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FACTORING NEWPARTS OF JACOBIANS OF CERTAIN MODULAR CURVES

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ABSTRACT. We prove a conjecture of Yamauchi which states that the level N for which the new part of $J_0(N)$ is $\mathbb{Q}$-isogenous to a product of elliptic curves is bounded. We also state and partially prove a higher-dimensional analogue of Yamauchi's conjecture. In order to prove the above results, we derive a formula for the trace of Hecke operators acting on spaces $S^{new}(N,k)$ of newforms of weight k and level N. We use this trace formula to study the equidistribution of eigenvalues of Hecke operators on these spaces. For any $d \geq 1$, we estimate the number of normalized newforms of fixed weight and level, whose Fourier coefficients generate a number field of degree less than or equal to d.

1. INTRODUCTION

For a positive integer N, let $\Gamma_0(N)$ denote the set of all matrices in $SL_2(\mathbb{Z})$ such that N divides the lower left entry and let $X_0(N)$ denote the quotient of the extended upper half plane by the action of $\Gamma_0(N)$. It can be viewed as an algebraic curve defined over $\mathbb{Q}$. Let $J_0(N)$ denote the Jacobian variety of $X_0(N)$. For a positive divisor M of N, let New_M denote the set of normalized newforms of weight 2 with respect to $\Gamma_0(M)$ and for each $f(z) = \sum_{n=1}^{\infty} a_n(f)e^{2\pi i n z} \in New_M$, let $K_f = \mathbb{Q}(\{a_n(f)_{n\geq 1}\})$. By the work of Shimura [21], one can associate to each $f \in New_M$, an abelian variety quotient A_f of $J_0(N)$ such that if f and $f' \in New_M$ are Galois conjugates, then A_f is $\mathbb{Q}$-isogenous to $A_{f'}$. (We denote this as $A_f \sim_{\mathbb{Q}} A_{f'}$.) Later, Ribet [17] showed that each A_f is $\mathbb{Q}$-simple. Ribet also observed (see Proposition 3.2 of [2]) that the converse is true, that is, for f and $f' \in New_M$, $A_f \sim_{\mathbb{Q}} A_{f'}$ only if f and f' are Galois conjugates. Thus, by the work of Ribet and Shimura, we have the decomposition

$$J_0(N) \sim_{\mathbb{Q}} \bigoplus_{M|N} \bigoplus_{f\in New_M/G_{\mathbb{Q}}} A_f^{n_f},$$

where $G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, n_f denotes the number of positive divisors of N/M, and the dimension of A_f is equal to $[K_f : \mathbb{Q}]$. For a fixed positive integer d, the problem of determining all levels N for which all $\mathbb{Q}$-simple factors of $J_0(N)$ are of dimension less than or equal to d has been well investigated. By a result of Serre (see Theorem 7 of [20]), we know that there are only finitely many levels N for which this happens.

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In [5], Cohen listed out all the odd N's for the case $d = 1$, that is, all the N's such that all $\mathbb{Q}$-simple factors of $J_0(N)$ are elliptic curves. Building up on Cohen's work, Yamauchi [22] listed out all the N's (even and odd) such that all $\mathbb{Q}$-simple factors of $J_0(N)$ are elliptic curves. In [15], we made Serre's result effective, that is, for any $d \geq 1$, we found an effectively computable constant $B(d)$, which depends only on d such that if all $\mathbb{Q}$-simple factors of $J_0(N)$ are of dimension $\leq d$, then $N \leq B(d)$.

In this article, our focus is on the new part of $J_0(N)$, denoted as $J_0^{new}(N)$, for which we have the decomposition

$$J_0^{new}(N) \sim_{\mathbb{Q}} \bigoplus_{f \in New_N/G_{\mathbb{Q}}} A_f.$$

In [22], Yamauchi conjectured that if $J_0^{new}(N)$ is $\mathbb{Q}$-isogenous to a product of elliptic curves, then N is bounded above by 1800. In section 5 of this paper, we essentially prove this conjecture in the following theorem:

Theorem 1. *If $J_0^{new}(N)$ is $\mathbb{Q}$-isogenous to a product of elliptic curves, then N is bounded above by an absolute and effectively computable constant.*

We can formulate a higher dimensional analogue of Yamauchi's conjecture, as follows:

Conjecture 2. *For any $d \geq 1$, there are only finitely many positive integers N such that $J_0^{new}(N)$ is $\mathbb{Q}$-isogenous to a product of $\mathbb{Q}$-simple abelian varieties of dimension less than or equal to d.*

Let p be a prime number. In [19], Royer proved that for any sufficiently large number N coprime to p, $J_0^{new}(N)$ has a $\mathbb{Q}$-simple factor of dimension $\gg \sqrt{\log\log N}$, where the implied constant depends on p. This immediately implies that for any $d \geq 1$, there are only finitely many N's not divisible by p such that all $\mathbb{Q}$-simple factors of $J_0^{new}(N)$ are of dimension less than or equal to d. This result for the case $p = 2$ was also independently proved by Lim [12]. However, neither Royer's nor Lim's results are effective. In the direction of Conjecture 2, we prove the following theorem in section 5:

Theorem 3. *Let p be a fixed prime. For any integer $d \geq 1$, if N is coprime to p and $J_0^{new}(N)$ is isogenous to a product of $\mathbb{Q}$-simple abelian varieties of dimension less than or equal to d, then N is bounded above by a constant $B(p, d)$ that depends only on p and d. More precisely,*

$$log\, 2N \leq 311206 d^2 2^{4d^2} p^{d^2/2} \log p.$$

Remark 4. *The advantage of Theorem 3 over the previous results is that it clearly reveals the dependence of the bound for N on p and d. This explicit bound enables us to address Yamauchi's conjecture (for the case $d = 1$) for* **all** *N. The constant in Theorem 1 obtained by our current methods turns out to be much larger than the one conjectured by Yamauchi. We relegate the sharpening of this constant to future work.*

In the next section, we state some results that lead to Theorems 1 and 3 and are also of independent interest.

