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PRIME DIVISORS OF FOURIER COEFFICIENTS OF MODULAR FORMS

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§1. Introduction. The Ramanujan τ-function is defined by

$$\Delta = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

Ramanujan [6] investigated the divisibility properties of $\tau(n)$ and conjectured that $\tau(n) \equiv 0 \pmod{691}$ for almost all n. This was verified by Watson [12]. Serre [9] has strengthened this to the following assertion: given an integer d, we have $\tau(n) \equiv 0 \pmod{d}$ for almost all n (i.e., for all n excepting a set of density 0). In fact, Serre's result holds for the Fourier coefficients of modular forms of integral weight for any congruence subgroup of $SL_2(\mathbf{Z})$.

The purpose of this paper is to further investigate the divisibility properties of these coefficients. For definiteness, we shall state the results for τ, though they apply to more general multiplicative functions.

We first prove the following strengthening of Serre's result: given d as above, $\tau(n)$ is divisible by d^{ω}, where $\omega = [\delta \log\log n]$, for almost all n. (Here δ is a positive constant depending on d.) We then consider the effect of varying d. Denote by $\nu(n)$ the number of *distinct* prime divisors of n. Assuming the Generalized Riemann Hypothesis (GRH), we show that

$$\sum_{\substack{p \leq x \\ \tau(p) \neq 0}} (\nu(\tau(p)) - \log\log p)^2 \ll \pi(x) \log\log x$$

and

$$\sum_{\substack{n \leq x \\ \tau(n) \neq 0}} \left(\nu(\tau(n)) - \frac{1}{2} (\log\log n)^2 \right)^2 \ll x (\log\log x)^3 \log_4 x.$$

(Here, $\log_4 x = \log\log\log\log x$.) In particular, given $\epsilon > 0$, we have

$$|\nu(\tau(p)) - \log\log p| < (\log\log p)^{1/2+\epsilon}$$

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for almost all primes p. These estimates are the "modular analogues" of the classical result of Hardy and Ramanujan [4] that

$$\sum_{n \leqslant x} (\nu(n) - \log\log n)^2 \ll x \log\log x.$$

Observe that $\nu(\tau(n))$ is neither a multiplicative nor an additive function, and so seems to fall outside the scope of existing generalizations of the Hardy–Ramanujan estimate (see for example, Elliott [2]).

Using our estimates, it is possible to deduce that for any $\epsilon > 0$, we have

$$\log|\tau(p)| \geqslant we^{-2w}\log p \geqslant (\log p)^{1-\epsilon}$$

for almost all p, where $w = (\log\log p)^{1/2+\epsilon}$. Of course, this falls far short of what is expected. For example, the above bound does not give $|\tau(p)| > p^{\epsilon}$, whereas it is conjectured [9] that

$$|\tau(p)| \gg p^{9/2-\epsilon}$$

for *all* primes p.

Finally, we remark that the full strength of the GRH is not needed in the above results. Indeed, let

$$\pi^*(x,d) = \#\{p \leqslant x \,|\, p \text{ prime and } \tau(p) \equiv 0 \pmod d\}.$$

It is known that $\pi^*(x,d) \sim \delta(d)\pi(x)$ for some $\delta(d) > 0$. The GRH implies that uniformly in d,

$$\pi^*(x,d) = \delta(d)\pi(x) + O(d^3 x^{1/2}\log x)$$

and in particular, that

$$\sum_{d \leqslant x^{\theta}} |\pi^*(x,d) - \delta(d)\pi(x)| \ll x^{1/2+4\theta}\log x.$$

For our purposes, it would suffice to know that

$$\sum_{d \leqslant x^{1/F(x)}} |\pi^*(x,d) - \delta(d)\pi(x)| \ll \frac{x}{(\log x)^{1+\gamma}}$$

for some $\gamma > 0$, and some monotone increasing function F such that $F(x) > 1$ and $F(x) = o(\log\log x)$. It would be of interest to know whether this estimate can be proved by known techniques of analytic number theory.

As mentioned above, our methods work for a large class of multiplicative functions. We shall present the general case in Sections 2, 3 and 4. In Section 5, we shall specialize to modular forms. The applications to lower bounds are discussed in Section 6.

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Notation. Throughout, p, q, r, p_1, p_2 denote rational primes; $\pi(x)$ denotes the number of primes $\leqslant x$; $\nu(n)$ denotes the number of *distinct* prime divisors of the rational integer n. We say that a sequence $\{a_n\}_{n\in I}$ has *normal order* $\{b_n\}_{n\in I}$ (I some indexing set) if for any $\epsilon > 0$,

$$\frac{\#\{n \leqslant x, n \in I \text{ and } |a_n - b_n| < \epsilon b_n\}}{\#\{n \leqslant x, n \in I\}} \to 1 \quad \text{as} \quad x \to \infty.$$

§2. Divisibility by a fixed integer. Let f be a nonzero multiplicative function $f : \mathbf{N} \to \mathcal{O}$ from the natural numbers to the ring of integers of an algebraic number field. We define for each $d \in \mathbf{Z}$

$$\pi_f^*(x,d) = \#\{p \leqslant x \,|\, f(p) \equiv 0 \pmod d\}.$$

Suppose that the following holds:

$$\text{there is a function } \delta \text{ such that } \pi_f^*(x,d) \sim \delta(d)\pi(x) \quad \text{as} \quad x \to \infty. \qquad (*)$$

THEOREM 2.1. *Let d be a fixed positive integer and $\epsilon > 0$. Then, for almost all n, $f(n)$ is divisible by*

$$d^{[(1-\epsilon)\delta(d)\log\log n]}.$$

Proof. Define

$$\nu(d,n) = \#\{p^\alpha \,|\, p^\alpha \| n \text{ and } f(p^\alpha) \equiv 0 \pmod d\}.$$

