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The Partition Function Revisited

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Abstract. In 1918, Hardy and Ramanujan wrote their landmark paper deriving the asymptotic formula for the partition function. The paper however was fundamental for another reason, namely for introducing the circle method in questions of additive number theory. Though this method is powerful, it is often difficult and technically complicated to employ. In 2011, Bruinier and Ono discovered a new algebraic formula for the partition function obtained via the theory of weak Maass forms. This formula allows us to deduce the Hardy-Ramanujan formula using basic theory of Fourier expansions of Maass forms and the theory of positive definite binary quadratic forms. The Hardy-Ramanujan formula also leads to the asymptotics of Fourier coefficients of the j-function, a fact hitherto unnoticed. These asymptotics were obtained earlier by Petersson in 1932 and Rademacher in 1938 (independently) using the circle method. We report on our joint work with M. Dewar in this context.

Keywords. Partition function, j-function, saddle-point method, weak Maass forms.

1. Introduction

Almost a century ago, G. H. Hardy and Srinivasa Ramanujan wrote a landmark paper on the partition function. This paper was significant for two reasons. On the one hand, it was the first paper to have determined the asymptotic behaviour of the partition function and on the other, it was significant in that it introduced the circle method into number theory, which later became an effective tool to study additive questions. A few years ago, Bruinier and Ono [1] derived an algebraic formula for the partition function using the theory of harmonic weak Maass forms. In a recent paper [3], the author and M. Dewar derived the Hardy-Ramanujan asymptotic formula from this algebraic formula. In this way, we bypass the circle method. In a later paper, [4], we showed that the asymptotics of the partition function also leads to the asymptotics of coefficients of the j-function, something which was originally done by Petersson [19] in 1932 and later Rademacher [20] in 1938 using the circle method. This is particularly interesting from a historical perspective since our paper shows that the asymptotic behaviour of the coefficients of the j-function is almost an immediate consequence of the Hardy-Ramanujan paper [9] written in 1918. This paper is a short survey of these recent developments in the study of the asymptotics of the partition function and related functions.

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The partition function, denoted $p(n)$, is the number of ways of writing n as a non-decreasing sum of positive integers. Thus, $p(1) = 1$, $p(2) = 2$, $p(3) = 3$ and $p(4) = 5$ since

$$4, \quad 1+3, \quad 2+2, \quad 1+1+2, \quad 1+1+1+1+1$$

are the five partitions of 4. Thus, each partition can be "factored" uniquely as

$$1^{k_1}2^{k_2}\ldots$$

where the notation symbolizes

$$n = \underbrace{1+1+\cdots+1}_{k_1} + \underbrace{2+2+\cdots+2}_{k_2} + \cdots$$

The partition function is ubiquitous in mathematics. For instance, $p(n)$ is the number of conjugacy classes of the symmetric group S_n. Since (for any group) the number of conjugacy classes is equal to the number of irreducible representations, it is natural to ask if there is a canonical method of constructing all the irreducible representations from the knowledge of the partitions. Indeed, this is the case and the classical theory of Young diagrams and Specht modules comprises an aesthetically complete chapter in the representation theory of the symmetric group. Partitions also play a major role in the representation theory of GL_n. We refer the reader to [7] for further details.

The study of $p(n)$ has a long venerable history and there are many questions still unanswered about its behaviour. Euler was the first to begin a systematic study of $p(n)$. From the "unique factorization" alluded to above, one can write its generating function as

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{-1}, \quad |q| < 1.$$

This equation already suggests a "modular connection" since the famous Dedekind η-function given by

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad q = e^{2\pi i z},$$

is a modular form of weight $1/2$. Euler discovered that

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3n^2+n}{2}}, \tag{1}$$

the so-called pentagonal-number theorem. Later Jacobi discovered that

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(2n+1)q^{n(n+1)/2}. \tag{2}$$

Ramanujan considered the 24th power of the η-function:

$$\Delta(z) := \eta(z)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n, \quad q = e^{2\pi i z},$$

and showed that the coefficients $\tau(n)$ are of sufficient arithmetic interest. This motivated his celebrated conjectures regarding the τ-function and these conjectures had a pivotal role in the development of 20th century number theory.

In 1972, I. G. MacDonald [13] discovered a fascinating connection between powers of the η-function and properties of affine root systems in the theory of classical Lie algebras. In fact, MacDonald [13] discovered elegant formulas for the coefficients of powers of η, namely $\eta^{\dim \mathfrak{g}}$, where $\mathfrak{g}$ is a finite dimensional simple Lie algebra, and $\dim \mathfrak{g}$ denotes its dimension. Jacobi's formula is then a special case of MacDonald's identities specialized to the Lie algebra $\mathfrak{sl}_2$. In the particular case of the 24-th power of the η-function, MacDonald derives the following curious formula for the Ramanujan τ-function:

$$\tau(n) = \frac{1}{1!2!3!4!} \sum \prod_{i<j} (u_i - u_j), \tag{3}$$

where the sum is over integers $u_1, \ldots, u_5$ subject to the conditions,

$$u_i \equiv i (\bmod 5), \quad \sum_{i=1}^{5} u_i = 0, \quad \sum_{i=1}^{5} u_i^2 = 10n.$$

Apparently, this was discovered earlier by the physicist Freeman Dyson [5] using "pedestrian methods" or so he writes. In his entertaining and inspiring article, Dyson relates how he had come across this and writes that "it is rather surprising that Ramanujan did not think of it himself." He continues, "Pursuing these identities further by my pedestrian methods, I found that there exists a formula of the same degree of elegance as (3) for the d-th power of η whenever d belongs to the following sequence of integers:

$$d = 3, 8, 10, 14, 15, 21, 24, 26, 28, 35, 36, \ldots$$

There I stopped. I stared for a little while at this queer list of numbers. As I was, for the time being, a number theorist, they made no sense to me. My mind was so well compartmentalized that I did not remember that I had met these same numbers many times in my life as a physicist. If the numbers had appeared in the context of a problem in physics, I would certainly have recognized them as the dimensions of the finite dimensional Lie algebras." Since the work of Dyson and MacDonald, there has been a lot of activity and the relationship of these identities to Lie theory is now well understood. For instance, we refer the reader to the highly readable article [15] as well as the excellent Séminaire Bourbaki article by Demazure [2].