2. Preliminaries

Let $S(N,k)$ be the space of cusp forms of weight k ($k \geq 2$ is an even integer) with respect to $\Gamma_0(N)$ and for any integer $n \geq 1$, let $T_n(N,k)$ be the n-th Hecke operator acting on $S(N,k)$. Let $s(N,k)$ be the dimension of $S(N,k)$. Let p be a fixed prime and let (k_λ, N_λ) be a sequence of pairs of positive integers with $N_\lambda + k_\lambda \to \infty$ provided that p does not divide N_λ and k_λ is even. By the theorem of Deligne proving the Ramanujan-Petersson inequality, we know that the eigenvalues of $T_p(N_\lambda, k_\lambda)$ lie in the interval

$$[-2p^{\frac{k_\lambda-1}{2}}, 2p^{\frac{k_\lambda-1}{2}}].$$

In his celebrated 1997 paper [20], Serre proved that with $N_\lambda + k_\lambda \to \infty$ as above, the family of eigenvalues of the normalized p-th Hecke operator

$$T'_p(N_\lambda,\, k_\lambda) = \frac{T_p(N_\lambda,\, k_\lambda)}{p^{(k_\lambda-1)/2}}$$

is equidistributed in the interval $[-2,2]$ with respect to the measure

$$\mu_p = \begin{cases} \frac{p+1}{\pi}\frac{(1-x^2/4)^{1/2}}{(p^{1/2}+p^{-1/2})^2-x^2}dx & \text{if } x \in [-2,2] \\ 0 & \text{otherwise.} \end{cases}$$

In [15], we proved the following effective version of Serre's equidistribution theorem (see [15], p. 701): Let N be a positive integer and p be a prime not dividing N. Let $\{\lambda_{p,i}\}_{1\leq i\leq s(N,k)}$ denote the eigenvalues of $T_p(N,k)$. For an interval $[\alpha,\beta] \subset [-2,2]$ and for any positive integer $M \geq 1$,

$$\left| \left\{ \#i : \frac{\lambda_{p,i}}{p^{\frac{k-1}{2}}} \in [\alpha,\beta] \right\} - s(N,k)\int_\alpha^\beta \mu_p(x) \right|$$

$$\ll \frac{s(N,k)}{M+1} + (p^{3M/2}2^{\nu(N)}\sigma_0(N)\sqrt{N}\log p)M\log M,$$

where $\nu(N)$ denotes the number of prime divisors of N and $\sigma_0(N)$ denotes the number of positive divisors of N. This effective version of Serre's theorem has several applications. Most notably, in [15] we estimate for a given d and prime p not dividing N, the number of eigenvalues of T_p of degree less than or equal to d. We then determine an effectively computable constant B_d such that if $J_0(N)$ is isogenous to a product of $\mathbb{Q}$-simple abelian varieties of dimensions less than or equal to d, then $N \leq B_d$.

In this note, we restrict our attention to the eigenvalues of T_p acting on $S^{new}(N,k)$, the space of newforms of weight k and level N. We denote the dimension of $S^{new}(N,k)$ by $s^{new}(N,k)$. In [20], Serre also shows that the eigenvalues of T'_p acting on $S^{new}(N_\lambda,k_\lambda)$ are equidistributed with respect to μ_p. We obtain precise error terms in the effective equidistribution of eigenvalues of T_p acting on $S^{new}(N,k)$ and apply our effective results to study the factorization of $J_0^{new}(N)$.

We first compute a formula for the trace of T_n acting on $S^{new}(N,k)$, which we denote as $T_n^{new}(N,k)$. This is an important ingredient in obtaining effective equidistribution results. Although a formula for $s^{new}(N,k)$ is now known by the work of Martin [13], the trace formula for $T_n^{new}(N,k)$ in closed form has so far not been computed. It is therefore worthwhile to fill this gap in the literature. In section 3, we prove the following theorem:

Theorem 5. *Let n be a positive integer coprime to N. The trace of the Hecke operator $T_n^{new}(N,k)$ is given by*

$$\begin{cases} n^{(k/2-1)} \cdot \frac{k-1}{12} N B_1(N) & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise} \end{cases}$$

$$-\frac{1}{2} \sum_{t\in\mathbb{Z},\, t^2<4n} \frac{\varrho^{k-1} - \overline{\varrho}^{k-1}}{\varrho - \overline{\varrho}} \sum_f h_w\left(\frac{t^2-4n}{f^2}\right) B_2(N)_f$$

$$-{\sum}'_{\substack{d|n \\ 0<d\leq\sqrt{n}}} d^{k-1} B_3(N)_d + \begin{cases} \mu(N)\sum_{t|n} t & \text{if } k=2. \\ 0 & \text{otherwise.} \end{cases},$$

where

- *$B_1(N)$ is a multiplicative function such that for a prime power p^r,*

$$B_1(p^r) = \begin{cases} 1-\frac{1}{p} & \text{if } r=1, \\ 1-\frac{1}{p}-\frac{1}{p^2} & \text{if } r=2, \\ \left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right) & \text{if } r\geq 3. \end{cases}$$