Then, we have

$$\sum_{n \leqslant x} \nu(d,n) = \sum_{\substack{p^\alpha \leqslant x \\ f(p^\alpha) \equiv 0 \pmod d}} \sum_{\substack{n \leqslant x \\ p^\alpha \| n}} 1.$$

The contribution from all terms with $\alpha \geqslant 2$ is clearly $O(x)$. For the remaining terms, we have

$$\sum_{\substack{p \leqslant x \\ f(p) \equiv 0 \pmod d}} \sum_{\substack{n \leqslant x \\ p \| n}} 1 = \sum_{\substack{p \leqslant x \\ f(p) \equiv 0 \pmod d}} \left\{ \frac{x}{p} \cdot \frac{\phi(p)}{p} + O(1) \right\} = \sum_{\substack{p \leqslant x \\ f(p) \equiv 0 \pmod d}} \frac{x}{p} + O(x)$$

$$= (\delta(d) + o(1))x \log\log x$$

using (*) and partial summation. Next,

$$\sum_{n \leqslant x} \nu(d,n)^2 = \sum_{\substack{p_1, p_2 \leqslant x \\ f(p_1) \equiv f(p_2) \equiv 0 \pmod{d}}} \left[\sum_{\substack{n \leqslant x \\ p_1 \| n \\ p_2 \| n}} 1 \right] + O(x \log\log x)$$

$$= \sum_{\substack{p_1 p_2 \leqslant x \\ p_1 \neq p_2 \\ f(p_1) \equiv f(p_2) \equiv 0 \pmod{d}}} \left\{ \frac{x}{p_1 p_2} \cdot \frac{\phi(p_1 p_2)}{p_1 p_2} + O(1) \right\} + O(x \log\log x)$$

$$= x \left(\sum_{\substack{p \leqslant x \\ f(p) \equiv 0 \pmod{d}}} \frac{1}{p} \right)^2 + O(x \log\log x)$$

$$= \left(\delta(d)^2 + o(1)\right) x (\log\log x)^2.$$

Hence,

$$\sum_{n \leqslant x} (\nu(d,n) - \delta(d) \log\log x)^2 = o\left(x (\log\log x)^2\right).$$

It is easily deduced that

$$\sum_{n \leqslant x} (\nu(d,n) - \delta(d) \log\log n)^2 = o\left(x (\log\log x)^2\right).$$

Thus, given $\epsilon > 0$, we have $\nu(d,n) > (1-\epsilon)\delta(d)\log\log n$ for all but $o(x)$ of the $n \leqslant x$. This proves the result.

§3. Prime divisors of $f(q)$. In this section, we shall suppose that the function f of §2 takes values in the rational integers (i.e., $\mathcal{O} = \mathbf{Z}$). Our purpose is to study the normal order of $\nu(f(q))$.

Let $F: \mathbf{R} \to \mathbf{R}$ be a monotone increasing function such that $F(x) = O(\log x)$ and $F(x) \geqslant 2$. Let $y = y(x) = x^{1/F(x)}$. Define

$$Z_f(x) = \#\{p \leqslant x \mid f(p) = 0\}$$

and

$$\pi_f(x,d) = \#\{p \leqslant x, 0 \neq f(p) \equiv 0 \pmod{d}\}.$$

Throughout this section, we shall suppose that the following hold:

(C1) there is a function $\delta: \mathbf{N} \to \mathbf{R}_{>0}$ such that

$$\pi_f(x,d) \sim \delta(d)\pi(x) \qquad \text{as} \quad x \to \infty$$

and satisfying $\delta(pq) = \delta(p)\delta(q) + O(1/pq(p+q))$ for all large distinct primes p, q.

(C2) there exists a $\beta > 0$ such that $|f(n)| < n^{\beta}$ for *all* n

(C3) $\sum_{d \leqslant y} |\pi_f(x,d) - \delta(d)\pi(x)| = O(\pi(x))$

(C4) $Z_f(x) = o(\pi(x))$.

Note that given (C4), the δ of (C1) is the same as in (*). To simplify the exposition, we shall also assume that

$$\delta(p) = \frac{1}{p} + O\left(\frac{1}{p^2}\right)$$

for all sufficiently large p. Such a restriction is certainly not necessary, and the reader is referred to Remark 2 at the end of this section, where a more general case is briefly discussed.

THEOREM 3.1. *Let $\nu_u(n)$ denote the number of distinct prime divisors of n which are less than u. Then*

$$\sum_{q \leqslant x}{}' (\nu_u(f(q)) - \log\log u)^2$$
$$= Z_f(x)(\log\log u)^2 + O(\pi(x)\log\log u) + O\left(\pi(x)F(x)^2\right)$$

where the prime on the sum indicates that only those q with $f(q) \neq 0$ are included.

Remark. If $u^2 \leqslant y$, the second 0-term may be dropped. If $\sqrt{y} \leqslant u \leqslant y$, the second 0-term may be replaced with $O(\pi(x)F(x)\log\log u)$.

COROLLARY 3.2. *Suppose that $F(x) = o(\log\log x)$. Then*

$$\sum_{q \leqslant x}{}' (\nu(f(q)) - \log\log q)^2 = o\left(\pi(x)(\log\log x)^2\right).$$

In particular, $\nu(f(q))$ has normal order $\log\log q$.

The proof depends on several lemmas. First, it is convenient to define the set

$$\Omega(d) = \{q \mid 0 \neq f(q) \equiv 0 \pmod{d}\}.$$

Thus $\pi_f(x,d) = \#\{q \leqslant x \mid q \in \Omega(d)\}$.

