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Some Ω-results for Ramanujan's τ-function

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§ 1. Introduction

I take great pleasure and pride in speaking to you, on this sacred land of India, about a very important arithmetical function, first discovered by the famous Indian mathematician Srinivasa Ramanujan. He had the foresight and intuition to recognize the importance of modular forms in the theory of numbers. He investigated [9], in rather great detail, one modular form, now called Ramanujan's cusp form, defined by

$$\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}, \quad q = e^{2\pi i z},$$

This can be expanded in a power series in q and we have

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n$$

and τ is called Ramanujan's τ-function. We have

$$\tau(1) = 1, \quad \tau(2) = -24, \quad \tau(3) = 252$$

$$\tau(4) = -1472, \quad \tau(5) = 4830 .$$

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Ramanujan made two conjectures:

(i) τ is a multiplicative function,

(ii) for each prime p, $|\tau(p)| \leq 2p^{\frac{11}{2}}$.

The first conjecture was proved by Mordell [7] in 1917 and generalized in a beautiful way by Hecke [5]. The second conjecture was generalized by Petersson [8] to include other modular forms. The full Ramanujan-Petersson conjecture was settled by Deligne [2] in 1976.

It is known that

$$\tau(p^{\alpha}) = p^{11\alpha/2} \frac{\sin(\alpha+1)\theta_p}{\sin\theta_p}$$

where $\theta_p \in (0, 2\pi)$, by the result of Deligne. As τ is multiplicative, it follows that

$$\tau(n) = O(n^{11/2} \exp(\frac{c \log n}{\log \log n})).$$

It is conjectured that this bound is sharp. More precisely,

<u>Conjecture</u>:

$$\tau(n) = \Omega(n^{11/2} \exp(\frac{c \log n}{\log \log n})).$$

for some $c > 0$.

There is a conjecture of Sato-Tate that the angles θ_p are equidistributed, as p varies, with respect to the measure

$$\frac{2}{\pi} \sin^2 \theta \ d\theta,$$

This is an unproved conjecture.

Even the weaker conjecture: there exists a $\varphi < \frac{\pi}{3}$ and a $\delta > 0$ such that card $(p \leq x : \ 0 \leqslant \theta_p \leqslant \varphi \ ) \gg x^{\delta}$, is unknown.

We shall prove below:

<u>Theorem 1</u> (modulo weak Sato-Tate conjecture)

$$\tau(n) = \Omega(n^{11/2} \exp(\frac{c \log n}{\log \log n})).$$

With respect to unconditional results, Rankin [10] showed

$$\limsup_{n \to \infty} \frac{|\tau(n)|}{n^{11/2}} = +\infty$$

and Joris [6] improved this to

$$\tau(n) = \Omega(n^{11/2} \exp(c(\log n)^{\frac{1}{22}-\epsilon})).$$

R. Balasubramanian and I [1], succeeded in showing:

<u>Theorem 2</u>

$$\tau(n) = \Omega(n^{11/2} \exp(c(\log n)^{2/3-\epsilon})).$$

§ 2. Preliminaries

First, let me prove that $\tau(n) = O(n^6)$, so that the audience may gain some familiarity with the ideas involved in the latter part of this talk. The cusp form $\Delta(z)$ enjoys some properties, namely,

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\,\Delta(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, and

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 , \quad a, b, c, d \in \mathbb{Z} \right\}.$$

$SL_2(\mathbb{Z})$ acts on $\mathfrak{h}$, the upper half plane. The standard fundamental domain for the action of $SL_2(\mathbb{Z})$ on $\mathfrak{h}$ is $-\frac{1}{2} \leq x \leq \frac{1}{2}$ and $x^2 + y^2 \geq 1$, where $z = x + iy$. The mapping $z \to -\frac{1}{z}$ corresponding to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ transforms the standard fundamental domain to another fundamental domain.

Now $y^6 \,|\Delta(z)|$ is easily checked to be invariant under the action of $SL_2(\mathbb{Z})$ and bounded. As

$$\int_0^1 \Delta(x+iy)\, e^{-2\pi imx}\, dx = \tau(m)\, e^{-2\pi my},$$

we find

$$y^6\ \tau(m)\ e^{-2\pi my} = O(1).$$

Choosing $y = \frac{1}{m}$, gives $\tau(m) = O(m^6)$.

For the sake of brevity, let us write

$$\tau_n = \tau(n)/n^{11/2}$$

and set

$$f(s) = \sum_{1}^{\infty} \frac{\tau_n^2}{n^s} .$$

Proof of theorem 1. Now suppose the Sato-Tate conjecture is true. Then card $(p \leq x : 0 \leq \theta_p \leq \frac{\pi}{6})$ is $\sim \frac{cx}{\log x}$.

Take $p \leq x$, such that $0 \leq \theta_p \leq \frac{\pi}{6}$. Then if N denotes the product of all such primes $\leq x$,

we have

$$\log N \sim cx$$

and so

$$\tau_N \geq (\sqrt{3})^{cx/\log x}$$

$$\geq \exp\left(\frac{c \log N}{\log \log N}\right),$$

as desired. The proof assuming the weak Sato-Tate conjecture is similar and left to the reader as an exercise.