- *ϱ and $\overline{\varrho}$ are the zeroes of the polynomial $x^2 - tx + n$*
- *The inner sum in the second term runs over all positive divisors f of t^2-4n such that $(t^2-4n)/f^2 \in \mathbb{Z}$ is congruent to 0 or $1 \bmod 4$*
- *$h_w(\Delta)$ is the class number of the imaginary quadratic order of discriminant Δ divided by 2 (resp. 3) if the discriminant is -4 (resp. -3)*
- *For a positive integer f, $B_2(N)_f$ is a multiplicative function of N such that*

$$B_2(p)_f = \begin{cases} p-1 & \text{if } p|f, \\ -1+\left(\frac{t^2-4n}{p}\right) & \text{otherwise} \end{cases},$$

  where $\left(\frac{*}{p}\right)$ denotes the Legendre symbol. If $N = p^r$ for some $r \geq 2$ and $p^b||f$, then

$$B_2(p^r)_f = \sum_{i=r-2}^{r} \sigma_0^{-1}(p^{r-i}) \frac{\psi(p^i)}{\psi(p^{i-\min\{i,b\}})} M(t,n,p^{i+\min\{i,b\}}),$$

  where

$$\psi(N) = N\prod_{p|N}\left(1+\frac{1}{p}\right),$$

  $\sigma_0^{-1}(N)$ denotes the Dirichlet inverse of $\sigma_0(N)$ and $M(t,n,p^{i+\min\{i,b\}})$ denotes the number of elements of $(\mathbb{Z}/p^i\mathbb{Z})^*$ which lift to solutions of $x^2 - tx + n \equiv 0 \bmod (p^{i+\min\{i,b\}})$
- *The dash on top of the summation in the third term of $Tr\, T_n^{new}(N,k)$ indicates that if there is a contribution from the term $d = \sqrt{n}$, it should be multiplied by $1/2$*
- *$B_3(N)_d$ is a multiplicative function of N such that for a prime power p^r,*

$$B_3(p^r)_d = \begin{cases} -\phi(p^{\frac{r-2}{2}}), & \text{if } r \text{ is even and } p^{\frac{r-2}{2}} || \left(\frac{n}{d}-d\right), \\ \phi(p^{\frac{r}{2}}) - \phi(p^{\frac{r-2}{2}}), & \text{if } r \text{ is even and } p^{\frac{r}{2}} | \left(\frac{n}{d}-d\right), \\ 0, & \text{otherwise} \end{cases}$$

Henceforth, let p be a prime not dividing N. For any closed interval $[\alpha,\beta]\subset[-2,2]$, let $E^{new}(p,N,k,[\alpha,\beta])$ denote the number of eigenvalues (counted with multiplicity) of the normalized Hecke operator $T_p'^{new}(N,k)$ lying in the interval $[\alpha,\beta]$. Also, for any $x\in[-2,2]$, let us define $\theta_x\in[0,\pi]$ such that $2\cos\theta_x=x$. In section 4, we prove the following theorems:

Theorem 6. *For any interval $[\alpha,\beta]\subset[-2,2]$, and for any positive integer M, we have*

$$\left|E^{new}(p,N,k,[\alpha,\beta])-s^{new}(N,k)\int_\alpha^\beta\mu_p\right|\leq 4\frac{s^{new}(N,k)}{M+1}+19p^{2M+1}4^{\nu(N)}$$

$$+2\sum_{1\leq m\leq M}\min\left(\frac{\theta_\alpha-\theta_\beta}{2\pi},\frac{1}{m\pi}\right)\left(10p^{2m+1}4^{\nu(N)}+\left|s^{new}(N,k)-NB_1(N)\frac{k-1}{12}\right|\mathcal{C}_m\right),$$

where, for every $m\geq 1$,

$$\mathcal{C}_m=\begin{cases}0, & \text{if } m \text{ is odd,}\\ \frac{1}{p}, & \text{if } m=2,\\ \frac{1}{p^{m-2}}-\frac{1}{p^m}, & \text{if } m\geq 4 \text{ is even.}\end{cases}$$

Theorem 7. *Let $\{a_{p,i}\}$, $1\leq i\leq s^{new}(N,k)$ denote the family of eigenvalues of $T_p^{new}(N,k)$. For any $\alpha\in[-2p^{(k-1)/2},2p^{(k-1)/2}]$ and for any $c>3$,*

$$\#\{1\leq i\leq s^{new}(N,k):\, a_{p,i}=\alpha\}\leq 8cs^{new}(N,k)\frac{\log p}{\log kN}+237\left(\frac{6c}{(c-3)e}\right)^2\frac{kN}{(\log kN)^2}.$$

Theorem 8. *For any positive integer d, let*

$$s^{new}(N,k,p)_d=\#\{1\leq i\leq s^{new}(N,k):\,[\mathbb{Q}(a_{p,i}):\mathbb{Q}]\leq d\}.$$

For any $c>3$, we have

$$s^{new}(N,k,p)_d\leq\mathcal{C}_{d,p,k}\left\{8cs^{new}(N,k)\frac{\log p}{\log kN}+237\left(\frac{6c}{(c-3)e}\right)^2\frac{kN}{(\log kN)^2}\right\},$$

where

$$C_{d,p,k}=d^2\prod_{i=1}^d\left(2\binom{d}{i}\left(2p^{\frac{k-1}{2}}\right)^i+1\right).$$