LEMMA 3.3. $\pi_f(x,d) = 0$ *unless* $d < x^{\beta}$.

Proof. If $q \leqslant x$ and $q \in \Omega(d)$, then $d \leqslant |f(q)| < q^{\beta} \leqslant x^{\beta}$.

LEMMA 3.4. *If $u \leqslant y$, then*

$$\sum_{p \leqslant u} \pi_f(x,p) = \pi(x)\log\log u + O(\pi(x)).$$

If $u > y$, then

$$\sum_{p \leqslant u} \pi_f(x, p) = \pi(x)\log\log x + O(\pi(x)F(x)).$$

Proof. If $u \leqslant y$, it follows from (C3) that

$$\sum_{p \leqslant u} \pi_f(x, p) = \sum_{p \leqslant u} \pi(x)\delta(p) + O(\pi(x)) = \pi(x)\log\log u + O(\pi(x)).$$

If $u > y$, then

$$\sum_{y < p \leqslant u} \pi_f(x, p) = \sum_{q \leqslant x}{}' \sum_{\substack{y < p \leqslant u \\ p \mid f(q)}} 1 \ll \sum_{q \leqslant x}{}' \frac{\log x}{\log y} \ll \pi(x)F(x).$$

Remark 3.5. More generally, it can be proved by essentially the same method that

$$\sum_{\substack{d \leqslant x^\beta \\ \nu(d) = k}} \mu^2(d)\pi_f(x, d) = \frac{1}{k!}\pi(x)(\log\log x)^k + O\big(\pi(x)F(x)(\log\log x)^{k-1}\big).$$

We shall use this, in the case $k = 2$, in §4.

Proof of Theorem 3.1. First suppose that $u^2 \leqslant y$. Then

$$\sum_{q \leqslant x}{}' \nu_u(f(q)) = \pi(x)\log\log u + O(\pi(x)). \tag{3.1}$$

To see this, write the left hand side as

$$\sum_{q \leqslant x}{}' \sum_{\substack{p \mid f(q) \\ p < u}} 1 = \sum_{p < u} \pi_f(x, p)$$

and apply Lemma 3.4. Next, we shall check that

$$\sum_{q \leqslant x}{}' \nu_u^2(f(q)) = \pi(x)(\log\log u)^2 + O(\pi(x)\log\log u). \tag{3.2}$$

Indeed, the sum on the left is

$$\sum_{q \leqslant x}{}' \sum_{\substack{p_1, p_2 < u \\ p_1, p_2 \mid f(q)}} 1 = \sum_{\substack{p_1, p_2 < u \\ p_1 \neq p_2}} \pi_f(x, p_1 p_2) + O(\pi(x)\log\log u).$$

Using (C1) and (C3), the first sum may be written as

$$\sum_{\substack{p_1, p_2 < u \\ p_1 \neq p_2}} \delta(p_1 p_2)\pi(x) + O(\pi(x)) = \pi(x)(\log\log u)^2 + O(\pi(x)\log\log u).$$

Now (3.1) and (3.2) together imply the theorem. If $u^2 > y$, then by the Schwarz inequality

$$\sum_{q \leqslant x}{}' (\nu_u(f(q)) - \log\log u)^2 \ll \sum_{q \leqslant x}{}' (\nu_{\sqrt{y}}(f(q)) - \log\log u)^2 + \sum_{q \leqslant x}{}' (\nu_u(f(q)) - \nu_{\sqrt{y}}(f(q)))^2.$$

Since $\nu_u(f(q)) - \nu_{\sqrt{y}}(f(q)) = O(F(x))$, the second sum on the right is $O(\pi(x) F(x)^2)$. This completes the proof.

Corollary 3.2 follows from the theorem by using the Schwarz inequality and (C4).

Remarks. 1. (C4) was only used to deduce Corollary 3.2. We observe that if f vanishes on a set of primes of positive density, $\nu(f(q))$ does not have a normal order. This follows from Theorem 3.1.

2. Set $D(x) = \sum_{p \leqslant x} \delta(p)$. Under some suitable hypotheses, the above method will show that $\nu(f(q))$ has normal order $D(q)$. For example, the following assumptions would suffice in place of the assumption $\delta(p) = 1/p + O(1/p^2)$:

(i) $D(x) - D(x^{1/2}) = O(1)$

(ii) $\sum_{p>x} \delta(p)^2 = O(1/x)$

(iii) (C3) holds with $F(x) = o(D(x))$.

3. There should be no difficulty in removing the assumption that $\mathcal{O} = \mathbf{Z}$, but we have not carried it out.

§4. Prime divisors of $f(n)$. As before, f is a nonzero integer valued multiplicative function. We shall suppose that it satisfies hypotheses (C1), (C2) and the following strengthening of (C3) and (C4):

(C3′) $$\sum_{d \leqslant y} |\pi_f(x,d) - \delta(d)\pi(x)| \ll \frac{x}{(\log x)^{1+\gamma}} \quad \text{for some} \quad \gamma > 0$$

(C4′) $$Z_f(x) \ll \frac{x}{(\log x)^{1+\gamma}}\, .$$

As before, we shall continue to assume that $\delta(p) = (1/p) + O(1/p^2)$ for all large p. Our aim in this section is to prove the following.

THEOREM 4.1.

$$\sum_{n \leqslant x}{}' \left(\nu(f(n)) - \frac{1}{2}(\log\log n)^2\right) \ll x(F(x) + \log_4 x)(\log\log x)^3.$$

The proof will require several lemmas.