§ 3. Real zeroes of $f(s)$

Let

$$\psi(s) = (2\pi)^{-2(s+11)} \Gamma(s+11)\, \Gamma(s)\, \zeta(2s)\, f(s)\, s(s-1).$$

Then, it is known [4] that

$$\psi(s) = \iint_{\mathcal{D}} y^{12} \left|\Delta(z)\right|^2 \phi(z,s) \frac{dx\,dy}{y^2},$$

where $\mathcal{D}$ denotes the standard fundamental domain and

$$\phi(z, s) = \frac{s(s-1)}{z} \left(\frac{y}{\pi}\right)^3 \Gamma(s) \sum{}' \left| mz + n \right|^{-2s},$$

the dash on the summation indicating that the sum over all pairs of integers $(m, n) \neq (0, 0)$.

We shall sketch the proof that $\psi(s)$ has no real zeroes. It is known that $\phi(z, s) > 0$ for $\frac{1}{2} \leq s \leq 1$ and $y \leq 7$, $z = x + iy$. Therefore, we split the fundamental domain into two regions: $y \leq A$ and $y > A$ with $A \leq 7$. In $y \leq A$, $\phi(z, s) > 0$ and the total contribution is > 0. In $y > A$, the contribution is small in virtue of the fact that $\Delta(z) = O(e^{-2\pi y})$. The total integral is therefore > 0.

A more precise proof requires the use of Kronecker's limit formula, and we refer the reader to [1] for the details.

§ 4. Proof of theorem 2.

We already saw that if we had $\tau_p > 1$ for a large proportion of the primes, the Ω-result would follow. This leads us to consider $\tau_p^2 - 1$ and to study

$$\sum_p \frac{\tau_p^2 - 1}{p^s} .$$

If we let

$$\phi(s) = \frac{\zeta(2s)}{\zeta(s)} f(s) ,$$

we find

$$\log \phi(s) = \sum_p \frac{\tau_p^2 - 1}{p^s} \left(1 + \frac{\tau_p^2 - 3}{2p^s} + \ldots\right)$$

We need:

Lemma 1 $$\sum_{\substack{p \\ \tau_p^2 > 1}} \frac{\tau_p^2 - 1}{p^\beta} = +\infty \quad \text{if} \quad \beta < \tfrac{1}{2} .$$

Proof Suppose not. Then, for some $\beta_0 < \frac{1}{2}$,

$$\sum_{\tau_p^2 > 1} \frac{\tau_p^2 - 1}{p^{\beta_0}} < \infty$$

and so if we let

$$\log \phi(s) = f_+(s) - f_-(s),$$

where

$$f_+(s) = \sum_{\substack{p \\ \tau_p^2 > 1}} \frac{\tau_p^2 - 1}{p^s} \left(1 + \frac{\tau_p^2 - 3}{2p^s} + \dots\right)$$

and

$$f_-(s) = -\sum_{\substack{p \\ \tau_p^2 < 1}} \frac{\tau_p^2 - 1}{p^s} \left(1 + \frac{\tau_p^2 - 3}{2p^s} + \dots\right),$$

we find that the abscissa of convergence of $f_+(s)$ is $\leq \beta_0$. Hence, $f_+(s)$ converges absolutely in $\text{Re } s \geqslant \beta_0$ and in particular, for $\text{Re } s \geqslant \frac{1}{2}$. Therefore, the singularities of $\log \phi(s)$ coincide with the singularities of $f_-(s)$ in $\text{Re } s \geqslant \frac{1}{2}$. By a standard method, it is possible to show that if $N(T)$ denotes the number of zeroes of $\psi(s)$ in the critical strip, up to height T, we find

$$N(T) = \frac{2}{\pi} T \log T + O(T).$$

As $\zeta(s)$ has only $\frac{T}{2\pi} \log T + O(T)$ zeroes in this region, we find that $\log \phi(s)$ has singularities in $\text{Re } s \geqslant \frac{1}{2}$. But $f_-(s)$ has non-negative Dirichlet coefficients and so, by a famous theorem of

Landau, one of these singularities is real. This contradicts the fact that $\phi(s)$ has no real zeroes. This proves the lemma.

Now we know

$$\sum_{\tau_p^2 > 1} \frac{\tau_p^2 - 1}{p^\beta} = +\infty \qquad \text{for} \quad \beta < \tfrac{1}{2}.$$

In a standard way, it is now possible to deduce that for some $0 \leq \gamma \leq \beta$,

$$\text{card}\,(e^m < p < e^{m+1} \;;\; p^{-\gamma-\epsilon} < \tau_p^2 - 1 < p^{-\gamma})$$

$$\geq \frac{(e^m)^{\beta+\gamma}}{m^2} .$$

Setting

$$N = \prod_p p$$

where the primes range over the primes in the above set, we find

$$\tau_N \geq \prod_p \left(1 + \frac{1}{p^{\gamma+\epsilon}}\right)$$

$$\gg \exp\left(\sum_p \frac{1}{p^{\gamma+\epsilon}}\right)$$

$$\gg \exp\left((\log N)^{\frac{\beta-\epsilon}{\beta+\gamma}}\right)$$

As $\gamma \leq \beta$, we get $\tau_N = \Omega(\exp(\log N)^{\frac{1}{2}-\epsilon})$.

