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On the p-Adic Series $\sum_{n=1}^{\infty} n^k \cdot n!$

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1. Introduction

Let $\mathbb{Q}_p$ be the field of p-adic numbers. It is well known that the sum

$$\sum_{n=1}^{\infty} a_n,$$

with $a_n \in \mathbb{Q}_p$ converges if and only if $|a_n|_p \to 0$ as $n \to \infty$ (see for example, [**6**, p. 87]). Since $|n!|_p \to 0$, we see that $\sum_{n=1}^{\infty} n!$ converges in $\mathbb{Q}_p$, as well as the related sums $\sum_{n=1}^{\infty} n^k \cdot n!$ for k a non-negative integer. Our goal in this paper is to investigate these p-adic sums.

Our first question is whether

$$\alpha = \sum_{n=1}^{\infty} n!$$

is a p-adicirrational or not, a question first asked by Schikhof [**9**, p. 17]. We conjecture that it is. Observe however that

$$\sum_{n=1}^{\infty} n \cdot n! = -1.$$

A simple proof by induction shows that $1\cdot 1! + 2\cdot 2! + 3\cdot 3! + \cdots + m\cdot m! = \sum_{n=0}^{m} n\cdot n! = (m+1)! - 1$. Since $\lim_{m\to\infty} |(m+1)!|_p = 0$, the desired result follows. On the other hand, we show below that

$$\alpha_k = \sum_{n=1}^{\infty} n^k \cdot n! = v_k - u_k \alpha,$$

where $v_k, u_k \in \mathbb{Z}$. We will prove using ideas from combinatorics and number theory that $u_k \neq 0$ for $k \equiv 0$ or $2 \pmod 3$. Thus, the irrationality of α_k hinges on the irrationality of α in these cases. This strengthens the observation made by Dragovich [**5**] that if α is irrational, then so is α_k. We suspect $u_k \neq 0$ for every $k \geq 2$. In this paper, we will assume that α is irrational and study the sequences u_k and v_k.

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The sequence of integers u_k defined below is of independent combinatorial interest. We hope to address the question of the non-vanishing of u_k for $k \geq 2$ in a future paper. Below, we establish some relationships with Stirling numbers of the second kind that allows us to prove non-vanishing results in some cases. It should be clear from the context that in the summations being considered, which ones are p-adic sums and which are summations in the field of real numbers.

2. Preliminaries

We must first recall some notation and elementary lemmas from combinatorics. The reader who wishes a more detailed review of this material is advised to consult Comtet [**3**].

The rising factorial is

$$\langle x \rangle_n = x(x+1)(x+2)\cdots(x+n-1).$$

The Stirling number of the first kind, denoted by $s(k,j)$, is defined by the rule that $(-1)^{k+j}s(k,j)$ is the number of permutations of $\{1,\ldots,k\}$ with j cycles. The Stirling number of the second kind, denoted by $S(k,j)$, is the number of partitions of $\{1,\ldots,k\}$ into j non-empty parts. Stirling numbers of the first kind are related to the rising factorial by the following relation [**3**, p. 213].

Lemma 1.

$$\langle x \rangle_j = \sum_{j=1}^{k} |s(k,j)| x^j.$$

We will also have need for a very useful inversion lemma [**3**, p. 144].

Lemma 2. *Let $\{f_k\}$ and $\{g_k\}$ be sequences of real numbers. Then,*

$$f_k = \sum_{j=1}^{k} S(k,j) g_j \Leftrightarrow g_k = \sum_{j=1}^{k} s(k,j) f_j.$$

We recall the following basic fact which will be used below:

$$\left(\sum_{n=0}^{\infty} \frac{a_n T^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{b_n T^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{c_n T^n}{n!}$$

where

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

Bell numbers will appear below, and will be denoted B_k. Since B_{k+1} is just the total number of all partitions of the set $\{1,2,\ldots,k+1\}$ [**3**, p. 210], we deduce:

Lemma 3.

$$B_{k+1} = \sum_{j=1}^{k+1} S(k+1,j).$$

3. A generating function for u_k

We must first demonstrate that $\sum_{n=1}^{\infty} n^k \cdot n! = v_k - u_k\alpha$, where $\alpha = \sum_{n=1}^{\infty} n!$ and v_k and u_k are integers. Notice that $\sum_{n=1}^{\infty}\big((n+k)! - n!\big)$ is a telescoping sum that equals $-\sum_{n=1}^{k} n!$. For $k = 2$ this becomes,

$$\sum_{n=1}^{\infty}\big((n+2)! - n!\big) = \sum_{n=1}^{\infty} n!\big((n+2)(n+1) - 1\big),$$
$$\sum_{n=1}^{\infty} n^2 \cdot n! = -\sum_{n=1}^{\infty} n!.$$

An inductive argument now shows that $\sum_{n=1}^{\infty} n^k \cdot n!$ can be written as $v_k - u_k\alpha$ where $v_k, u_k \in \mathbb{Z}$.