Let d be a positive integer, and define

$$B_d(x) = \#\left\{ n \leqslant x \mid d \mid n, \left(d, \frac{n}{d}\right) = 1 \text{ and } f(n) \neq 0 \right\}.$$

We shall need an asymptotic formula for $B_d(x)$ which is uniform in d. The case $d = 1$ is described in Serre [10, §6.3] and we follow his approach.

Let $f_0(n) = 1$ if $f(n) \neq 0$ and $f_0(n) = 0$ if $f(n) = 0$. Define a function c_d by

$$c_d(n) = \sum_{\substack{e \mid n \\ (e,d)=1}} \mu\left(\frac{n}{e}\right) f_0(e).$$

Then c_d is multiplicative and

$$\sum_{e \mid n} c_d(e) = \begin{cases} f_0(n) & \text{if } (n,d) = 1 \\ 0 & \text{else.} \end{cases}$$

We have

$$1 + \sum_{m=1}^{\infty} c_d(p^m) p^{-ms} = \begin{cases} \left(1 - \frac{1}{p^s}\right)\left(1 + \sum_{m=1}^{\infty} f_0(p^m) p^{-ms}\right) & \text{if } p \nmid d \\ \left(1 - \frac{1}{p^s}\right) & \text{if } p \mid d. \end{cases} \tag{4.1}$$

We find that

$$\left(1 - \frac{1}{p^s}\right)\left(1 + \sum_{m=1}^{\infty} f_0(p^m) p^{-ms}\right) = \begin{cases} 1 + O(p^{-2s}) & \text{if } f(p) \neq 0 \\ 1 - \frac{1}{p^s} + O(p^{-2s}) & \text{if } f(p) = 0. \end{cases}$$

LEMMA 4.2. *If $f(d) \neq 0$, then*

$$B_d(x) = c\lambda_d \cdot \frac{x}{d} + O\left(2^{\nu(d)} \frac{x}{d} \cdot \frac{1}{(1 + \log(x/d))^{1+\gamma}}\right)$$

uniformly for $d \leqslant x$. Here

$$c = \sum_{m=1}^{\infty} \frac{c_1(m)}{m}$$

is the density of integers n such that $f(n) \neq 0$ and

$$\lambda_d = \frac{\phi(d)}{d} \prod_{p \mid d} \left(1 + \sum_{m=1}^{\infty} c_1(p^m) p^{-m}\right)^{-1}. \tag{4.2}$$

Proof. We have

$$B_d(x) = \sum_{m \leqslant x/d} \sum_{e \mid m} c_d(e) = \frac{x}{d} \sum_{e \leqslant x/d} \frac{c_d(e)}{e} + O\left(\sum_{e \leqslant x/d} |c_d(e)|\right).$$

For each integer e, let e_1 be the largest divisor of e which is coprime to d, and write $e = e_1 e_2$. Then $(e_1, e_2) = 1$ and $c_d(e_1) = c_1(e_1)$, $c_d(e_2) = \mu(e_2)$. Hence

$$\sum_{e \leqslant x/d} |c_d(e)| = \sum_{e_2 \leqslant x/d}^{*} |c_d(e_2)| \sum_{\substack{e_1 \leqslant x/de_2 \\ (e_1, d) = 1}} |c_d(e_1)| \leqslant 2^{\nu(d)} \sum_{e_1 \leqslant x/d} |c_1(e_1)|$$

where the * on the sum indicates that $p \mid e_2 \Rightarrow p \mid d$. From (C4′), it follows that

$$\sum_{n \leqslant x} |c_1(n)| \ll \frac{x}{(\log x)^{1+\gamma}}$$

(see, for example, Wirsing [13, p. 89]). Hence,

$$\sum_{e \leqslant x/d} |c_d(e)| \ll 2^{\nu(d)} \frac{x}{d} \cdot \frac{1}{(1 + \log(x/d))^{1+\gamma}}.$$

By a similar method

$$\sum_{e > x/d} \frac{|c_d(e)|}{e} \ll 2^{\nu(d)} \frac{x}{d} \cdot \frac{1}{(1 + \log(x/d))^{1+\gamma}}.$$

The lemma is proved if we observe that $\sum_{e=1}^{\infty} c_d(e)/e = c\lambda_d$, as follows from (4.1).

We also note the following as it will be useful later on.

LEMMA 4.3. *We have* $\lambda_d = \pi_{p \mid d}(1 + O(1/p))$.

Proof. This follows from (4.1) and (4.2).

For an integer n, let $n_1 = \pi_{p \| n} p$ and write $n = n_1 n_2$. Let $\nu_1(n) = \nu(f(n_1))$ and $\nu_2(n) = \nu(f(n_2))$.

LEMMA 4.4.

$$\sum_{n \leqslant x}' \nu(f(n)) = \sum_{n \leqslant x}' \nu_1(n) + O(x)$$

$$\sum_{n \leqslant x}' \nu^2(f(n)) = \sum_{n \leqslant x}' \nu_1^2(n) + O\left(x^{1/2}\Big(\sum_{n \leqslant x}' \nu_1^2(n)\Big)^{1/2}\right) + O(x).$$

Proof. We have $\nu_1(n) \leqslant \nu(f(n)) \leqslant \nu_1(n) + \nu_2(n)$. Also,

$$\sum_{n \leqslant x}' \nu_2(n) \leqslant \sum_{p < x^{\beta}} \sum_{\substack{q^{\alpha} \leqslant x \\ \alpha \geqslant 2 \\ f(q^{\alpha}) \equiv 0 \ (\mathrm{mod}\ p)}} B_{q^{\alpha}}(x) \ll \sum_{q \leqslant \sqrt{x}} \sum_{\substack{\alpha = 2 \\ f(q^{\alpha}) \neq 0}}^{\infty} \frac{x}{q^{\alpha}} \nu(f(q^{\alpha})).$$

Using the trivial estimate $\nu(n) \leqslant \log n$, and condition (C2), we find that this sum is $O(x)$. A similar argument also shows that $\sum'_{n<x} \nu_2^2(n) = O(x)$. Now the first

statement of the lemma follows immediately, and the second follows from an application of the Schwartz inequality.