Returning to Euler's formula (1), we see that the coefficients of the power series expansion of

$$\left(\sum_{n=0}^{\infty} p(n) q^n \right)^{-1}$$

are "lacunary" and equal to 0 or ± 1. It is conjectured that the only odd powers of η which are lacunary are the cases of Euler and Jacobi cited above, that is, $d = 1, 3$. For positive even values of d, Serre [23] has determined that the complete list is

$$\{2, 4, 6, 8, 10, 14, 26\}.$$

That the coefficients of η^d are lacunary for d equal to 2, 4, 6, and 8 is due to Ramanujan and appears in his celebrated paper [21] in which he makes his famous conjectures about the τ-function. (Curiously, there is a typo in (104) in the statement of Ramanujan's conjecture on the τ-function.)

Positive powers of η are modular forms either of integral weight or half-integral weight. In both cases, the growth of the Fourier coefficients is polynomial. By contrast, the coefficients of the negative powers of η are of exponential growth and this is one of the corollaries of the Hardy-Ramanujan work.

2. The Hardy-Ramanujan formula

The question of the asymptotic behaviour of $p(n)$ was first answered in the 1918 paper of Hardy and Ramanujan [9]. They proved that

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}, \quad n \to \infty. \tag{4}$$

In their proof, they discovered a new method called *the circle method* which made fundamental use of the modular property of the Dedekind η-function. We see from the Hardy-Ramanujan formula that $p(n)$ has exponential growth.

As pointed out by Selberg [22], the circle method has its origins in Ramanujan's first letter to G. H. Hardy written on January 11, 1913 from India. There, he wrote "the coefficient of q^n in

$$(1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \cdots)^{-1},$$

is very nearly

$$n\frac{1}{4n}\left(\cosh \pi\sqrt{n} - \frac{\sinh \pi\sqrt{n}}{\pi\sqrt{n}}\right)."$$

Ramanujan seems to have discovered the asymptotic behaviour of the n-th term of a modular form of weight $-1/2$. As noted above, the generating function for the partition function is also (apart from a factor of $q^{-1/24}$) a modular form of weight $-1/2$ and thus, the asymptotic behaviour of $p(n)$ is closely akin to the formula given in Ramanujan's letter. Twenty years after the publication of the Hardy-Ramanujan paper, Rademacher discovered an explicit formula for $p(n)$:

$$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty}\sqrt{k}A_k(n)\frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}(n - \frac{1}{24})}\right)}{\sqrt{n - \frac{1}{24}}}\right)$$

where

$$A_k(n) = \sum_{0 \le m < k, (m,k)=1} e^{\pi i (s(m,k) - 2mn/k)},$$

and

$$s(b,c) = \frac{1}{4c} \sum_{j=1}^{c} \cot \frac{\pi j}{c} \cot \frac{\pi j b}{c},$$

is the so-called Dedekind sum. Notice that when $k = 1$, the first term resembles Ramanujan's term in his letter of January 11, 1913.

It seems that Selberg [22] arrived at this formula independently and he wrote, "In the summer of 1937, I had actually myself been studying the paper by Hardy and Ramanujan and had arrived at Rademacher's formula.... It always seemed strange to me that Hardy and Ramanujan did not end up with this formula... and I believe firmly that the responsibility for this rests with Hardy."

3. The arithmetical-algebraic formula of Bruinier-Ono

Recently, Bruinier and Ono [1] discovered a new "exact formula" for $p(n)$. This formula expresses $p(n)$ as a finite sum of special values of a particular "weak" Maass form evaluated at CM points in the upper half-plane that have discriminant $-24n + 1$.

To be precise, let

$$E_2(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n, \quad \sigma(n) = \sum_{d|n} d, \quad q = e^{2\pi i z},$$

be the "quasi-modular" Eisenstein series of weight 2. Let η be the Dedekind η-function. Put

$$\begin{aligned} F(z) &= \frac{E_2(z) - 2E_2(2z) - 3E_2(3z) + 6E_2(6z)}{2\eta(z)^2 \eta(2z)^2 \eta(3z)^2 \eta(6z)^2} \\ &= \frac{1}{q} - 10 - 29q - 104q^2 + \cdots \end{aligned} \tag{5}$$

One can show that $F(z)$ is a weight -2 meromorphic modular form on $\Gamma_0(6)$. Let

$$P(z) = -\left(\frac{1}{2\pi i} \frac{d}{dz} + \frac{1}{2\pi y} \right) F(z).$$

This is a *weak Maass form* of weight zero. It is an eigenfunction of the hyperbolic Laplacian:

$$-y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

with eigenvalue -2. Now consider all positive definite integral binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ with discriminant $b^2 - 4ac = -24n + 1$, and $6|a$,

