int_{\sqrt{U}}^{\infty} \frac{t^2}{t^{3+2\delta}}\,dt\right) = O\left(\frac{1}{U^\delta}\right).$$

Hence we obtain, $D = O\left(\frac{N}{U^\delta}\right)$.

Now, for B, interchanging the summation and applying Lemma 2 (for $h = 0$) we get,

$$B \ll N\sum_{rs>U} \frac{d(r)d(s)\sqrt{rs}}{(rs)^{1+\delta}}$$
$$= N\sum_{rs>U} \frac{d(r)d(s)}{(rs)^{1+(\delta-1/2)}}.$$

Note that, $\sum_{rs=t} d(r)d(s) = d_4(t)$. Using Theorem 3 and Proposition 3 we get,

$$O\left(\sum_{t>U}\frac{d_4(t)}{t^{1+(\delta-1/2)}}\right)=O\left(\frac{U(\log U)^3}{U^{1+(\delta-1/2)}}+\int_U^\infty \frac{t(\log t)^3}{t^{2+(\delta-1/2)}}\,dt\right)$$

$$=O\left(\frac{(\log U)^3}{U^{(\delta-1/2)}}+\int_U^\infty \frac{(\log t)^2}{t^{1+(\delta-1/2)}}\,dt\right),\quad \text{integrating by parts}$$

$$=O\left(\frac{(\log U)^3}{U^{(\delta-1/2)}}\right),\quad \text{integrating by parts multiple times.}$$

Hence we end up getting, $B=O\left(\frac{N(\log U)^3}{U^{(\delta-1/2)}}\right)$ and thus

$$\sum_{n\le N} f(n)g(n)=N\sum_r \hat{f}(r)\hat{g}(r)\phi(r)+O(U\log U)+O\left(\frac{N(\log U)^3}{U^{(\delta-1/2)}}\right).$$

To optimize the error term we choose, $U=N^{\frac{2}{1+2\delta}}(\log N)^{\frac{4}{1+2\delta}}$ and for this choice of U we get

$$\sum_{n\le N} f(n)g(n)=N\sum_r \hat{f}(r)\hat{g}(r)\phi(r)+O\left(N^{\frac{2}{1+2\delta}}(\log N)^{\frac{5+2\delta}{1+2\delta}}\right).$$

This concludes the proof of Theorem 1.

3.2. Proof of Theorem 2

This proof also starts off similarly and we get

$$\sum_{n\le N} f(n)g(n+h)=A+B,\quad \text{where}$$

$$A=\sum_{n\le N}\sum_{\substack{r,s\\ rs\le U}}\hat{f}(r)\hat{g}(s)c_r(n)c_s(n+h)\quad\text{and}\quad B=\sum_{n\le N}\sum_{\substack{r,s\\ rs>U}}\hat{f}(r)\hat{g}(s)c_r(n)c_s(n+h).$$

Likewise, the interchange of the summations and Lemma 1 yield, $A=C+D+O(U\log U)$, where $C=N\sum_r \hat{f}(r)\hat{g}(r)c_r(h)$, the main term and $D=-N\sum_{r^2>U}\hat{f}(r)\hat{g}(r)c_r(h)$. This time we have,

$$D=O\left(N\sum_{r>\sqrt{U}}\frac{c_r(h)}{r^{2+2\delta}}\right).$$

To apply Theorem 3 we need to know about $\sum_{r\le x} c_r(h)$. We write,

$$\sum_{r\le x} c_r(h)=\sum_{r\le x}\sum_{d|r,d|h}\mu(r/d)d=\sum_{\substack{k,d\\ dk\le x,d|h}} d\mu(k)=\sum_{d|h} d\sum_{k\le x/d}\mu(k).$$

The innermost sum is $M(x/d)$. Now due to Proposition 2 we get,

$$\sum_{r\le x} c_r(h) = O\left(\sum_{d|h} x e^{-c\sqrt{\log(x/d)}}\right) = O\left(x e^{-c\sqrt{\log x}}\epsilon(h)\right),$$

for some function $\epsilon(\cdot)$ of h which is bounded above by $e^{c\sqrt{\log h}}d(h)$. Hence using Theorem 3 we obtain,

$$D = O\left(N\epsilon(h)\left[\frac{\sqrt{U}e^{-c\sqrt{\log\sqrt{U}}}}{U^{1+\delta}} + \int_{\sqrt{U}}^{\infty}\frac{te^{-c\sqrt{\log t}}}{t^{3+2\delta}}\,dt\right]\right)$$
$$= O\left(N\epsilon(h)\left[\frac{1}{U^{1/2+\delta}} + \int_{\sqrt{U}}^{\infty}\frac{1}{t^{2+2\delta}}\,dt\right]\right) = O\left(\frac{N\epsilon(h)}{U^{1/2+\delta}}\right).$$

For B, we apply Lemma 2. A similar calculation yields, $B = O\left(\frac{\sqrt{N(N+h)}(\log U)^3}{U^{(\delta-1/2)}}\right)$. Hence for fixed h we can write,

$$\sum_{n\le N} f(n)g(n+h) = N\sum_r \hat{f}(r)\hat{g}(r)c_r(h) + O(U\log U) + O\left(\frac{N(\log U)^3}{U^{(\delta-1/2)}}\right)$$

and then similarly as before, choosing $U = N^{\frac{2}{1+2\delta}}(\log N)^{\frac{4}{1+2\delta}}$ we conclude,

$$\sum_{n\le N} f(n)g(n+h) = N\sum_r \hat{f}(r)\hat{g}(r)c_r(h) + O\left(N^{\frac{2}{1+2\delta}}(\log N)^{\frac{5+2\delta}{1+2\delta}}\right).$$

3.3. Proof of Corollary 1

Note that using (2) we get,

$$\sum_r \frac{c_r(h)}{r^{s+t+2}} = \frac{1}{\zeta(s+t+2)}\frac{\sigma_{s+t+1}(h)}{h^{s+t+1}} = \frac{\sigma_{-(s+t+1)}(h)}{\zeta(s+t+2)}.$$

This completes the proof of Corollary 1.

3.4. Proof of Corollary 2

Since $\mu(r)$, $\phi_s(r)$, $c_r(h)$ are multiplicative functions of r and the Möbius function is supported at the square free numbers, we get,

$$\sum_r \frac{\mu^2(r)}{\phi_{s+1}(r)\phi_{t+1}(r)}c_r(h) = \prod_{p\ prime}\left(1 + \frac{\mu^2(p)}{\phi_{s+1}(p)\phi_{t+1}(p)}c_p(h)\right).$$

Now, $c_p(h) = \begin{cases} p-1 & \text{if } p|h, \\ -1 & \text{if } p \nmid h. \end{cases}$ Hence we obtain,

$$\frac{1}{\zeta(s+1)\zeta(t+1)} \sum_r \frac{\mu^2(r)}{\phi_{s+1}(r)\phi_{t+1}(r)} c_r(h)$$
$$= \prod_{p|h} \left[\left(1 - \frac{1}{p^{s+1}}\right)\left(1 - \frac{1}{p^{t+1}}\right) + \frac{p-1}{p^{s+t+2}} \right]$$
$$\times \prod_{p\nmid h} \left[\left(1 - \frac{1}{p^{s+1}}\right)\left(1 - \frac{1}{p^{s+1}}\right) - \frac{1}{p^{s+t+2}} \right].$$

This completes the proof of Corollary 2.

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