Remark 9. *The above bound is non-trivial only if we can choose p sufficiently small so that $C_{d,p,k}\ll(\log kN)^a$ for some $a<1$.*

3. Eichler-Selberg Trace Formula and modifications

The Eichler-Selberg trace formula describes the trace of T_n acting on $S(N,k)$. Following the presentation of this formula in ([11], p. 370), for every integer $n\geq 1$, Tr $T_n(N,k)=\sum_{i=1}^4 A_i(n,N,k)$, where A_i's are as follows :

$$A_1(n,N,k)=\frac{k-1}{12}\psi(N)\begin{cases}n^{(k/2-1)} & \text{if } n \text{ is a square,}\\ 0 & \text{otherwise.}\end{cases}$$

$$A_2(n,N,k)=-\frac{1}{2}\sum_{t\in\mathbb{Z},\,t^2<4n}\frac{\varrho^{k-1}-\overline{\varrho}^{k-1}}{\varrho-\overline{\varrho}}\sum_f h_w\left(\frac{t^2-4n}{f^2}\right)\mu(t,f,n).$$

Here, ϱ, $\overline{\varrho}$, f and $h_w(\Delta)$ are as in Theorem 5 and

$$\mu(t,f,n) = \frac{\psi(N)}{\psi\left(\frac{N}{N_f}\right)} M(t,n,NN_f),$$

where $N_f = \gcd\,(N,f)$ and $M(t,n,NN_f)$ denotes the number of elements of $(\mathbb{Z}/N\mathbb{Z})^*$ which lift to solutions of $x^2 - tx + n \equiv 0 \bmod NN_f$.

$$A_3(n,N,k) = -{\sum}'_{\substack{d|n,\\ 0<d\leq\sqrt{n}}} d^{k-1} F(N)_d,$$

where $F(N)_d$ is a multiplicative function of N defined as

$$F(N)_d = \sum_{\substack{c|N \\ \gcd\,(c,\frac{N}{c})|\,\frac{n}{d}-d}} \phi\left(\gcd\left(c, \frac{N}{c}\right)\right).$$

The dash on top of the summation defining $A_3(n,N,k)$, just as in Theorem 1, indicates that if there is a contribution from the term $d = \sqrt{n}$, it should be multiplied by $\frac{1}{2}$.

$$A_4(n,N,k) = \begin{cases} \sum_{t|n,\, t>0} t & \text{if } k=2, \\ 0 & \text{otherwise.} \end{cases}$$

By the Atkin- Lehner decomposition [1], we know that

$$S(N,k) = \bigoplus_{d|N} \bigoplus_{a|\left(\frac{N}{d}\right)} i_{a,d}(S^{new}(d,k)), \tag{1}$$

where, for positive integers a and d such that $ad|N$, $i_{a,d}$ denotes the embedding $f(z) \mapsto f(az)$ of $S(d,k)$ into $S(N,k)$. Thus,

$$s(N,k) = \sum_{d|N} \sigma_0\left(\frac{N}{d}\right) s^{new}(d,k).$$

From this, we deduce that

$$s^{new}(N,k) = \sum_{d|N} \sigma_0^{-1}\left(\frac{N}{d}\right) s(d,k),$$

where $\sigma_0^{-1}(m)$ denotes the inverse of $\sigma_0(m)$ with respect to Dirichlet convolution, that is $\sigma_0^{-1}(N)$ is a multiplicative function defined as follows on prime powers:

$$\sigma_0^{-1}(p^r) = \begin{cases} 1, & \text{if } r = 0 \text{ or } 2, \\ -2 & \text{if } r = 1, \\ 0 & \text{if } r > 2. \end{cases}$$

This idea was utilised by Martin ([13], Theorem 1) to derive a closed-form and computationally efficient formula for $s^{new}(N,k)$ from previously known formulae for $s(N,k)$. This inversion technique can also be used to derive an explicit formula for Tr $T_n^{new}(N,k)$, using the Eichler-Selberg trace formula for Tr $T_n(N,k)$, provided n is coprime to N. We observe that if $(n,N) = 1$ and $f(z) \in S^{new}(d,k)$ for some

$d|N$, then T_n has the same eigenvalue on $f(z)$ as it does on $i_{a,d}(f)$. Thus, by the Atkin-Lehner decomposition (1), we get that

$$\text{Tr } T_n(N,k) = \sum_{d|N} \sigma_0\left(\frac{N}{d}\right) \text{Tr } T_n^{new}(d,k). \tag{2}$$

Thus,

$$\text{Tr } T_n^{new}(N,k) = \sum_{d|N} \sigma_0^{-1}\left(\frac{N}{d}\right) \text{Tr } T_n(d,k).$$

The following can be easily verified by checking them at prime powers:

$$\sum_{d|N} \sigma_0^{-1}(N/d)\psi(d) = NB_1(N), \sum_{d|N} \sigma_0^{-1}(N/d)\mu(t,f,d) = B_2(N)_f,$$

$$\sum_{r|N} \sigma_0^{-1}(N/r)F(r)_d = B_3(N)_d \text{ and } \sum_{d|N} \sigma_0^{-1}(N/d) = \mu(N).$$

Combining the above facts, we are able to obtain $\sum_{d|N} \sigma_0^{-1}(N/d)A_i(n,d,k)$ for each i. This proves Theorem 5.