$b \equiv 1 \pmod{12}$. The group $\Gamma_0(6)$ acts on such forms in the obvious way and we let $\mathcal{Q}_n^6$ be the finite set of orbit representatives under this action. Let

$$\alpha_Q = \frac{-b + \sqrt{1-24n}}{2a}.$$

Then the Bruinier-Ono formula for $p(n)$ is

$$p(n) = \frac{1}{24n-1} \sum_{Q \in \mathcal{Q}_n^6} P(\alpha_Q). \tag{6}$$

The natural question that arises from this result is if one can derive both the Hardy-Ramanujan formula and the Rademacher formula for $p(n)$ from (6). Since the derivation of (6) uses the theory of weak harmonic Maass forms and not the circle method, we would then have a new derivation of (4) if we can derive (4) from (6). Indeed, this is the case as shown in [3]. We proved that for any natural number N,

$$(24n-1)p(n) = \sum_{m=1}^{N} c_m \left(1 - \frac{6m}{\pi\sqrt{24n-1}}\right) e^{\pi\sqrt{24n-1}/6m} + O\left(h(1-24n)e^{\frac{\pi\sqrt{24n-1}}{6(N+1)}}\right),$$

where the c_m's depend on the congruence class of $n \pmod{m}$ and can be given explicitly (see [3] for the precise formula). For instance,

$$c_1(n) = 2\sqrt{3}, \quad c_2(n) = 2(-1)^n \left(\cos\frac{\pi}{12} + \cos\frac{5\pi}{12}\right), \quad \text{etc.}$$

and $h(1-24n)$ is the class number of $\mathbb{Q}(\sqrt{1-24n})$. In [3], we show that both the Hardy-Ramanujan formula and a formula related to the Rademacher formula can be deduced.

To elaborate, an integral binary quadratic form

$$Q(x, y) = ax^2 + bxy + cy^2, \quad a, b, c \in \mathbb{Z}$$

with discriminant $b^2 - 4ac$ is said to be *primitive* if $(a, b, c) = 1$. We write $Q = [a, b, c]$ to indicate this quadratic form. Given a natural number n, we henceforth consider such positive definite quadratic forms with discriminant $1 - 24n$. It is easily checked that for such forms $0 \equiv ac \not\equiv b \pmod 2$ and that $3|b$ if and only if $ac \equiv 2 \pmod 6$ and $3 \nmid b$ if and only if $ac \equiv 0 \pmod 6$. The *principal root* of $Q = [a, b, c]$ is α_Q which lies in the upper half-plane $\mathfrak{h}$. We have a right action of $SL_2(\mathbb{Z})$ on integral binary quadratic forms with fixed discriminant in the following way: given

$$g := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}),$$

$$\left[Q \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right](x, y) := Q(\alpha x + \beta y, \gamma x + \delta y). \tag{7}$$

Of course, we also have a left action of $SL_2(\mathbb{Z})$ on the upper half-plane:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot z := \frac{\alpha z + \beta}{\gamma z + \delta}.$$

For any $f : \mathfrak{h} \to \mathbb{C}$, and $g \in SL_2(\mathbb{Z})$ given by (7), we define the "slash" operator $|_k$ by

$$(f|_k g)(z) := (\gamma z + \delta)^{-k} f\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right).$$

It is routine to check that for $g \in SL_2(\mathbb{Z})$,

$$\alpha_{Q \circ g^{-1}} = g \cdot \alpha_Q.$$

We say $Q = [a, b, c]$ is *reduced* if $|b| \le a \le c$. One can show that a form is reduced if and only if α_Q lies in the standard fundamental domain:

$$\mathcal{D} = \{z \in \mathfrak{h} : -1/2 \le \Re(z) \le 1/2, \quad \text{and} \quad |z| \ge 1\}.$$

We denote by $\mathcal{Q}_n^1$ the set of primitive forms (that is, $\gcd(a, b, c) = 1$) of discriminant $1 - 24n$ with $\alpha_Q \in \mathcal{D}$. As mentioned before, the group

$$\Gamma_0(6) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) : \quad 6|\gamma \right\}$$

acts on the set of primitive forms $[a, b, c]$ of discriminant $1 - 24n$ with $6|a$ and $b \equiv 1 \pmod{12}$. Let $\mathcal{Q}_n^6$ be the set of equivalence classes of forms for this action. Gross, Kohnen and Zagier [8] proved there is a bijection between $\mathcal{Q}_n^1$ and $\mathcal{Q}_n^6$ and this bijection can be given explicitly (see [3] as well as the proposition on page 505 of [8]).

Now, one can write down the q-expansion of $P(z)$:

$$P(z) = \left(1 - \frac{1}{2\pi y}\right) q^{-1} + \frac{5}{\pi y} + \left(29 + \frac{29}{2\pi y}\right) q + \cdots$$

Using this q-expansion and standard theory of modular forms, we proceed to derive the Hardy-Ramanujan formula from the Bruinier-Ono formula. The first step is to transform the sum in (6) to a sum over $\mathcal{Q}_n^1$ so that the α_Q lie in the standard fundamental domain. To do this, we use the explicit bijection provided by the Gross-Kohnen-Zagier theorem. Indeed, the group $\Gamma_0(6)$ has index 12 in $SL_2(\mathbb{Z})$ and we can choose an explicit set of right coset representatives:

$$\begin{aligned}
\gamma_\infty &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\gamma_{\frac{1}{3},r} &= \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \text{ for } r = 0, 1 \\
\gamma_{\frac{1}{2},s} &= \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \text{ for } s = 0, 1, 2 \\
\gamma_{0,t} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ for } t = 0, 1, 2, 3, 4, 5.
\end{aligned} \tag{8}$$

Table 1. The matrix γ_Q for $Q = [a, b, c] \in \mathcal{Q}_n^1$. Triples $(\overline{a}, \overline{b}, \overline{c})$ which can never occur are omitted or left blank.