Observe now that for the same sum,

$$\begin{aligned}\sum_{n=1}^{\infty}\big((n+k)! - n!\big) &= \sum_{n=1}^{\infty}(n-1)!\{(n+k)(n+k-1)\cdots n - n\}\\ &= \sum_{n=1}^{\infty}(n-1)!\langle n\rangle_{k+1} - \sum_{n=1}^{\infty} n!.\end{aligned}$$

Hence applying Lemma 1,

$$\begin{aligned}-\sum_{n=1}^{k} n! + \alpha &= \sum_{n=1}^{\infty}(n-1)!\langle n\rangle_{k+1}\\ &= \sum_{n=1}^{\infty}(n-1)!\sum_{j=1}^{k+1}|s(k+1,j)|n^j\\ &= \sum_{j=1}^{k+1}|s(k+1,j)|\sum_{n=1}^{\infty} n^{j-1}n!.\end{aligned}$$

Letting $\sum_{n=1}^{\infty} n^{j-1}n! = v_{j-1} - u_{j-1}\alpha$ gives

$$-\sum_{n=1}^{k} n! + \alpha = \sum_{j=1}^{k+1}|s(k+1,j)|\{v_{j-1} - u_{j-1}\alpha\}.$$

If we now suppose that α is irrational, then this yields

$$\sum_{j=1}^{k+1}|s(k+1,j)|v_{j-1} = -\sum_{n=1}^{k} n!, \tag{3.1}$$

$$\sum_{j=1}^{k+1}|s(k+1,j)|u_{j-1} = -1. \tag{3.2}$$

Removing absolute values from equation (3.2) we get

$$\sum_{j=1}^{k+1}(-1)^j s(k+1,j)u_{j-1} = (-1)^k. \tag{3.3}$$

Applying Lemma 2 to equation (3.3) yields one generating function. Let $f_{k+1} = (-1)^{k+1}u_k$ and $g_j = (-1)^{j-1}$ to get

$$\begin{aligned} f_{k+1} = (-1)^{k+1}u_k &= \sum_{j=1}^{k+1} S(k+1,j)g_j \\ &= \sum_{j=1}^{k+1}(-1)^{j-1}S(k+1,j). \end{aligned}$$

Thus,

LEMMA 4.

$$(-1)^k u_k = \sum_{j=1}^{k+1}(-1)^j S(k+1,j).$$

This is a simplification of the method of solving $k+1$ linear equations to determine each u_k outlined by Dragovich [**5**, p. 101]. Using this relation, the first few terms in the sequence $\{u_k\}$ can be easily calculated:

$$\{0, 1, -1, -2, 9, -9, -50, 267, -413, -2180, 17731, -50533, -110176, \\ 1966797, -9938669, 8638718, 278475061, -2540956509, 9816860358\dots\}$$

This is the negative of sequence A014182 of Sloane [**10**]. Lemma 4 implies the crucial result

LEMMA 5.

$$u_k \equiv \sum_{j=1}^{k+1} S(k+1,j) \pmod 2.$$

Other generating functions for the u_k also exist. Now we caution the reader that below, the series being considered are not p-adic series but usual series involving real numbers.

LEMMA 6.

$$(-1)^k u_k = e\sum_{n=0}^{\infty}\frac{n^{k+1}}{n!}(-1)^n.$$

PROOF. Recall the known identity [**3**, p. 204]

$$S(k+1,j) = \frac{1}{j!}\sum_{r=0}^{j}(-1)^r\binom{j}{r}(j-r)^{k+1}.$$

Now,

$$\begin{aligned} (-1)^k u_k &= \sum_{j\geq 1}(-1)^j S(k+1,j) \\ &= \sum_{j\geq 1}\frac{(-1)^j}{j!}\sum_{0\leq r\leq j}(-1)^r\binom{j}{r}(j-r)^{k+1} \\ &= \sum_{r\geq 0}\frac{(-1)^r}{r!}\sum_{j\geq r}\frac{(-1)^j}{j!}\frac{j!}{(j-r)!}(j-r)^{k+1} \end{aligned}$$

$$= \sum_{r\geq 0} \frac{1}{r!} \sum_{j\geq r} \frac{(-1)^{j-r}}{(j-r)!}(j-r)^{k+1}$$

$$= \sum_{r\geq 0} \frac{1}{r!} \sum_{n=0}^{\infty} \frac{n^{k+1}}{n!}(-1)^n$$

$$(-1)^k u_k = e \sum_{n=0}^{\infty} \frac{n^{k+1}}{n!}(-1)^n. \quad \square$$

Lemma 7.

$$\sum_{k=0}^{\infty} \frac{(-1)^k u_k T^k}{k!} = -e^{-e^T+T+1}$$

Proof.