By utilizing prime powers, it is possible to improve the exponent $\frac{1}{2}$ to $\frac{2}{3}$. The key idea is supplied by the lemma:

Lemma 2. There is a constant $c > 1$ such that for every prime p, there is an $m(p)$

$$m(p) \ll \frac{1}{|\tau_p^2 - 1|}$$

and $$\tau_{p^{m(p)}} \geq c > 1, \text{ whenever } \tau_p^2 > 1.$$

We leave the proof of this lemma as an exercise to the reader.

§. 5 Concluding remarks

There is nothing special about Ramanujan's cusp form, with respect to Ω-theorems. We have:

Theorem 3. If $f(z) = \sum_1^\infty a(n) e^{2\pi i n z}$ is a normalized Hecke eigen form of weight k, then

$$a(n) = \Omega\left(n^{\frac{k-1}{2}} \exp\left(\frac{c(\log n)^{\frac{1}{k}}}{\log \log n}\right)\right).$$

If similar results concerning real zeroes of the Rankin convolution were known, this theorem could be improved. We remark that the Sato-Tate conjecture is equivalent to showing that each of the Dirichlet series

$$\sum_{1}^{\infty} \frac{\tau_n^k}{n^s}$$

has an analytic continuation for $\operatorname{Re} s \geq 1$ and no zeroes on $\sigma = 1$ for $k \geq 3$. At present, this is only known for $k = 3$.

We have shown that $\tau(n)$ becomes very large infinitely often. But there is a conjecture of Elliott [3] that this should not happen too often. More precisely, Elliott showed either

(i) $$\sum_{n \leq x} |\tau_n| = o(x) \quad \text{as} \quad x \to \infty$$

or

(ii) $$\sum_{p} \frac{(\tau_p - 1)^2}{p} < +\infty \,, \quad \text{but not both.}$$

He could not decide which was true. We shall show

$$\sum_{p} \frac{(\tau_p - 1)^2}{p} = +\infty \,.$$

Consider

$$g(s) = \zeta(s) \left(\sum_{1}^{\infty} \frac{\tau_n^2}{n^s}\right) \left(\sum_{1}^{\infty} \frac{\tau_n}{n^s}\right)^{-2}.$$

Then $g(s)$ has an analytic continuation to $\operatorname{Re} s \geq \frac{1}{2}$ except for a pole of order 2 at $s = 1$.

Moreover,

$$g(s) = \prod_p \left(1 + \frac{(\tau_p - 1)^2}{p^s} + \dots\right)$$

and it is easily seen that

$$\log g(s) = \sum_p \frac{(\tau_p - 1)^2}{p^s} + g_1(s)$$

where $g_1(s)$ is analytic for $\operatorname{Re} s > \frac{1}{2}$. Therefore, we find

$$\sum_p \frac{(\tau_p - 1)^2}{p^s} = 2 \log \left(\frac{1}{s-1}\right) + g_2(s),$$

where $g_2(s)$ is analytic as $s \to 1^+$. Choosing $s = 1 + \frac{1}{\log x}$ reveals, in a straightforward manner,

$$\sum_{p \leq x} \frac{(\tau_p - 1)^2}{p} \gg \log \log x.$$

In fact, one can show

$$\sum_{p \leq x} \frac{(\tau_p - 1)^2}{p} = (2 + \underline{o}(1)) \log \log x,$$

by standard methods.

Shortly after this conference, Professor K. Ramachandra suggested that it may be possible to show effectively,

$$\tau_n = \Omega(\exp(\frac{c(\log n)^{2/3}}{(\log \log n)^A}))$$

for some $A > 0$, by utilising the theorem in [11].

By the theorem of Landau, it follows that

$$\sum_{p \leq x} (\tau_p^2 - 1) \log p \geq cx^{\frac{1}{2}}$$

for some $c > 0$. This, however, is ineffective, but can be made effective by an elaborate averaging argument carried out in [11], which we shall not discuss here. By decomposing the interval $[1, x]$ into $O(\log x)$ intervals, it follows that for some u,

$$\sum_{\substack{u \leq p \leq 2u \\ \tau_p^2 > 1}} (\tau_p^2 - 1) > \frac{c\, u^{\frac{1}{2}}}{(\log u)^2}$$

As before, it is easily deduced that for some $m \leq \frac{1}{2} \log u$,

$$\text{card } (u \leq p \leq 2u : e^{-m-1} < \tau_p^2 - 1 \leq e^{-m}) \geq \frac{c\, u^{\frac{1}{2}}\, e^m}{(\log u)^3} .$$

Proceeding as before, we find finally,

$$\tau_n = \Omega(\exp(\frac{c(\log n)^{2/3}}{(\log \log n)^{5/3}})) .$$

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