Remark 10. *The idea of performing Möbius inversion on equation (2) was also utilised by Hamer* [9] *to obtain the trace of $T_n^*(N,k)$, the Hecke operator T_n acting on*

$$S^*(N,k) = \bigoplus_{d|N} S^{new}(N,k)$$

provided $(n,N)=1$. She observes that equation (2) can also be written as

$$Tr\ T_n(N,k) = \sum_{d|N} Tr\ T_n^*(d,k).$$

She then obtains a trace formula for $T_n^*(N,k)$ for squarefree N by Möbius inversion.

Remark 11. *For $n=1$, the trace of $T_n^{new}(N,k)$ is equal to $s^{new}(N,k)$. Thus, Proposition 5 for $n=1$ gives the same formula as Theorem 1 of* [13]*. Since $B_3(N)_1 = 0$ if N is not a square, we immediately deduce that:*

$$\left| s^{new}(N,k) - NB_1(N)\frac{k-1}{12} \right| \leq \begin{cases} \frac{\sqrt{N}}{2} + \frac{7}{12}2^{\nu(N)} + 1 & \text{if } N \text{ is a square} \\ \frac{7}{12}2^{\nu(N)} + 1 & \text{otherwise} \end{cases} \tag{3}$$

We now state two results which will help us to obtain explicit bounds on the terms of the trace formula.

Proposition 12. *For a positive integer $N > 0$, let*

$$H(N) = \sum_{d^2|N} h_w\left(\frac{-N}{d^2}\right),$$

where the sum runs over all positive divisors d of N such that $-N/d^2 \in \mathbb{Z}$ is congruent to 0 or $1 \pmod 4$. Then,

$$\sum_{t^2<4n} H(4n-t^2) = 2\sigma_1(n) - \lambda(n) + \frac{1}{6},$$

where

$$\lambda(n) = \sum_{d|n} min\left(d, \frac{n}{d}\right) \text{ and } \sigma_1(n) = \sum_{\substack{d|n \\ d>0}} d.$$

Proof. The above recursion formula is due to Kronecker and Gierster (see [6], pages 108 and 127). It was also proved by Eichler (see [7], equation (6)) by interpreting the numbers $H(N)$ in terms of the number of fixed points of Hecke's correspondences on the Riemann surface $X_0(2)$. We have stated this formula as it appears in Theorem 5.3.8 of [4]. □

Proposition 13. *Suppose a and b are integers such that $a^2 - 4b \neq 0$. Given an integer K, the number of solutions $\bmod K$ of the congruence $x^2 - ax + b \equiv 0 \bmod K$ is less than or equal to $2^{\nu(K)}\sqrt{|a^2-4b|}$.*

Proof. This proposition forms the content of Huxley's paper [10]. □

From Propositions 12 and 13, we deduce the following:

Proposition 14. *Let p be a prime not dividing N. Then, for any $m > 0$,*

$$\left| Tr\, {T'_{p^m}}^{new}(N,k) - NB_1(N)\frac{k-1}{12}\begin{cases} p^{-m/2} & \text{if } m \text{ is even,} \\ 0 & \text{otherwise.}\end{cases} \right| \leq 8p^{2m+1}4^{\nu(N)} + p^{m/2+1}.$$

Proof. Inserting Proposition 13 in $B_2(N)_f$, we deduce that

$$|B_2(N)_f| \leq 4^{\nu(N)}\psi(f)\sqrt{4n-t^2}.$$

Combining this estimate for $B_2(N)$ with Proposition 12, we get that

$$\left| \frac{1}{2}\sum_{t\in\mathbb{Z},\, t^2<4n} \frac{\varrho^{k-1}-\bar{\varrho}^{k-1}}{\varrho-\bar{\varrho}} \sum_f h_w\left(\frac{t^2-4n}{f^2}\right) B_2(N) \right| \leq 8n^{\frac{k+1}{2}} 4^{\nu(N)}\sigma_0(n).$$

We also observe that for any $n > 1$,

$$\sideset{}{'}\sum_{\substack{d|n \\ 0<d\leq\sqrt{n}}} d^{k-1}B_3(N)_d \leq \sum_{\substack{d|n \\ 0<d\leq\sqrt{n}}} d^k.$$

In particular, taking $n = p^m$, $m \geq 1$ and dividing by $(p^m)^{\frac{k-1}{2}}$, we prove Proposition 14. □

4. Effective equidistribution results

Let us consider the compact set $[0,1]$ of $\mathbb{R}$. Let $A_1, A_2, \cdots$ be a sequence of finite nonempty multisets of $[0,1]$ with $\#A_n \to \infty$ as $n \to \infty$, where $\#A_n$ denotes the cardinality of A_n. We say that $\{A_n\}$ is equidistributed with respect to a measure μ if for every $A \subseteq [0,1]$,

$$\lim_{n\to\infty} \frac{\#\{t \in A_n : t \in A\}}{\#A_n} = \mu(A).$$

Let $e(x) := e^{2\pi i x}$. Suppose that the *Weyl limits* of this sequence

$$c_m = \lim_{n\to\infty} \frac{1}{\#A_n}\sum_{t\in A_n} e(mt),$$

exist for every $m \in \mathbb{Z}$ and $\sum_{m=1}^{N}|c_m|^2 = \mathrm{o}(N)$. Then, by a generalization of the Wiener-Schoenberg theorem, (see [14], Theorem 11.3.3, p.181) the measure μ is given by $F(-x)dx$, where $F(x) = \sum_{m=-\infty}^{\infty} c_m e(mx)$. Let $||\mu||$ be the supremum of $|F(x)|$ for $x \in [0,1]$. In [15], we proved the following all-purpose effective equidistribution theorem:

Theorem 15. *For any* $I = [a, b] \subseteq [0, 1]$, *let* $N_I(V) := \#\{t \in A_V : t \in I\}$ *and let* $D_{I,V}(\mu) := |N_I(V) - V\mu(I)|$. *Then,*

$$D_{I,V}(\mu) \leq \frac{V||\mu||}{M+1} + \sum_{\substack{m=-M \\ m\neq 0}}^{M} \left(\frac{1}{M+1} + \ min\ \left(b-a, \frac{1}{\pi|m|}\right)\right) \left|\sum_{n=1}^{V} e(mx_n) - Vc_m\right|,$$

if V *and* M *are natural numbers.*

For a prime p not dividing N, let $\left\{\frac{a_{p,i}}{p^{\frac{k-1}{2}}}\right\}$, $1 \leq i \leq s^{new}(N, k)$ denote the family of eigenvalues of $T'^{new}_p(N, k)$. For each i, choose $\theta_{p,i} \in [0, \pi]$ such that

$$\frac{a_{p,i}}{p^{\frac{k-1}{2}}} = 2\cos\theta_{p,i}.$$

We study the distribution of the sequence $\pm\frac{\theta_{p,i}}{2\pi} \pmod 1$ $(1 \leq i \leq s^{new}(N, k))$. The Weyl limits, in this case, are

$$\begin{aligned} c_m &= \lim_{\substack{N+k\to\infty \\ (p,N)=1 \\ k \text{ even}}} \frac{1}{2s^{new}(N,k)} \sum_{i=1}^{s^{new}(N,k)} \left\{ e\left(\frac{m\theta_{p,i}}{2\pi}\right) + e\left(-\frac{m\theta_{p,i}}{2\pi}\right)\right\} \\ &= \lim_{\substack{N+k\to\infty \\ (p,N)=1 \\ k \text{ even}}} \frac{1}{2s^{new}(N,k)} \sum_{i=1}^{s^{new}(N,k)} 2\cos(m\theta_{p,i}). \end{aligned}$$

For $m = 1$,

$$\sum_i 2\cos(m\theta_{p,i}) = \text{Tr } T'^{new}_p(N, k),$$

and for $m \geq 2$,

$$\sum_i 2\cos(m\theta_{p,i}) = \text{Tr } {T'_{p^m}}^{new}(N, k) - \text{Tr } {T'_{p^{m-2}}}^{new}(N, k).$$

Thus, from Proposition 14 and equation (3), we deduce that the Weyl limits c_m's are given by $c_0 = 1$, $c_m = 0$ for m odd and for m even,

$$c_m = \frac{1}{2}\left(\frac{1}{p^{|m|/2}} - \frac{1}{p^{(|m|-2)/2}}\right).$$

We also observe that for any $m \geq 1$,

$$\sum_i 2\cos m\theta_{p,i} - 2s^{new}(N, k)c_m$$

$$= \begin{cases} \text{Tr } T'^{new}_p(N, k), & \text{if } m = 1, \\ \text{Tr } {T'_{p^2}}^{new}(N, k) - \frac{s^{new}(N,k)}{p}, & \text{if } m = 2, \\ \text{Tr } {T'_{p^m}}^{new}(N, k) - \text{Tr } {T'_{p^{m-2}}}^{new}(N, k) + \mathcal{C}_m s^{new}(N, k), & \text{if } m \geq 3, \end{cases}$$

where the $\mathcal{C}_m$'s are as defined in Theorem 6.

By substituting the above information in Theorem 15, we get that for any $[\alpha, \beta] \subset [-2, 2]$ and for any $M \geq 1$,

$$\left|E^{new}(p, N, k, [\alpha, \beta]) - s^{new}(N, k)\int_\alpha^\beta \mu_p\right| \leq \frac{2s^{new}(N, k)}{M+1}$$

$$+\sum_{1\le |m|\le M}\left(\frac{1}{M+1}+\min\left(\frac{\theta_\alpha-\theta_\beta}{2\pi},\frac{1}{\pi|m|}\right)\right)\left|\sum_i 2\cos m\theta_{p,i}-2s^{new}(N,k)c_m\right|$$

$$\le \frac{2s^{new}(N,k)}{M+1}+19p^{2M+1}4^{\nu(N)}+\frac{2}{M+1}\left(\frac{2}{p}-\frac{1}{p^{M/2}}\right)\left|s^{new}(N,k)-NB_1(N)\frac{k-1}{12}\right|$$

$$+2\sum_{m=1}^{M}\min\left(\frac{\theta_\alpha-\theta_\beta}{2\pi},\frac{1}{m\pi}\right)\left(11p^{2m+1}4^{\nu(N)}+\left|s^{new}(N,k)-NB_1(N)\frac{k-1}{12}\right|\right)\mathcal{C}_m.$$

This proves Theorem 6. Thus, for a fixed $\alpha\in[-2p^{(k-1)/2},2p^{(k-1)/2}]$ and for any $M\ge 1$,

$$\#\{1\le i\le s^{new}(N,k):a_{p,i}=\alpha\}\le 4\frac{s^{new}(N,k)}{M+1}+19p^{2M+1}4^{\nu(N)}.$$

By a result of Ramanujan (see equation (200) of [16]), we know that

$$\sigma_0(N)\le 8\left(\frac{3N}{35}\right)^{1/3}.$$

Since $2^{\nu(N)}\le\sigma_0(N)$, we deduce that

$$4^{\nu(N)}\le 64\left(\frac{3N}{35}\right)^{2/3}.$$

For any $c>3$ we choose

$$M+1=\left[\frac{\frac{1}{c}\log kN}{2\log p}\right].$$

By elementary calculus, we know that $\log x\le x^a/ae$, for any $a>0$ and $x\ge 1$. Thus,

$$(kN)^{\frac{2}{3}+\frac{1}{c}}\le\left(\frac{6c}{(c-3)e}\right)^2\frac{kN}{(\log kN)^2}.$$