Define the sums

$$S(x)=\sum_{p<x^{\beta}}\sum_{\substack{q\leqslant x\\ q\in\Omega(p)}}B_q(x)$$

$$T(x)=\sum_{p<x^{\beta}}\sum_{\substack{m=q_1q_2\leqslant x\\ q_1<q_2\\ q_1,q_2\in\Omega(p)}}B_m(x).$$

As is easily verified, these sums are related to the average of ν_1 as follows:

$$S(x)-T(x)\leqslant\sum_{n\leqslant x}{}'\,\nu_1(n)\leqslant S(x). \tag{4.3}$$

Next, we find estimates for S and T.

LEMMA 4.5.

$$\sum_{p<x^{\beta}}\sum_{\substack{q\leqslant x\\ q\in\Omega(p)}}\frac{1}{q}=\frac{1}{2}(\log\log x)^2+O(F(x)\log\log x).$$

Proof. By partial summation, the sum in question is

$$\sum_{p\leqslant x^{\beta}}\left\{\frac{1}{x}\pi_f(x,p)+\int_2^x\pi_f(t,p)\frac{dt}{t^2}\right\}=\Sigma_1+\Sigma_2\quad\text{(say)}.$$

By Lemma 3.4,

$$\Sigma_1\ll\frac{1}{\log x}\{\log\log x+F(x)\}.$$

Using also Lemma 3.3,

$$\begin{aligned}\Sigma_2&=\int_2^x\left\{\sum_{p\leqslant x^{\beta}}\pi_f(t,p)\right\}\frac{dt}{t^2}=\int_2^x\left\{\sum_{p\leqslant t^{\beta}}\pi_f(t,p)\right\}\frac{dt}{t^2}\\&=\int_2^x\{\pi(t)\log\log t+O(\pi(t)F(t))\}\frac{dt}{t^2}\\&=\frac{1}{2}(\log\log x)^2+O(F(x)\log\log x),\end{aligned}$$

proving the lemma.

LEMMA 4.6. $S(x) = (1/2)cx(\log\log x^2) + O(xF(x)\log\log x)$.

Proof. Using Lemma 4.2, we find that for any $w < x/3$,

$$S(x) = \sum_{p<x^\beta} \sum_{\substack{q \leqslant w \\ q \in \Omega(p)}} \left\{ c\lambda_q \cdot \frac{x}{q} + O\left(\frac{x}{q(\log(x/q))^{1+\gamma}} \right) \right\} + O\left(x \log\left(\frac{x}{w} \right) \right). \quad (4.4)$$

By Lemma 4.3, $\lambda_q = 1 + O(1/q)$. Hence, the first sum is

$$cx \sum_{p<x^\beta} \sum_{\substack{q \leqslant x \\ q \in \Omega(p)}} \frac{1}{q} + O\left(x \sum_p \sum_q \frac{1}{q^2} \right) + O\left(x \log\left(\frac{x}{w} \right) \right).$$

The first 0-term is easily checked to be $O(x)$ by an argument similar to that used in Lemma 4.4. Finally, the second sum in (4.4) is

$$\ll \frac{x}{(\log(x/w))^{1+\gamma}} \sum_{p<x^\beta} \sum_{\substack{q \leqslant x \\ q \in \Omega(p)}} \frac{1}{q}.$$

The lemma now follows if we apply Lemma 4.5, and choose, for example, $w = x/\log x$.

LEMMA 4.7. $T(x) \ll x(F(x) + \log_4 x)\log\log x$.

Proof. Let $z = \log\log x$. We write $T(x) = \Sigma_1 + \Sigma_2$ where in Σ_1, $p < z$ and in Σ_2, $z < p < x^\beta$. Now,

$$\Sigma_1 \leqslant \sum_{p<z} \sum_{\substack{q \leqslant x \\ p \in \Omega(p)}} B_q(x) \ll x \sum_{p<z} \sum_{\substack{q \leqslant x \\ q \in \Omega(p)}} \frac{1}{q}$$

and using the method of Lemma 4.5, we find that

$$\Sigma_1 \ll x(\log\log x)(\log\log z + F(x)).$$

As for Σ_2, we use partial summation to write

$$\Sigma_2 \ll x \sum_{z<p<x^\beta} \sum_{\substack{q_1 \leqslant \sqrt{x} \\ q_1 \in \Omega(p)}} \frac{1}{q_1} \left\{ \frac{\pi_f\left(\frac{x}{q_1}, p \right)}{\left(\frac{x}{q_1} \right)} + \int_{q_1}^{x/q_1} \pi_f(t, p) \frac{dt}{t^2} \right\} = x(\Sigma_3 + \Sigma_4) \quad \text{(say)}.$$

By Lemma 4.5,

$$\Sigma_3 \ll \frac{1}{\log x} \sum_{z<p<x^\beta} \sum_{\substack{q_1 \leqslant \sqrt{x} \\ q_1 \in \Omega(p)}} \frac{1}{q_1} \ll \frac{(\log\log x)^2}{\log x}.$$

