

Remark. Admittedly, these two suggestions for evaluating the infinite series might, for some readers, not qualify as *proof*. For the sceptical reader we recommend the exercise of evaluating the infinite series as a partial fraction expansion. In fact, by analogy with the elementary, real analysis derivation of $\sum_{k=-\infty}^{\infty} 1/(k\pi + x)^2 = 1/\sin^2 x$ presented in [3, p. 198], we get

$$\begin{aligned} \sum_{k=-\infty}^{\infty} \frac{\epsilon}{\epsilon^2 + (k\pi + x)^2} &\leftarrow \frac{1}{n} \sum_{k=-n/2}^{n/2-1} \frac{\sinh(2\epsilon/n)/2}{\sinh^2(\epsilon/n) + \sin^2((k\pi + x)/n)} \\ &= \frac{\sinh(2\epsilon)/2}{\sinh^2 \epsilon + \sin^2 x} \end{aligned}$$

as $n = 2^v \rightarrow \infty$. Here, the identity follows recursively from the simple case $v = 1$. A little endurance in massaging the hyperbolic and trigonometric functions helps us to conclude the proof with the identity

$$\frac{1}{1 + \eta^2} \cdot \frac{\sinh(2\epsilon)/2}{\sinh^2 \epsilon + \sin^2(\arctan \eta)} = \frac{\tanh \epsilon}{\tanh^2 \epsilon + \eta^2}.$$

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Pick's Theorem via Minkowski's Theorem

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In 1899, Georg Alexander Pick published one of his most beautiful theorems [12]. This theorem provided a formula for easily calculating the area of a planar polygon P whose vertices have integer coordinates. Such a polygon is called a *lattice polygon*, since the

points in the plane with integer coordinates are sometimes called *lattice points*. In fact, Pick's theorem states that if I is the number of lattice points in the interior of P and B is the number of lattice points on its boundary, then the area A of P is given by

$$A = I + \frac{B}{2} - 1.$$

The beauty of this formula stems from its simplicity and depth. It has been successfully explained to twelve-year-olds, and yet mathematicians are still researching some of its consequences today.

Pick was born into a Jewish family in Vienna on August 10, 1859. He received his Ph.D. from the University of Vienna under the supervision of Leo Koenigsberger in 1880. He spent most of his working life at the University of Prague, where his colleagues and students praised his excellence at both research and teaching. In 1910, Albert Einstein applied to become a professor of theoretical physics at the University of Prague. Pick found himself on the appointments committee and was the driving force in getting Einstein accepted. For the brief period that Einstein was at Prague, he and Pick were the closest friends. They were both talented violinists and frequently played together. In 1929, Pick retired and moved back to his hometown of Vienna. Nine years later, Austria was annexed by Germany. In an attempt to escape the Nazi regime, Pick returned to Prague. However, on July 13, 1942, he was captured and transported to the Theresienstadt concentration camp. He passed away there thirteen days later, at the age of 82 [8].

Pick's formula first came to popular attention in 1969 (seventy years after Pick published it) in Steinhaus's book *Mathematical Snapshots* [15]. Since then, mathematicians have come up with a variety of different proofs using tools ranging from Euler's formula to the Weierstrass \wp -function [1], [2], [4], [9], [16]. We have also seen important connections between this formula (and its generalization) and ideas in combinatorics, algebraic geometry, and complex analysis [2], [4].

In this note, we present a new proof of Pick's theorem via Minkowski's convex body theorem. We begin by reviewing Minkowski's theorem for the benefit of the reader (see [10] or [11, p. 76] giving Siegel's proof of it). Recall that a *region* R is an open, connected set; R is a *symmetric region* if x in R implies that $-x$ is in R .