		$b \pmod{12}$					
		1	3	5	7	9	11
$a \equiv 0 \pmod 6$	$c \equiv 0 \pmod 6$	γ_∞		$\gamma_{\frac{1}{2},0}$	$\gamma_{\frac{1}{3},1}$		$\gamma_{0,0}$
	$c \equiv 1 \pmod 6$	γ_∞		$\gamma_{\frac{1}{2},2}$	$\gamma_{\frac{1}{3},0}$		$\gamma_{0,5}$
	$c \equiv 2 \pmod 6$	γ_∞		$\gamma_{\frac{1}{2},1}$	$\gamma_{\frac{1}{3},1}$		$\gamma_{0,4}$
	$c \equiv 3 \pmod 6$	γ_∞		$\gamma_{\frac{1}{2},0}$	$\gamma_{\frac{1}{3},0}$		$\gamma_{0,3}$
	$c \equiv 4 \pmod 6$	γ_∞		$\gamma_{\frac{1}{2},2}$	$\gamma_{\frac{1}{3},1}$		$\gamma_{0,2}$
	$c \equiv 5 \pmod 6$	γ_∞		$\gamma_{\frac{1}{2},1}$	$\gamma_{\frac{1}{3},0}$		$\gamma_{0,1}$
$a \equiv 1 \pmod 6$	$c \equiv 0 \pmod 6$	$\gamma_{0,1}$		$\gamma_{0,3}$	$\gamma_{0,4}$		$\gamma_{0,0}$
	$c \equiv 2 \pmod 6$		$\gamma_{0,2}$			$\gamma_{0,5}$	
$a \equiv 2 \pmod 6$	$c \equiv 0 \pmod 6$	$\gamma_{\frac{1}{2},2}$		$\gamma_{\frac{1}{2},0}$	$\gamma_{0,2}$		$\gamma_{0,0}$
	$c \equiv 1 \pmod 6$		$\gamma_{0,1}$			$\gamma_{\frac{1}{2},1}$	
	$c \equiv 3 \pmod 6$	$\gamma_{\frac{1}{2},2}$		$\gamma_{\frac{1}{2},0}$	$\gamma_{0,5}$		$\gamma_{0,3}$
	$c \equiv 4 \pmod 6$		$\gamma_{0,4}$			$\gamma_{\frac{1}{2},1}$	
$a \equiv 3 \pmod 6$	$c \equiv 0 \pmod 6$	$\gamma_{\frac{1}{3},0}$		$\gamma_{0,3}$	$\gamma_{\frac{1}{3},1}$		$\gamma_{0,0}$
	$c \equiv 2 \pmod 6$	$\gamma_{\frac{1}{3},0}$		$\gamma_{0,1}$	$\gamma_{\frac{1}{3},1}$		$\gamma_{0,4}$
	$c \equiv 4 \pmod 6$	$\gamma_{\frac{1}{3},0}$		$\gamma_{0,5}$	$\gamma_{\frac{1}{3},1}$		$\gamma_{0,2}$
$a \equiv 4 \pmod 6$	$c \equiv 0 \pmod 6$	$\gamma_{\frac{1}{2},1}$		$\gamma_{\frac{1}{2},0}$	$\gamma_{0,4}$		$\gamma_{0,0}$
	$c \equiv 2 \pmod 6$		$\gamma_{0,2}$			$\gamma_{\frac{1}{2},2}$	
	$c \equiv 3 \pmod 6$	$\gamma_{\frac{1}{2},1}$		$\gamma_{\frac{1}{2},0}$	$\gamma_{0,1}$		$\gamma_{0,3}$
	$c \equiv 5 \pmod 6$		$\gamma_{0,5}$			$\gamma_{\frac{1}{2},2}$	
$a \equiv 5 \pmod 6$	$c \equiv 0 \pmod 6$	$\gamma_{0,5}$		$\gamma_{0,3}$	$\gamma_{0,2}$		$\gamma_{0,0}$
	$c \equiv 4 \pmod 6$		$\gamma_{0,4}$			$\gamma_{0,1}$	

Here is the bijection between $\mathcal{Q}_n^1$ and $\mathcal{Q}_n^6$. For each $Q \in \mathcal{Q}_n^1$, there exists a unique coset representative γ_Q from the above list such that the equivalence class of $Q \circ \gamma_Q^{-1}$ belongs to $\mathcal{Q}_n^6$. The explicit value of γ_Q is given in Table 1 of [3] which we reproduce above.

Using the bijection between $\mathcal{Q}_n^6$ and $\mathcal{Q}_n^1$, the Bruinier-Ono formula now becomes

$$(24n - 1)p(n) = \sum_{Q \in \mathcal{Q}_n^1} P(\gamma_Q \cdot \alpha_Q)$$

To determine the asymptotics of the right hand side, we exploit the modular connection. The Maass raising operator R_k acts on complex-valued functions on the upper half-plane and is given by

$$R_k = 2i\frac{\partial}{\partial z} + \frac{k}{y} = i\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) + \frac{k}{y}.$$

and this intertwines with the slash operator as:

$$R_k(F|_k\gamma) = (R_kF)|_{k+2}\gamma,$$

as is easily verified. From the preceding, we see that

$$P = \frac{1}{4\pi}R_{-2}F.$$

Thus, by the intertwining property,

$$P|_0\gamma = \left(\frac{1}{4\pi}R_{-2}F\right)|_0\gamma = \frac{1}{4\pi}R_{-2}(F|_{-2}\gamma).$$

Let $\zeta_6 = e^{2\pi i/6}$. The Fourier expansion of F at the cusp $i\infty$ is given by (5). The modular curve $X_0(6)$ has four cusps given by $0, 1/3, 1/2, i\infty$. At the other cusps, $1/3$, $1/2$ and 0, the expansions are given by

$$\begin{aligned} F\ |_{-2}\ \gamma_{\frac{1}{3},r} &= 2(-1)^r q^{-1/2} + 20 - (-1)^r 34q^{1/2} + \cdots \\ F\ |_{-2}\ \gamma_{\frac{1}{2},s} &= 3\zeta_6^{3-2s} q^{-1/3} + 30 - 87\zeta_6^{3+2s} q^{1/3} + \cdots \\ F\ |_{-2}\ \gamma_{0,t} &= 6\zeta_6^{-t} q^{-1/6} - 60 - 174\zeta_6^{t} q^{1/6} - \cdots . \end{aligned} \tag{9}$$