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k u_k T^k}{k!} &= e \sum_{k=0}^{\infty} \frac{T^k}{k!} \sum_{n=0}^{\infty} \frac{(-1)^n n^{k+1}}{n!} \\ &= e \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{T^k n^k}{k!} \frac{(-1)^n n}{n!} \\ &= e \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} n e^{Tn} \\ &= -e \sum_{n=1} \frac{(-1)^{n-1}}{(n-1)!} e^{T(n-1)} e^T \\ &= -e \cdot e^{-e^T} \cdot e^T \\ \sum_{k=0}^{\infty} \frac{(-1)^k u_k T^k}{k!} &= -e^{-e^T+T+1}. \end{aligned}$$

Since $e^{e^T-1} = \sum B_k T^k/k!$ [**3**, p. 211], applying the remark made earlier about the multiplication of exponential generating functions and Lemma 7, we see that

$$\left(\sum_{k=0}^{\infty} \frac{B_k T^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{(-1)^k u_k T^k}{k!}\right) = -\sum_{k=0}^{\infty} \frac{T^k}{k!}$$

which proves

Lemma 8.

$$\sum_{k=0}^{n} \binom{n}{k} B_k (-1)^{n-k} u_{n-k} = -1.$$

4. Congruences for $\{u_k\}$

There are several interesting congruence relations for $\{u_k\}$.

Lemma 9. *For every prime p, $u_{p-1} \equiv -2 \pmod{p}$.*

Proof. Recall that $S(p,1) = S(p,p) = 1$ and that for $2 \le k \le p-1$, $S(p,k) \equiv 0 \pmod p$, [**3**, p. 219]. Observe,

$$\begin{aligned}(-1)^{p-1}u_{p-1} &= \sum_{j=1}^{p}(-1)^j S(p,j),\\ (-1)^{p-1}u_{p-1} &\equiv (-1)S(p,1) + (-1)S(p,p) \pmod p,\\ u_{p-1} &\equiv -2 \pmod p. \quad \square\end{aligned}$$

Lemma 10. *For every prime p, $u_p \equiv -1 \pmod p$.*

Proof. Substitute the recurrence $S(k+1,j) = S(k,j-1)+jS(k,j)$, [**3**, p. 208] into

$$\begin{aligned}(-1)^p u_p &= \sum_{j=1}^{p+1}(-1)^j S(p+1,j),\\ -u_p &= \sum_{j=1}^{p+1}(-1)^j S(p,j-1) + \sum_{j=1}^{p+1}(-1)^j jS(p,j),\\ -u_p &\equiv -u_{p-1} + (-1)S(p,1) \pmod p,\\ u_p &\equiv u_{p-1} + 1 \pmod p,\\ u_p &\equiv -2+1 \pmod p,\\ u_p &\equiv -1 \pmod p. \quad \square\end{aligned}$$

Using the fact that for every prime p,

$$\binom{p^r}{k} \equiv 0 \pmod p$$

whenever $1 \le k \le p^r - 1$ along with Lemma 8, we conclude

Lemma 11. *Let p be a prime number. For every $r \ge 1$,*

$$B_{p^r} - 1 \equiv (-1)^{p^r} u_{p^r} \pmod p.$$

5. A general theorem

Theorem 1. *If $k \equiv 0$ or $2 \pmod 3$ then $u_k \neq 0$, and hence in these cases α_k will be irrational provided α is.*

Since we have already shown in Lemma 5 that

$$u_k \equiv \sum_{j=1}^{k+1} S(k+1,j) \equiv B_{k+1} \pmod 2,$$

Theorem 1 will follow from the following lemma:

Lemma 12. *If $k \equiv 2 \pmod 3$ then B_k is even, otherwise B_k is odd.*

Proof. We will proceed by induction. $B_0 = 1$, $B_1 = 1$ and $B_2 = 2$, so the base cases are clear. Suppose the lemma is true for all $j \le k$. Recall the recursion for B_{k+1},

$$B_{k+1} = \sum_{j=0}^{k}\binom{k}{j}B_j.$$

By the induction hypothesis, if $j \equiv 2 \pmod 3$ then B_j is even, and otherwise B_j is odd. Hence the recursive formula becomes

$$\sum_{j \not\equiv 2 \pmod 3} \binom{k}{j} \equiv \sum_{j \equiv 0 \pmod 3} \binom{k}{j} + \sum_{j \equiv 1 \pmod 3} \binom{k}{j} \pmod 2.$$

