From:
Number Theory
Proceedings of the Third Matscience Conference Held at Mysore, India, June 3–6, 1981
edited by K. Alladi, Springer Lecture Notes, Volume 938, (1982), 172-176

PROBLEMS IN ENUMERATION OF FINITE GROUPS

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Let $G(n)$ be the number of non-isomorphic groups of order n. It is trivial to see that $G(n) \leq n^{n^2}$. Since any finite group of order n can be generated by $O(\log n)$ elements, it follows that $G(n) \leq n^{cn \log n}$, for some constant $c > 0$. The first non-trivial upper bound was obtained by Gallagher [2] who showed

$$G(n) \leq n^{cn^{2/3}(\log n)^2}.$$

Sims [8] conjectured that

(1) $$G(n) \leq n^{c(\log n)^3}$$

for some $c > 0$, and showed that (1) is true if we restrict our attention to solvable groups of order n. P. Neumann [7] showed that if $S(n)$ denotes the number of simple groups of order n and

$$S(n) \leq n^{c'(\log n)^3}$$

then (1) is true for some constant $c > 0$. With the recent classification of all finite simple groups, (1) is now known to be true. But (1) can be improved further, if we restrict our attention to other sets of integers of positive density. It was shown in [4] that for squarefree n,

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$$G(n) \leq \varphi(n),$$

where φ is Euler's totient function. Moreover, on a set of density $\geq \frac{2}{5}$,

$$G(n) \leq (\text{Log } n)^2.$$

It is not true that for n squarefree, $G(n) \leq (\log n)^A$ for some $A > 0$. In fact, one can show for squarefree n,

$$G(n) = \Omega(\exp(\frac{c \log n}{\log \log n})).$$

On the other hand, it is known [6] that,

$$\sum_{n \leq x} \mu^2(n) \log G(n) = (1 + \underline{o}(1))\ c\ x \log \log x \quad \text{as} \quad x \to \infty,$$

for some constant $c > 0$.

<u>Question 1.</u> Is it true that for n squarefree,

$$G(n) = O(\exp(\frac{c \log n}{\log \log n}))$$

for some $c > 0$?

With respect to the distribution of the values of $G(n)$ much less is known. It is an interesting result of Burnside that $G(n) = 1$ if and only if $(n, \varphi(n)) = 1$. Erdős [1] gave an asymptotic formula for the number of $n \leq x$ such that $(n, \varphi(n)) = 1$ as

$$= (1 + \underline{o}(1)) \frac{x e^{-\gamma}}{\log \log \log x} ,$$

where γ is Euler's constant. Let $F_A(x)$, $F_N(x)$, $F_{SS}(x)$, $F_S(x)$ denote the number of $n \leq x$ such that all groups of order n are abelian, nilpotent, supersolvable and solvable respectively. It has been shown that [5]

$$F_A(x) = (1 + \underline{o}(1)) \frac{x e^{-\gamma}}{\log \log \log x} ,$$

$$F_N(x) = (1 + \underline{o}(1)) \frac{x e^{-\gamma}}{\log \log \log x} ,$$

$$F_{SS}(x) = (1 + \underline{o}(1)) G x, \ G \geq \frac{6}{\pi^2} ,$$

$$F_S(x) = (1 + \underline{o}(1)) c_2 x, \ c_2 \geq 0.9.$$

If $F_k(x)$ is the number of $n \leq x$ such that $G(n) = k$, we know from above that

$$F_1(x) = \frac{(1 + \underline{o}(1)) x e^{-\gamma}}{\log \log \log x} .$$

One can show that $G(n) = 2$ if and only if

(i) $n = 2p$, p a prime or

(ii) $n = p_1 p_2 m$, $(p_1 m, \varphi(p_1 m)) = 1$,

$(p_2 m, \varphi(p_2 m)) = 1$, $p_2 \equiv 1 \pmod{p_1}$,

or

(iii) $n = p^2m,\ (pm, \varphi(pm)) = 1, \quad (p+1, m) = 1.$

Using Brun's sieve, it can be proved [4] that

$$F_2(x) = O\left(\frac{x \log\log\log\log x}{(\log\log\log x)^2}\right).$$

Question 2. Is it true that

$$F_2(x) = \frac{(1+o(1))\,cx}{(\log\log\log x)^2}$$

for some $c > 0$?

It is too early to predict a general trend. As k increases, the arithmetical conditions on n so that $G(n) = k$, become more complicated. One can show, with some difficulty,

$$F_3(x) = O\left(\frac{x}{\log\log\log x}\right).$$

References

1 P. Erdős, Some asymptotic formulas in number theory, J. Indian Math. Soc. 12 (1948) 75-78.

2 P.X. Gallagher, Counting finite groups of given order, Math. Zeitschrift, 102 (1967) 236-237.

3 A. Hughes, On automorphisms of nilpotent groups and supersolvable orders, Proceedings of Santa Cruz Conference on Finite Groups, AMS (1981).

4 M. Ram Murty and V. Kumar Murty, On the number of groups of a given order (to appear).

5 M. Ram Murty and V. Kumar Murty, On the density of various classes of groups (to appear).

6 M. Ram Murty and V. Kumar Murty, On groups of squarefree order (to appear)

7 P. Neumann, An enumeration theorem for finite groups, Quart. J. Math. Oxford Ser. (2) 20 (1969) 395-401.

8 C. Sims, Enumerating p-groups, Proc. London Math. Soc., 15(1965) 151-166.

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MRM.Kc.23.6.81