For $Q \in \mathcal{Q}_n^1$, let h_Q be the width of the cusp $\gamma_Q(i\infty)$ on the modular curve $X_0(6)$. Then $h_Q \in \{1, 2, 3, 6\}$. Define ζ_Q to be the sixth root of unity such that

$$F|_{-2}\gamma_Q = h_Q\zeta_Q q^{-1/h_Q} + O(1).$$

Since

$$P|_0\gamma_Q = \left(\frac{1}{4\pi}R_{-2}F\right)|_0\gamma_Q = \frac{R_{-2}}{4\pi}(F|_{-2}\gamma_Q),$$

we write this as

$$\frac{R_{-2}}{4\pi}(h_Q\zeta_Q q^{-1/h_Q}) + \frac{R_{-2}}{4\pi}(F|_{-2}\gamma_Q - h_Q\zeta_Q q^{-1/h_Q}).$$

Now, the first term is easily computed to be

$$\zeta_Q e^{-2\pi i z/h_Q}\left(1 - \frac{h_Q}{2\pi y}\right).$$

Thus,

$$P(\gamma_Q z) = (P|_0\gamma_Q)(z) = \zeta_Q\left(1 - \frac{h_Q}{2\pi y}\right)e^{-2\pi i z/h_Q} + \frac{R_{-2}}{4\pi}(F|_{-2}\gamma_Q - h_Q\zeta_Q q^{-1/h_Q}).$$

It is not difficult to show that the second term is bounded by an absolute constant (see [3] for details). Thus, the Bruinier-Ono formula leads to

$$(24n-1)p(n) = \sum_{Q\in\mathcal{Q}_n^1}\zeta_Q\left(1 - \frac{h_Q}{2\pi\Im(\alpha_Q)}\right)e^{-2\pi i\alpha_Q/h_Q} + O(h(1-24n)).$$

It is now easy to derive the Hardy-Ramanujan formula from this formula. First, since the dominant term in each summand is $e^{-2\pi i\alpha_Q/h_Q}$, we see that for $Q = [a, b, c]$ this is

$$\zeta e^{\pi\sqrt{24n-1}/ah_Q} \quad \text{where} \quad \zeta = e^{\pi bi/ah_Q}. \tag{10}$$

It is clear that the main contribution will arise when ah_Q is minimal. Now the widths of the cusps $i\infty = \gamma_\infty(i\infty)$, $\frac{1}{3} = \gamma_{\frac{1}{3},r}(i\infty)$, $\frac{1}{2} = \gamma_{\frac{1}{2},s}(i\infty)$ and $0 = \gamma_{0,t}(i\infty)$ are 1, 2, 3 and 6 respectively. For each $Q = [a, b, c] \in \mathcal{Q}_n^1$, we use Table 1 to identify γ_Q and as h_Q is the width of the cusp $\gamma_Q(i\infty)$, it is directly verified that in all cases, we have

$$ah_Q \equiv 0 (\bmod 6).$$

We therefore write our sum as

$$(24n-1)p(n) = \sum_{m\geq 1}\left(\sum_{\substack{Q=[a,b,c,]\in\mathcal{Q}_n^1\\ ah_Q=6m}} \zeta_{[a,b,c]}e^{\pi ib/6m}\right)$$

$$\times\left(1 - \frac{6m}{\pi\sqrt{24n-1}}\right)e^{\frac{\pi\sqrt{24n-1}}{6m}} + O(h(1-24n)).$$

The dominant term arises when $ah_Q = 6$ and then the exponential term in (10) is

$$\sim e^{\pi\sqrt{2n/3}},$$

which begins to agree with the exponential term in the Hardy-Ramanujan asymptotic formula. The forms Q that give rise to $ah_Q = 6$ are quickly identified and are four in number:

$$[1, 1, 6n], \quad [2, 1, 3n], \quad [3, 1, 2n], \quad \text{and} \quad [6, 1, n].$$

For each form, the corresponding root of unity ζ_Q is $\zeta_6^{-1}, \zeta_6^5, 1, 1$ respectively (see [3] for precise details). Since forms corresponding to $m \geq 2$ in the sum, that is, with $ah_Q \geq 12$ contribute at most

$$O(h(1-24n)e^{\sqrt{n/6}})$$

and since $h(1-24n) = O(\sqrt{n})$, we see that this term is

$$O(\sqrt{n}e^{\sqrt{n/6}}).$$

Thus, to derive the Hardy-Ramanujan asymptotic formula, we need only focus on the contribution from the term $m = 1$. From the four forms listed above, we obtain

$$e^{\pi i/6}(\zeta_6^{-1} + \zeta_6^5 + 1 + 1)e^{\frac{\pi\sqrt{24n-1}}{6}} = 2\sqrt{3}e^{\pi\sqrt{24n-1}/6}.$$

Putting everything together, we deduce

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}},$$

as n tends to infinity. As indicated in [3], a formula related to the Rademacher-Selberg explicit formula can also be derived from this analysis.

4. The coefficients of the j-function

It is interesting to note that once the asymptotic for $p(n)$ is known, one can determine the asymptotics of coefficients of other modular functions using some basic calculus. In particular, one can do this for the coefficients of the j-function. This is of historical interest in the sense that both Petersson in 1932 and Rademacher in 1938 used the circle method to derive the asymptotic behaviour of the coefficients of the j-function. Thus, this could have been done in 1918.

At the end of their paper, Hardy and Ramanujan indicate other implications of their method but do not seem to have noticed that Theorems 1, 2 and 4 below follow relatively easily from their work, without recourse to the circle method. We indicate this below.