Also, since $z < p < q_1^\beta$, we have

$$\Sigma_4 = \sum_{q_1 \leqslant \sqrt{x}} \frac{1}{q_1} \int_{q_1}^{x/q_1} \Bigg\{ \sum_{\substack{z<p<t^\beta \\ p \mid f(q_1)}} \pi_f(t, p) \Bigg\} \frac{dt}{t^2} .$$

The inner sum can be written as

$$\sum_{\substack{z<p<w \\ p \mid f(q_1)}} \delta(p)\pi(t) + O\left(\frac{\pi(t)}{(\log t)^\gamma} \right) + \sum_{\substack{w \leqslant p<t^\beta \\ p \mid f(q_1)}} \pi_f(t, p)$$

where $w = \max(t^{1/F(t)}, z)$. (Here we have used (C3′).) Thus we can write

$$\Sigma_4 = \Sigma_{41} + \Sigma_{42} + \Sigma_{43}$$

in an obvious way. Now, $\Sigma_{42} = O(\log\log x)$. Also

$$\Sigma_{41} \ll \left[\sum_{z<p<x^\beta} \delta(p) \sum_{\substack{q_1 \leqslant \sqrt{x} \\ q_1 \in \Omega(p)}} \frac{1}{q_1} \right] \int_2^x \pi(t) \frac{dt}{t^2} .$$

The term in brackets is

$$\sum_{z<p<x^\beta} \delta(p) \left\{ \frac{\pi_f(\sqrt{x}\,, p)}{\sqrt{x}} + \int_2^{\sqrt{x}} \pi_f(t, p) \frac{dt}{t^2} \right\}$$

$$= O\left(\frac{\log\log x}{\log x} \right) + \int_2^{\sqrt{x}} \left(\left\{ \sum_{z<p<w} \delta(p)^2 \pi(t) \right\} \right.$$

$$\left. + O\left(\frac{\pi(t)}{z(\log t)^\gamma} \right) + O\left(\frac{\pi(t)F(t)}{z} \right) \right) \frac{dt}{t^2} ,$$

where we have used (C3′). The integral is $\ll z^{-1}(\log\log x)(1 + F(x)) \ll F(x)$. Hence $\Sigma_{41} = O(F(x))$. Finally, the number of primes p such that $p \geqslant w$ and $p \mid f(q_1)$ is

$$O\left(\frac{\log f(q_1)}{\log w} \right) = O\left(\frac{(\log q_1)F(t)}{\log t} \right).$$

Hence,

$$\Sigma_{43} \ll \sum_{q_1 \leqslant \sqrt{x}} \frac{1}{q_1} \int_{q_1}^{x/q_1} \pi(t) F(t) \frac{\log q_1}{\log t} \frac{dt}{t^2}$$

$$\ll F(x) \sum_{q_1 \leqslant \sqrt{x}} \frac{\log q_1}{q_1} \int_{q_1}^{\infty} \frac{dt}{t(\log t)^2} \ll F(x) \log\log x.$$

Hence $\Sigma_4 = O(F(x)\log\log x)$ and this completes the proof of the lemma.

Combining Lemma 4.4, (4.3), Lemmas 4.6 and 4.7, we deduce that

$$\sum_{n \leqslant x}{}' \nu(f(n)) = \frac{1}{2} cx(\log\log x)^2 + O(x(F(x) + \log_4 x)\log\log x). \tag{4.4}$$

We next investigate the average order of $\nu^2(f(n))$.

LEMMA 4.8.

$$\sum_{n \leqslant x}{}' \nu^2(f(n)) = \frac{1}{4} cx(\log\log x)^4 + O\big(x(F(x) + \log_4 x)(\log\log x)^3\big).$$

Proof. Using (4.4) and the Schwarz inequality, we have

$$\sum_{n \leqslant x}{}' \nu^2(f(n)) \geqslant \frac{1}{4} cx(\log\log x)^4 + O\big(x(F(x) + \log_4 x)(\log\log x)^3\big),$$

so it suffices to prove an asymptotic upper bound. We can replace $\nu(f(n))$ by $\nu_1(n)$ by Lemma 4.4.

We define the sums

$$T_1(x,d) = \sum_{\substack{r \leqslant x \\ r \in \Omega(d)}} \frac{\lambda_r}{r}$$

$$T_2(x,d_1,d_2) = \sum_{\substack{m \leqslant x \\ m = r_1 r_2 \\ r_1 \neq r_2 \\ r_1 \in \Omega(d_1),\, r_2 \in \Omega(d_2)}} \frac{\lambda_m}{m}$$

where d_1, d_2 are distinct. Then, using Lemma 4.2, and (4.4),

$$\sum_{n \leqslant x}{}' \nu_1^2(n) \leqslant cx \sum_{\substack{p,q \leqslant x^\beta \\ p \neq q}} T_2(x,p,q) + O\left[x \sum_{\substack{pq \leqslant x^\beta \\ p \neq q}} \sum_{\substack{r \leqslant x \\ r \in \Omega(pq)}} \frac{1}{r}\right]$$

$$+ O\big(x(\log\log x)^2\big) + O(E)$$

where

$$E = x \sum_{\substack{pq \leqslant x^\beta \\ p \neq q}} \sum_m \frac{1}{m(1+\log(x/m))^{1+\gamma}},$$

the inner sum ranging over the same set of integers m as in $T_2(x, pq)$.