Let $\zeta = e^{2\pi i/3}$ be a cube root of unity. From the binomial theorem, we see

$$\sum_{j=0}^{k} \binom{k}{j} x^j = (1+x)^k,$$
$$\sum_{j=0}^{k} \binom{k}{j} \zeta^j x^j = (1+\zeta x)^k,$$
$$\sum_{j=0}^{k} \binom{k}{j} \zeta^{2j} x^j = (1+\zeta^2 x)^k.$$

Adding these together we get

$$\sum_{j=0}^{k} \binom{k}{j} x^j (1+\zeta^j+\zeta^{2j}) = (1+x)^k + (1+\zeta x)^k + (1+\zeta^2 x)^k.$$

Let $x = 1$.

$$\sum_{j=0}^{k} \binom{k}{j} (1+\zeta^j+\zeta^{2j}) = 2^k + (1+\zeta)^k + (1+\zeta^2)^k.$$

Recall

$$1+\zeta^j+\zeta^{2j} = \begin{cases} 3 & \text{if } j \equiv 0 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}$$

So

$$\begin{aligned} 3 \sum_{j \equiv 0 \pmod 3} \binom{k}{j} &= 2^k + (1+\zeta)^k + (1+\zeta^2)^k \\ &= 2^k + (1+\zeta)^k + (1+\zeta^{-1})^k \\ &= 2^k + 2\Re(1+\zeta)^k \\ &= 2^k - 2\Re(\zeta^{2k}). \end{aligned}$$

Let us note that

$$\zeta^{2k} = \begin{cases} 1 & \text{if } k \equiv 0 \pmod 3 \\ \zeta & \text{if } k \equiv 2 \pmod 3 \\ \zeta^2 & \text{if } k \equiv 1 \pmod 3). \end{cases} \tag{5.1}$$

Now for the other sum. Consider

$$\sum_{j=0}^{k} \binom{k}{j} x^j = (1+x)^k,$$

$$\sum_{j=0}^{k} \binom{k}{j} \zeta^{j-1} x^j = \zeta^{-1}(1+\zeta x)^k,$$

$$\sum_{j=0}^{k} \binom{k}{j} \zeta^{2j-2} x^j = \zeta^{-2}(1+\zeta^2 x)^k.$$

Adding these together gives

$$\sum_{j=0}^{k} \binom{k}{j} x^j (1+\zeta^{j-1}+\zeta^{2j-2}) = (1+x)^k + \zeta^{-1}(1+\zeta x)^k + \zeta^{-2}(1+\zeta^2 x)^k.$$

Let $x = 1$

$$\sum_{j=0}^{k} \binom{k}{j} (1+\zeta^{j-1}+\zeta^{2j-2}) = 2^k + \zeta^{-1}(1+\zeta)^k + \zeta^{-2}(1+\zeta^2)^k.$$

Now

$$1+\zeta^{j-1}+\zeta^{2j-2} = \begin{cases} 3 & \text{if } j-1 \equiv 0 \pmod 3 \\ 0 & \text{otherwise.} \end{cases}.$$

Hence,

$$\begin{aligned} 3 \sum_{j\equiv 1 \pmod 3} \binom{k}{j} &= 2^k + \zeta^{-1}(1+\zeta)^k + \zeta(1+\zeta^{-1})^k \\ &= 2^k + 2\Re\zeta(1+\zeta^{-1})^k \\ &= 2^k + 2\Re\zeta(1+\zeta^2)^k \\ &= 2^k + 2\Re\zeta(-\zeta)^k \\ &= 2^k + (-1)^k 2\Re(\zeta^{k+1}). \end{aligned}$$

Now

$$\zeta^{k+1} = \begin{cases} 1 & \text{if } k+1 \equiv 0 \pmod 3 \\ \zeta & \text{if } k+1 \equiv 1 \pmod 3 \\ \zeta^2 & \text{if } k+1 \equiv 2 \pmod 3). \end{cases} \tag{5.2}$$

Combining the information in equations (5.1) and (5.2), we see that if $k \equiv 1 \pmod 3$ then $B_{k+1} \equiv 0 \pmod 2$, and if $k \not\equiv 1 \pmod 3$ then $B_{k+1} \equiv 1 \pmod 2$ proving Theorem 1. □

6. Concluding remarks

The non-vanishing of u_k is a conjecture of Wilf (see [**7**]). In [**11**], it is proved that the number of $k \le x$ with $u_k = 0$ is $O(x^{2/3})$.

Other questions deserving attention linked to the study of $\{u_k\}$ concern the existence and properties of p-adic interpolation of functions of the form

$$f(s) = \sum n^s \cdot n!.$$

If the sum is taken over all odd numbers n, then it can be shown that $f(s)$ has a 2-adic interpolation, which raises the question of whether $\{u_k\}$ and $\{v_k\}$ have 2-adic limits or not. Similar questions can be raised about p-adic interpolation and limits for odd primes, which seem to be more involved cases.

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