Recall that the normalized Eisenstein series E_k of weight k is given by

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \quad \sigma_{k-1}(n) = \sum_{d|n} d^{k-1}, \quad q = e^{2\pi iz},$$

where B_k denotes the k-th Bernoulli number. For $k \geq 4$, this is a modular form of weight k for the full modular group $SL_2(\mathbb{Z})$. Recall also that the j-function is given by

$$j(z) = \frac{E_4(z)^3}{\Delta(z)}.$$

Let us write

$$j(z) = \frac{1}{q} + \sum_{n=0}^{\infty} c(n)q^n, \quad q = e^{2\pi iz}.$$

Then, Petersson [19] and Rademacher [20] showed that

$$c(n) \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}n^{3/4}}.$$

It would be of interest if one could derive a Bruiner-Ono type formula for $c(n)$ and thus obtain an exact formula for these coefficients. They are of substantial interest since they are related to the representation theory of the Monster group, as was discovered by McKay and Borcherds. Initially, Dewar and I were looking for such a formula, but soon realized that the asymptotic beahviour of $c(n)$ can be deduced from the asymptotics of $p(n)$. Here is a sketch of how this is done. For the details, we refer the reader to [4].

Let $M_k^!$ be the space of weight k weakly holomorphic modular forms on $SL_2(\mathbb{Z})$. These are meromorphic modular forms whose only poles (if any) are at $i\infty$. If $f \in M_k^!$ has $\mathrm{ord}_{i\infty} f = -m < 0$, then

$$f = \sum_{j=0}^{[k/12]+m} b_j E_{k+12(m-j)}\Delta^{j-m},$$

for some $b_j \in \mathbb{C}$, $b_0 \neq 0$. This is because $f\Delta^m$ is an ordinary (holomorphic) modular form of weight $k + m$ and one has an explicit basis in terms of E_k's and powers of Δ. Using the asymptotics of $p(n)$, we proved in [4] the following:

Theorem 1. *Suppose* $k \in 2\mathbb{Z}$*,* $f \in M_k^!$ *and* $\text{ord}_{i\infty} f = -m < 0$ *and that*

$$f(z) = \sum_{n=-m}^{\infty} \lambda_f(n) q^n, \quad q = e^{2\pi i z}.$$

Then, as $n \to \infty$*,*

$$\lambda_f(n) \sim i^k \frac{\lambda_f(-m)}{\sqrt{2n}} \left(\frac{n}{m}\right)^{k/2-1/4} e^{4\pi\sqrt{nm}}, \quad i = \sqrt{-1}.$$

This theorem will be used to prove the following theorem of Petersson [19] and Rademacher [20]:

Theorem 2. *The* j*-function is weakly holomorphic of weight zero with* $\text{ord}_{i\infty} j = -1$ *so that*

$$c(n) \sim \frac{1}{\sqrt{2}n^{3/4}} e^{4\pi\sqrt{n}},$$

which is the Petersson-Rademacher formula.

The proof makes use of Laplace's saddle point method. In the course of our studies, we derived the following theorem which is of independent interest and is really a theorem concerning power series.

Theorem 3. *Suppose*

$$f(z) = \sum_{n=0}^{\infty} \lambda_f(n) q^n,$$

$$g(z) = \sum_{n=0}^{\infty} \lambda_g(n) q^n,$$

where

$$\lambda_f(n) \sim c_f n^{\alpha} e^{A\sqrt{n}}, \quad \lambda_g(n) \sim c_g n^{\beta} e^{B\sqrt{n}}$$

with $\alpha, \beta, A, B, c_f, c_g \in \mathbb{R}$ *and* $A, B, c_f, c_g > 0$*. Then, for*

$$(fg)(z) = \sum_{n=0}^{\infty} \lambda_{fg}(n) q^n,$$

we have

$$\lambda_{fg}(n) \sim c_f c_g 2\sqrt{2\pi} \frac{A^{2\alpha+1} B^{2\beta+1}}{(A^2 + B^2)^{\alpha+\beta+5/4}} n^{\alpha+\beta+3/4} e^{\sqrt{(A^2+B^2)n}},$$

as $n \to \infty$*.*

The idea of the proof of this theorem is easy to explain. Clearly,

$$\lambda_{fg}(n) = \sum_{r=0}^{n} \lambda_f(r) \lambda_g(n-r). \tag{11}$$

Intuitively, this is approximated by

$$c_f c_g \sum_{r=0}^{n} r^{\alpha}(n-r)^{\beta} e^{A\sqrt{r}+B\sqrt{n-r}}$$

which can be re-written as

$$c_f c_g n^{\alpha+\beta} \sum_{r=0}^{n} G(r/n) e^{\sqrt{n}F(r/n)},$$

where

$$G(x) = x^{\alpha}(1-x)^{\beta}$$

and

$$F(x) = A\sqrt{x} + B\sqrt{1-x},$$

both being continuous functions on $(0, 1)$. The function $F(x)$ is easily seen to be an increasing function on $(0, c)$ and a decreasing function on $(c, 1)$, where

$$c = \frac{A^2}{A^2 + B^2}.$$

Note that $F(c) = \sqrt{A^2 + B^2}$. It is evident that the major contribution to the sum will come from values of j such that j/n is close to c. By the continuity of G, for any $\epsilon > 0$, there is a $\delta > 0$ such that for $|x - c| < 2\delta$

$$(1-\epsilon)G(c) < G(x) < (1+\epsilon)G(c)$$

so that our sum (11) in question can be split into three parts:

$$S_0 + S_1 + S_2$$

according as $r \le (c-\delta)n$; $(c-\delta)n < r < (c+\delta)n$; $r \ge (c+\delta)n$, respectively. The first and last sums are easily shown to be negligible. Indeed,

$$\begin{aligned} S_0 &= O\left(n^{|\alpha|+|\beta|} \sum_{r=0}^{[(c-\delta)n]} e^{\sqrt{n}F(r/n)}\right) \\ &= O(n^{|\alpha||\beta|+1} e^{\sqrt{n}F(c-\delta)}) \\ &= o(n^{\alpha+\beta+3/4} e^{\sqrt{n}F(c)}). \end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned} S_2 &= O(n^{|\alpha|+|\beta|+1} e^{\sqrt{n}F(c+\delta)}) \\ &= o(n^{\alpha+\beta+3/4} e^{\sqrt{n}F(c)}). \end{aligned} \tag{13}$$