First, we consider the main term. We have

$$T_2(x, p, q) = T_1(x^{1/2}, p) T_1(x^{1/2}, q) + O\Big(\Sigma \frac{1}{m}\Big)$$

where the sum in the O-term is over $m \leqslant x$ such that $m = r_1 r_2$, $r_1 \in \Omega(p)$, $r_2 \in \Omega(q)$, $r_1 \neq r_2$, and either r_1 or r_2 is $> x^{1/2}$. Using Lemma 4.5, we see then that

$$\sum_{\substack{p,q \leqslant x^\beta \\ p \neq q}} \sum \frac{1}{m} \ll \Bigg[\sum_{p \leqslant x^\beta} \sum_{\substack{x^{1/2} < r_1 \leqslant x \\ r_1 \in \Omega(p)}} \frac{1}{r_1}\Bigg]\Bigg(\sum_{q \leqslant x^\beta} \sum_{\substack{r_2 \leqslant x \\ r_2 \in \Omega(q)}} \frac{1}{r_2}\Bigg)$$

$$\ll (\log\log x)(\log\log x)^2 = (\log\log x)^3.$$

Also, by Lemma 4.5,

$$\sum_{p \leqslant x^\beta} T_1(x^{1/2}, p)^2 \ll \sum_{r_1 \leqslant x} \frac{1}{r_1} \sum_{p \leqslant x^\beta} \sum_{\substack{r_2 \leqslant x \\ r_2 \in \Omega(p)}} \frac{1}{r_2} \ll (\log\log x)^3.$$

Hence,

$$\begin{aligned} \sum_{\substack{p,q \leqslant x^\beta \\ p \neq q}} T_2(x, p, q) &= \Big(\sum_{p \leqslant x^\beta} T_1(x^{1/2}, p)\Big)^2 - \sum_{p \leqslant x^\beta} T_1(x^{1/2}, p)^2 + O\big((\log\log x)^3\big) \\ &= \Big[\frac{1}{2}(\log\log x)^2 + O(F(x)\log\log x)\Big]^2 \\ &\quad + O\big((\log\log x)^3\big) \\ &= \frac{1}{4}(\log\log x)^4 + O\big(F(x)(\log\log x)^3\big). \end{aligned}$$

Next, using Remark 3.5 with $k = 2$, and the method of Lemma 4.5, it is easy to check that

$$\sum_{\substack{pq \leqslant x^\beta \\ p \neq q}} \sum_{\substack{r \leqslant x \\ r \in \Omega(pq)}} \frac{1}{r} \ll (\log\log x)^3.$$

And finally, we write $E = E_1 + E_2$, where in E_1, m is less than $w = x/\log x$, and in E_2, $w \leqslant m \leqslant x$. By Lemma 4.5,

$$E_1 \ll \frac{x}{(\log\log x)^{1+\gamma}} \Bigg(\sum_{p \leqslant x^\beta} \sum_{\substack{r \leqslant x \\ r \in \Omega(p)}} \frac{1}{r} \Bigg)^2 \ll x(\log\log x)^{3-\gamma}.$$

Also, by Lemma 4.5,

$$E_2 \ll x \sum_{p \leqslant x^\beta} \sum_{\substack{r_1 \leqslant x \\ r_1 \in \Omega(p)}} \frac{1}{r_1} \sum_{\frac{w}{r_1} \leqslant r_2 \leqslant \frac{x}{r_1}} \frac{\nu(f(r_2))}{r_2} \ll x(\log\log x)^2 \log\Big(\frac{x}{w}\Big) = x(\log\log x)^3.$$

This completes the proof of Lemma 4.8.

Proof of Theorem 4.1. This follows from (4.4) and Lemma 4.8.

Theorem 4.1 implies that if $F(x) = o(\log\log x)$, $\nu(f(n))$ has normal order $\frac{1}{2}(\log\log n)^2$. More generally, under suitable hypotheses, $\nu(f(n))$ should have normal order

$$\int_2^n \frac{D(t)}{t \log t}\, dt.$$

§5. **Applications to modular forms.** Let f be a cusp form (of integral weight $k \geqslant 2$) for $\Gamma_0(N)$, which is a normalized eigenform for the Hecke operators, and let χ be its Nebentypus character. (This usage of the symbol f should not be confused with the multiplicative function of the previous sections.) Write $f = \sum_{n \geqslant 1} a_n e^{2\pi i n z}$ for its Fourier expansion at ∞, and suppose that the a_n are rational integers. (This forces χ to be real and χ is nontrivial if and only if f has complex multiplication (cf. [7]).) Then, the function $n \mapsto a_n$ is multiplicative.

Let $G = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and let d be a positive integer. Then by Deligne [1], there is a representation

$$\rho_d : G \to \mathrm{GL}_2\Big(\prod_{l \mid d} \mathbf{Z}_l \Big)$$

(where the product is over distinct prime divisors of d) with the following essential property: if p is a prime not dividing dN and σ_p a Frobenius element at p in G, then ρ_d is unramified at p and

$$\mathrm{tr}\, \rho_d(\sigma_p) = a_p, \qquad \det \rho_d(\sigma_p) = p^{k-1}\chi(p).$$

Denote by $\tilde{\rho}_d$ the reduction mod d of ρ_d:

$$\tilde{\rho}_d : G \xrightarrow{\rho_d} \mathrm{GL}_2\Big(\prod_{l \mid d} \mathbf{Z}_l \Big) \twoheadrightarrow \mathrm{GL}_2(\mathbf{Z}/d).$$

Let H_d be the kernel of $\tilde{\rho}_d$, K_d the subfield of $\overline{\mathrm{Q}}$ fixed by H_d, and $G_d = G/H_d = \mathrm{Gal}(K_d/\mathrm{Q})$. Let C_d be the subset of $\tilde{\rho}_d(G)$ consisting of elements of trace 0, and let $\delta(d) = |C_d|/|G_d|$.