Theorem (Minkowski's Convex Body Theorem). *A bounded, symmetric, convex region C in \mathbb{R}^n with volume greater than 2^n contains at least one nonzero lattice point.*

Proof. We first define a function φ on \mathbb{R}^n by $\varphi(x) = 1$ if x belongs to $C/2 = \{y/2 : y \in C\}$ and $\varphi(x) = 0$ otherwise. We then set

$$\Phi(x) = \sum_{\lambda \in \mathbb{Z}^n} \varphi(x + \lambda).$$

The function Φ is bounded and integrable, so

$$\begin{aligned} \int_{[0,1]^n} \Phi(x) dx &= \int_{[0,1]^n} \sum_{\lambda \in \mathbb{Z}^n} \varphi(x + \lambda) dx = \int_{\mathbb{R}^n} \varphi(x) dx \\ &= \text{vol} \left(\frac{C}{2} \right) = \frac{\text{vol}(C)}{2^n} > 1. \end{aligned}$$

Since Φ is integer-valued, we must have $\Phi(x) \geq 2$ for some x . In other words, there are two distinct points $P + \lambda_1$ and $P + \lambda_2$ in $C/2$ that differ by a lattice point. Call

Now let (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) be the coordinates of A , B , and C , respectively. Then the area of $\triangle ABC$ is the absolute value of

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Because the determinant here is a nonzero integer, this expression is always at least $1/2$. ■

We can now prove Pick's theorem in general, by showing that the formula $I + B/2 - 1$ is additive and decomposing a general polygon into elementary triangles.

Theorem (Pick's Theorem). *Suppose that P is a convex lattice polygon. If B is the number of lattice points on the boundary of P (including vertices) and I is the number of lattice points in the interior of P , then the area of P is given by the expression $I + B/2 - 1$.*

Proof. The lemma establishes that every elementary triangle satisfies this formula. Now suppose that the lattice polygon can be decomposed into two lattice polygons P_1 and P_2 for which the Pick formula holds. We show that P must satisfy this formula as well. Let I_X and B_X signify the number of lattice points in the interior and boundary, respectively, of a planar compact set X . Note that

$$I_P = I_{P_1} + I_{P_2} + D - 2,$$

where D is the number of points on the common boundary of P_1 and P_2 . Also,

$$B_P = B_{P_1} + B_{P_2} - 2D + 2.$$

From this it follows that

$$\begin{aligned} I_P + \frac{B_P}{2} - 1 &= I_{P_1} + \frac{B_{P_1}}{2} - 1 + I_{P_2} + \frac{B_{P_2}}{2} - 1 \\ &= \text{area of } P_1 + \text{area of } P_2 = \text{area of } P. \end{aligned}$$

Since any lattice polygon can be decomposed into elementary triangles, the proof is complete. ■

Minkowski's theorem is true for spaces of any dimension, and this leads us to hope that we might be able to generalize Pick's theorem to higher dimensions. However, there is no simple generalization of Pick's theorem even to three dimensions. To see this, notice that the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(0, 1, r)$ is an elementary tetrahedron for all positive integers r but has volume $r/6$. Thus, we cannot write down a formula for the volume of a polyhedron in terms of integer lattice points.

J. E. Reeve was able to generalize Pick's theorem to three dimensions with the help of auxiliary lattices [13]. One such auxiliary lattice is the lattice of points (a, b, c) in \mathbb{R}^3 such that $(2a, 2b, 2c)$ is in \mathbb{Z}^3 . Reeve was able to calculate the volume of a lattice polyhedron by counting the number of integer lattice points and auxiliary lattice points in the interior and boundary of that polyhedron.

Generalizations to even higher dimensions are possible by way of so-called Ehrhart polynomials [6], [7], [14]. If P is a d -dimensional polytope in \mathbb{R}^d , then define $F_P(n) = \#\{\text{lattice points in } nP\}$. It is a theorem of Ehrhart that $F_P(n) = a_d n^d + a_{d-1} n^{d-1} + \cdots + a_0$ is a polynomial in n of degree d , called the *Ehrhart polynomial* of P . Ehrhart was also able to determine some of the coefficients of this polynomial. He proved that $a_d = \text{vol}(P)$ is the volume of P , $a_{d-1} = (1/2)\text{vol}(\delta P)$ is half the surface area of P normalized with respect to the sublattice on each facet of P , and a_0 is the Euler characteristic of P . Note that in the case of a two-dimensional polytope in \mathbb{R}^2 , we can retrieve Pick's formula by evaluating $F_P(1) = \text{vol}(P) + B/2 + 1$ and noting that $F_P(1) = I + B$ in the notation of the paper. The other coefficients of the polynomial remained mysterious for some time. Recently, Morelli and others have been able to link the other coefficients to Todd classes of toric varieties [3], [5].

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