Thus, the main contribution comes from S_1. For n sufficiently large, we see that

$$(1-\epsilon)^2 c_f c_g n^{\alpha+\beta} T < S_1 < (1+\epsilon)^2 c_f c_g n^{\alpha+\beta} T,$$

where

$$T = \sum_{r=[(c-\delta)n]}^{[(c+\delta)n]} G(r/n) e^{\sqrt{nF(r/n)}}.$$

In the interval of summation,

$$(1-\epsilon)G(c) < G(r/n) < (1+\epsilon)G(c),$$

so that

$$(1-\epsilon)^3 c_f c_g G(c) n^{\alpha+\beta+1} R < S_1 < (1+\epsilon)^3 c_f c_g G(c) n^{\alpha+\beta+1} R$$

where

$$R = \sum_{r=[(c-\delta)n]}^{[(c+\delta)n]} \frac{1}{n} e^{\sqrt{nF(r/n)}}.$$

It remains to determine the asymptotics of R, which resembles a Riemann sum for the integral

$$\int_{c-\delta}^{c+\delta} e^{\sqrt{n}F(x)} dx.$$

This intuition is accurate and the latter integral can be studied using Laplace's saddle point method (sometimes called the method of stationary phase in the literature). The precise details of the transition from our sum to the integral are given in Lemma 4 of our paper [4]. For this exposition, it suffices to indicate the general philosopy of the method, which can be applied to study the asymptotic behaviour of

$$\int_a^b e^{tF(x)} dx,$$

as t tends to infinity. The idea is to replace $F(x)$ by its Taylor expansion at $x = c$. Since $F(x)$ has a maximum at $x = c$, we see that $F'(c) = 0$ and $F''(c) < 0$. By the extended mean value theorem, we have for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\left| F(x) - F(c) - \frac{1}{2}F''(c)(x-c)^2 \right| < \epsilon(x-c)^2,$$

provided $|x-c| < 2\delta$. Thus, our integral is

$$\sim e^{\sqrt{n}F(c)} \int_{c-\delta}^{c+\delta} e^{\sqrt{n}\frac{1}{2}F''(c)(x-c)^2} dx.$$

We change variables and put $v = n^{1/4}(x-c)$ so that this integral is

$$\sim e^{\sqrt{n}F(c)} n^{-1/4} \int_{-\delta n^{1/4}}^{\delta n^{1/4}} e^{\frac{1}{2}F''(c)v^2} dv$$

The integral now tends to the classical probability integral

$$\int_{-\infty}^{\infty} e^{\frac{1}{2}F''(c)v^2}\,dv,$$

keeping in mind that $F''(c) < 0$, so that convergence is assured. We finally conclude that

$$S_1 \sim c_f c_g G(c)\sqrt{\frac{2\pi}{|F''(c)|}}\, n^{\alpha+\beta+3/4} e^{\sqrt{n}F(c)},$$

from which our result follows. This completes our sketch of the proof of Theorem 3.

We apply this to derive asymptotics of the multipartition function (sometimes called the "colored partition" function), $p_k(n)$ which is given by the generating function

$$\sum_{n=0}^{\infty} p_k(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^{-k}.$$

We proceed inductively to show:

Theorem 4. *Let*

$$c_k = \frac{1}{\sqrt{2}}(k/24)^{\frac{k+1}{4}}, \quad \alpha_k = -k/4 - 3/4, \quad A_k = \pi\sqrt{2k/3}.$$

Then, for any fixed k,

$$p_k(n) \sim c_k n^{\alpha_k} e^{A_k\sqrt{n}},$$

as n tends to infinity.

Proof. We proceed by induction and use Theorem 3. For $k = 1$, this is the Hardy-Ramanujan formula. Applying Theorem 3 we easily find the recursions:

$$\alpha_{k+1} = \alpha_k - 1/4$$

$$A_{k+1} = \sqrt{A_k^2 + 2\pi^2/3}$$

from which the formulas for α_k and A_k easily follow. The only complicated term is the constant term c_{k+1} for which we have the recurrence:

$$c_{k+1} = 2\sqrt{2\pi}\,\frac{c_k}{4\sqrt{3}}\,\frac{A_k^{2\alpha_k+1}(\pi\sqrt{2/3})^{-1}}{(A_k^2+2\pi^2/3)^{-k/4-2/4}}$$

and a routine (but a bit tedious) calculation completes the proof. □

This theorem allows us to determine the asymptotic behaviour of the coefficients of powers of $1/\Delta$ where Δ is the Ramanujan cusp form. Indeed,

$$q^k\Delta^{-k} = \prod_{n=1}^{\infty}(1-q^n)^{-24k} = \sum_{n=0}^{\infty} p_{24k}(n)q^n.$$

By the previous theorem,

$$p_{24k}(n) \sim \frac{1}{\sqrt{2n}}(k/n)^{6k+1/4} e^{4\pi\sqrt{kn}}.$$

To derive the asymptotics of the coefficients of the j-function, we combine this with the following.