Thus, for any $c>3$,

$$\#\{1\le i\le s^{new}(N,k):a_{p,i}=\alpha\}\le 8cs^{new}(N,k)\frac{\log p}{\log kN}+237\left(\frac{6c}{(c-3)e}\right)^2\frac{kN}{(\log kN)^2}.$$

This proves Theorem 7. We now recall the following Proposition from [15]:

Proposition 16. *For a positive integer d, and a real number $K>0$, the number of algebraic integers α of degree d and $H(\alpha)\le K$ is at most*

$$\prod_{i=1}^{d}\left(2\binom{d}{i}K^i+1\right)$$

where $H(\alpha)$ is the maximum of the absolute values of all conjugates of α.

Proof. This is Proposition 30 of [15]. □

If α is an eigenvalue of $T_p^{new}(N,k)$, then the absolute values of all conjugates of α are bounded above by $2p^{(k-1)/2}$. Thus, taking $K=2p^{\frac{k-1}{2}}$ in Proposition 16, we deduce that if α is an eigenvalue of $T_p^{new}(N,k)$ such that $[\mathbb{Q}(\alpha):\mathbb{Q}]\le d$, then α can take at most

$$C_{d,p,k}=d^2\prod_{i=1}^{d}\left(2\binom{d}{i}\left(2p^{\frac{k-1}{2}}\right)^i+1\right)$$

values. This, combined with Theorem 7 proves Theorem 8.

By the work of Atkin and Lehner [1], $S^{new}(N,k)$ has a unique basis $\{f_i\}_{1\le i\le s^{new}(N,k)}$ consisting of normalized newforms. For any such form $f_i(z) = \sum_{n=1}^{\infty} a_n(f_i)e^{2\pi i n z}$, let

$$K_i = \mathbb{Q}(\{a_n(f_i)\}_{n\ge 1}).$$

K_i is a finite extension of $\mathbb{Q}$. For any integer $d\ge 1$, we define

$$s^{new}(N,k)_d = \#\{1\le i\le s^{new}(N,k) : [K_i:\mathbb{Q}] = d\}.$$

Clearly, for any prime p not dividing N,

$$\sum_{r=1}^{d} s^{new}(N,k)_r \le s^{new}(N,k,p)_d.$$

In particular, taking $c = 20$ in Theorem 8, we deduce that for every $d\ge 1$,

$$\sum_{r=1}^{d} s^{new}(N,k)_r \le C_{d,p,k}\left(160 s^{new}(N,k)\frac{\log p}{\log kN} + 1617\frac{kN}{(\log kN)^2}\right). \tag{4}$$

We observe that

$$\mathcal{C}_{d,p,k} \le d^2 2^{4d^2}\left(p^{\frac{k-1}{2}}\right)^{\frac{d(d+1)}{2}},$$

and for $N\ge 2$ (see, for example, Theorem 3.1(g) of [3]),

$$\phi(N) \ge \frac{N\log 2}{\log 2N}.$$

Moreover, by a result of Halberstadt and Kraus ([8], Proposition B.2.(b)), $3\phi(N)/200 \le s^{new}(N,2)$ for all $N\ge 61$. From this, we deduce that,

$$\frac{2N}{(\log 2N)^2} \le \frac{2}{\log 2}\frac{\phi(N)}{\log 2N} \le \frac{400}{3\log 2}\frac{s^{new}(N,2)}{\log 2N}.$$

Thus, putting $k = 2$ in equation (4), we get

$$\sum_{r=1}^{d} s^{new}(N,2)_r \le 311206 d^2 2^{4d^2}(\sqrt{p})^{d^2}\log p\frac{s^{new}(N,2)}{\log 2N}. \tag{5}$$

The inequality in (5), for the case $k = 2$ can be applied to prove Theorems 3 and 1. We describe this application in the next section.

5. New parts of Jacobians of modular curves

We know that the dimension of $J_0^{new}(N)$ is equal to $s^{new}(N,2)$. Also, by the work of Ribet and Shimura, the number of $\mathbb{Q}$-simple factors of $J_0^{new}(N)$ of dimension d is equal to $s^{new}(N,k)_d/d$. Thus, if $J_0^{new}(N)$ is $\mathbb{Q}$-isogenous to a product of $\mathbb{Q}$-simple abelian varieties of dimension less than or equal to d, then

$$s^{new}(N,2) = \sum_{r=1}^{d} r\frac{s^{new}(N,2)_r}{r}.$$

Combining this with equation (5), we deduce that if p does not divide N and all $\mathbb{Q}$-simple factors of $J_0^{new}(N)$ are of dimension less than or equal to d, then

$$s^{new}(N,2) \le 311206 d^2 2^{4d^2} p^{d^2/2}\log p\frac{s^{new}(N,2)}{\log 2N}. \tag{6}$$