Theorem 5. *Suppose $k \geq 4$ is even and*

$$f(z) = \sum_{n=0}^{\infty} \lambda_f(n) q^n, \quad \lambda_f(n) \geq 0, \quad q = e^{2\pi i z}.$$

Suppose that

$$\lambda_f(n) \sim c_f e^{A\sqrt{n}}/n^{\alpha}, \quad c_f, A, \alpha > 0.$$

Let E_k be the normalized Eisenstein series of weight k for the full modular group. Then, for

$$fE_k(z) = \sum_{n=0}^{\infty} \lambda_{fE_k}(n) q^n,$$

we have

$$\lambda_{fE_k}(n) \sim \frac{c_f e^{A\sqrt{n}}}{n^{\alpha-k/2}} \left(\frac{4\pi i}{A}\right)^k, \quad i = \sqrt{-1}.$$

We indicate the proof of Theorem 5. Recall that

$$E_k = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

is the Eisenstein series of weight k for the full modular group with

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1},$$

and B_k is the k-th Bernoulli number. Keeping in mind that k is even, it is also useful to recall that $i^k B_k < 0$ where $i = \sqrt{-1}$. In other words, the coefficients of the q-expansion of $g = i^k(E_k - 1)$ are all positive. Then,

$$fE_k = f + i^k fg,$$

so that

$$\lambda_{fE_k}(n) = \lambda_f(n) + i^k \lambda_{fg}(n).$$

Since the asymptotics of the first term on the right hand side is given, it suffices to determine the asymptotics of the second. As before,

$$\lambda_{fg}(n) = \sum_{r=0}^{n} \lambda_f(n-r)\lambda_g(r).$$

Because of the exponential growth of $\lambda_f(n)$, the bulk of the contribution to this sum comes from terms r with $r < \delta n$ with $\delta < 1$. Thus, putting $H(x) = (1-x)^{-\alpha}$ for $0 < x < 1$, our sum becomes

$$c_f n^{-\alpha} \sum_{r<\delta N} \lambda_g(r) H(r/n) e^{A\sqrt{n}\sqrt{1-r/n}}.$$

Again by Taylor's theorem, $\sqrt{1-r/n}$ can be approximated by $1 - r/2n$ for r/n sufficiently small and $H(r/n)$ is approximated by 1. Thus, our sum becomes

$$\sim \frac{c_f e^{A\sqrt{n}}}{n^{\alpha}} \sum_{r<\delta n} \lambda_g(r) e^{-Ar/2\sqrt{n}}.$$

It is not difficult to extend the sum to infinity and calculate the negligible error term that emerges. Our sum then becomes

$$\sim \frac{c_f e^{A\sqrt{n}}}{n^{\alpha}} \sum_{r=1}^{\infty} \lambda_g(r) e^{-Ar/2\sqrt{n}}.$$

From this, we see that the sum is a specialization of $g = i^k(E_k - 1)$ at $z = iA/4\pi\sqrt{n}$. By the modularity of $E_k(z)$, this is quickly determined and we find that the sum is

$$\sim \frac{c_f e^{A\sqrt{n}}}{n^{\alpha-k/2}} (4\pi/A)^k,$$

from which the main result of our theorem follows. There are of course, several technical details to be considered and the reader can refer to [4] for them. But we hope this discussion highlights the essential strategy.

We remark that to study $c(n)$, we first apply Theorem 3 to derive the asymptotics of Fourier coefficients of $1/\Delta$ which is essentially $1/\eta^{24}$. We then apply Theorem 5 to E_4^3/Δ which is the j-function. From this, Theorem 2 is easily deduced.

5. Concluding remarks

Clearly, the methods outlined in this paper have potential applications to other problems, most notably to those where the cumbersome circle method had been previously applied. The study of the asymptotics of the partition function is nearly a 100 years old and as there is an extensive literature, one can re-examine much of the literature to see where our methods are applicable.

For instance, in 1954, Meinardus [14] obtained using the saddle point method the asymptotic formula for the general coefficient of the power series expansion of general products of the form

$$\prod_{n=1}^{\infty} (1-q^n)^{-a_n},$$

where a_n are real and non-negative numbers. He obtained a general asymptotic formula of the type

$$\sim Cn^{\kappa} \exp(n^{\alpha/\alpha+1}(1 + 1/\alpha)(A\Gamma(\alpha + 1)\zeta(\alpha + 1))^{1/(\alpha+1)})$$

where α is the abscissa of convergence of the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

A is its residue at $s = \alpha$. C, κ are explicit constants and ζ denotes the Riemann zeta function. As an application of his method, he derives the asymptotic formula for the generalized partition function $p_{a,b}(n)$, whose generating function is given by

$$\sum_{n=0}^{\infty} p_{a,b}(n)q^n = \prod_{n=1}^{\infty}(1 - q^{an+b})^{-1}.$$

The general theorem of Meinardus also implies our Theorem 4.

A general Tauberian theorem for partition-type functions was derived by Ingham in [10]. Ingham's paper contains a study of $p_{a,b}(n)$ and derives an asymptotic formula for it. It would be interesting to explore if there is a Rademacher-Selberg style exact formula for it.

Several authors have attempted elementary approaches to the study of the Hardy-Ramanujan formula and its generalizations. In 1942, Erdös [6] derived the Hardy-Ramanujan asymptotic formula by showing that

$$p(n) \sim \frac{Ae^{\pi\sqrt{2n/3}}}{n},$$

but he could not determine the constant A by his method. This was done later by Newman [18]. With respect to the elementary study of $p_{a,b}(n)$, there are several readable accounts, a recent one being the article [11]. It would be interesting to re-examine all of these partition questions in the light of the theory of weak Maass forms.

Finally, this being a paper to commemorate the legacy of Srinivasa Ramanujan, it seems fitting to look back from our vantage point of the 21st century and see his contributions in the mathematical landscape of the 20th century. Certainly, the remarkable feature of his work is its interconnectedness with other parts of mathematics and other branches of science, most notably physics. Indeed, the relationship of his work to Lie groups and representation theory is illuminating and this is highlighted through the Langlands program (see for example, [16] as well as [17]). A further avenue to explore in this context are the recent works connecting the Rogers-Ramanujan identities with representation theory of Kac-Moody Lie algebras (see [12]). Surely, "what now we see is a shadow of what must come."

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