This proves Theorem 3. As an immediate consequence of this theorem, we observe that if d is the largest dimension of the $\mathbb{Q}$-simple factors of $J_0^{new}(N)$, then

$$\log 2N \leq (C_p)^{d^2},$$

where C_p is a constant depending on p. Thus, the largest dimension of the $\mathbb{Q}$-simple factors of $J_0^{new}(N)$ is $\gg_p \sqrt{\log\log N}$. This proves a result previously obtained by Royer ([19], Theorem 1.1) using different methods. However, Royer's result was not effective. The advantage of Theorem 3 is that it explicitly shows the dependence of this implied constant on p. This explicit determination has an important application. We note that if all $\mathbb{Q}$-simple factors of $J_0^{new}(N)$ are elliptic curves, then taking $d = 1$ in equation (6), we get that for any prime p not dividing N,

$$\log 2N \leq 4979296 p^{1/2} \log p.$$

How small a prime p can we choose which does not divide N? If N is odd, we choose $p = 2$ and deduce that $\log 2N \leq 4979296\sqrt{2}\log 2$. If N is even, the prime number theorem tells us that for a sufficiently large N, there is a prime $p < 2\log N$ not dividing N. Using effective bounds for the Chebyshev functions (see equation (5.2) in Theorem 6 of [18]), one can show that for $N > 1,319,007$, there is a prime $p < 1.1\log N$ not dividing N. Thus, we have

$$\frac{(\log N)^{1/2}}{\log\log N} \leq D,$$

for an absolute constant D. This proves Theorem 1. However, our bound for N is clearly much bigger than 1800. In future work, we would like to refine the constants in our estimates in order to yield a better bound.

We also observe that the above idea does not prove Conjecture 2 for $d > 1$. This is because if we choose $p \leq 1.1\log N$, the inequality

$$\log 2N \leq 311206 d^2 2^{4d^2} (1.1\log N)^{d^2/2} \log(1.1\log N)$$

gives us an upper bound for N only if $d^2/2 < 1$, that is, only if $d = 1$. We relegate addressing Conjecture 2 in full generality to future research.

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References

1. A. O. L. Atkin and J. Lehner, *Hecke operators on* $\Gamma_0(m)$, Math. Ann., Vol. 185 (1970), 134-160.
2. M. H. Baker, E. González-Jiménez, J. González and B. Poonen, *Finiteness results for modular curves of genus at least 2*, Amer. J. Math., Vol. 127 (2005), 1325–1387.

3. D. Bressoud and S. Wagon, *A course in computational number theory*, Key College Publishing, Emeryville, CA, 2000.
4. H. Cohen, *A course in computational algebraic number theory*, Graduate Texts in Mathematics, 138, Springer, 1993.
5. H. Cohen, *Sur les N tels que $J_0(N)$ soit $\mathbb{Q}$-Isogène à un produit de courbes elliptiques*, preprint, available on http://www.ufr-mi.u-bordeaux.fr/∼cohen
6. L. E. Dickson, *History of the theory of numbers, Vol. 3*, Chelsea Publishing Co, New York, 1966.
7. M. Eichler, *On the class number of imaginary quadratic fields and the sums of divisors of natural numbers*, J. Indian Math. Soc., Vol. 19 (1955), 153-180.
8. E. Halberstadt and A. Kraus, *Courbes de Fermat: résultats et problèmes*, J. Reine Angew. Math., Vol. 548 (2002), 167–234.
9. C. Hamer, *A formula for the traces of the Hecke operators on certain spaces of newforms*, Arch. Math. (Basel), Vol. 70 (1998), no. 3, 204-210.
10. M. N. Huxley, *A note on polynomial congruences*, Recent progress in Analytic Number Theory, Vol.1, Durham, 1979, Academic Press, 1981, 193-196.
11. A. Knightly and C. Li, *Traces of Hecke operators*, Mathematical Surveys and Monographs, 133, American Mathematical Society, Providence, RI, 2006.
12. C-W. Lim, *Decomposition of $S_k(\Gamma_0(N))$ over $\mathbb{Q}$ and variants of partial nim*, PhD thesis, University of California at Berkeley, 2005.
13. G. Martin, *Dimensions of the spaces of cusp forms and newforms on $\Gamma_0(N)$ and $\Gamma_1(N)$*, J. Number Theory, Vol. 112 (2005), no. 2, 197-240.
14. M. Ram Murty, *Problems in Analytic Number theory*, Graduate Texts in Mathematics, 206, Springer, 2007, Second edition.
15. M. Ram Murty and K. Sinha, *Effective equidistribution of eigenvalues of Hecke operators*, J. Number Theory, Vol. 129 (2009), no. 3, 681–714.
16. S. Ramanujan, *Highly composite numbers*, Collected Papers of Srinivasa Ramanujan, AMS Chelsea Publ., Providence, RI, 2000, 78-128.
17. K. A. Ribet, *Twists of modular forms and endomorphisms of abelian varieties*, Math. Ann. Vol. 253 (1980), no. 1, 43-62.
18. J. B. Rosser and L. Schoenfeld, *Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$*, Math. Comp., Vol. 29 (1975), 243-269.
19. E. Royer, *Facteurs $\mathbb{Q}$-simples de $J_0(N)$ de grande dimension et de grand rang*, Bull. Soc. Math. France., Vol. 128 (2000), no. 2, 219–248.
20. J-P Serre, *Répartition asymptotique des valeurs propres de l'opérateur de Hecke T_p*, J. Amer. Math. Soc., Vol. 10 (1997), no. 1, 75-102.
21. G. Shimura, *On the factors of the Jacobian variety of a modular function field*, J. Math. Soc. Japan., Vol. 25 (1973), 523-544.
22. T. Yamauchi, *On $\mathbb{Q}$-simple factors of Jacobian varieties of modular curves*, Yokohama Math. J., Vol. 53, no. 2 (2007), 149-160.

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