The condition $a_q \equiv 0 \pmod d$ (for q a prime not dividing dN) means that for any Frobenius element σ_q of q, $\tilde{\rho}_d(\sigma_q) \in C_d$. Hence by the Čebotarev density theorem applied to K_d/Q,

$$\pi_f^*(x,d) \overset{\mathrm{df}}{=} \#\{q \leqslant x \mid a_q \equiv 0 \pmod d\} \sim \frac{|C_d|}{|G_d|}\pi(x) = \delta(d)\pi(x).$$

As C_d contains the image of complex conjugations, it is nonempty.

PROPOSITION 5.1. *Let f be as above. Let m be a fixed positive integer and $\epsilon > 0$. Then for almost all n, a_n is divisible by*

$$m^{[(1-\epsilon)\delta(m)\log\log n]}.$$

Proof. The result follows from Theorem 2.1 since (5.1) shows that (*) of §2 holds.

By abuse of notation, we shall write

$$\pi_f(x,d) = \#\{q \leqslant x \mid 0 \neq a_q \equiv 0 \pmod d\}$$

$$Z_f(x) = \#\{q \leqslant x \mid a_q = 0\}.$$

By the Generalized Riemann Hypothesis (GRH), we shall mean the Riemann Hypothesis for all Artin L-series.

LEMMA 5.2. *If f has complex multiplication, $Z_f(x) \sim \frac{1}{2}\pi(x)$. If f does not have complex multiplication,*

$$Z_f(x) \ll \begin{cases} x/(\log x)^{3/2-\epsilon} & (\text{for all } \epsilon > 0) \text{ unconditionally} \\ x^{3/4} & \text{on the assumption of GRH}. \end{cases}$$

The first statement is implicit in Ribet [7] and the second in Serre [10, p. 175].

LEMMA 5.3. *Suppose that f does not have complex multiplication. Then $\pi_f(x,d) \sim \delta(d)\pi(x)$. If the* GRH *is assumed, then for $x \geqslant 2$,*

$$\pi_f(x,d) = \delta(d)\pi(x) + O\big(d^3 x^{1/2}\log(dNx)\big) + O(x^{3/4}).$$

Proof. The first assertion follows from (5.1) and Lemma 5.2. If the GRH is assumed,

$$\pi_f^*(x,d) = \delta(d)\pi(x) + O\big(\delta(d)x^{1/2}\log(|D_d|x^{g_d})\big)$$

where g_d is the degree and D_d is the discriminant of K_d/Q (cf. Lagarias and

Odlyzko [5]). Now from an inequality of Hensel, $\log|D_d| \leqslant g_d \log(dNg_d)$ (cf. Serre [10, §1.4]). Using the inequalities $g_d \leqslant d^4$ and $|C_d| \leqslant d^3$, we deduce that

$$\pi_f^*(x,d) = \delta(d)\pi(x) + O\big(d^3 x^{1/2}\log(dNx)\big).$$

Now using Lemma 5.2, we deduce the second assertion of the lemma.

Suppose from now on that f does not have complex multiplication.

LEMMA 5.4. *If f is of level 1 or of weight 2, then $\delta(l) = 1/l + O(1/l^2)$ for all sufficiently large primes l and $\delta(ll') = \delta(l)\delta(l')$ for all sufficiently large primes l, l'. In any case, $\delta(l) \ll 1/l$.*

Proof. If f is of level 1 or of weight 2, it has been shown by Swinnerton–Dyer [11] and Serre [8] respectively, that for l sufficiently large,

$$G_l = \left\{ g \in \mathrm{GL}_2(\mathbf{F}_l) \mid \det g \in (\mathbf{F}_l^*)^{k-1} \right\},$$

and for l, l' sufficiently large,

$$G_{ll'} = G_l \times G_{l'}.$$

From this, it is easily calculated that $|C_l| = l^3 + O(l^2)$. Hence $\delta(l) = 1/l + O(1/l^2)$. Also $\delta(ll') = \delta(l)\delta(l')$.

Now, in general, $\rho_l(G)$ is open in $\mathrm{GL}_2(\mathbf{Z}_l)$ (cf. Serre [10, Prop. 17]). Thus, $\rho_l(G)$ is a compact l-adic Lie group of dimension 4 and as in [10, §4.2], $|G_l| \gg l^4$. Since we clearly have $|C_l| \ll l^3$, it follows that $\delta(l) \ll 1/l$.

Remark. Ribet has pointed out to us that the lemma is true for weight $\geqslant 2$ without any restriction on level. This apparently follows from recent work of Carayol.

PROPOSITION 5.5. *Let f satisfy*

(i) *f is a normalized eigenform of the Hecke operators with a Fourier expansion $f = \sum_{n \geqslant 1} a_n e^{2\pi i n z}$, $a_n \in \mathbf{Z}$*

(ii) *f does not have complex multiplication.*

Suppose also that the GRH *is true. Then*

$$\sum_{\substack{q \leqslant x \\ a_q \neq 0}} (\nu(a_q) - \log\log q)^2 \ll \pi(x)\log\log x$$

and

$$\sum_{\substack{n \leqslant x \\ a_n \neq 0}} \left(\nu(a_n) - \frac{1}{2}(\log\log n)^2\right)^2 \ll x(\log\log x)^3(\log_4 x).$$

Proof. Condition (C1) and the assumption on δ hold by Lemmas 5.3 and 5.4. By estimates of Hecke, $|a_n| \leqslant n^{(1/2)(k+1)}$. Hence (C2) holds with $\beta = \frac{1}{2}(k+1)$ for example. Condition (C3′) may be verified with $y = x^{1/10}$ for example, and any $\gamma > 0$, using Lemma 5.3. And (C4′) follows from Lemma 5.2. Thus the proposition follows from Theorems 3.1 and 4.1.

§6. Other applications. We can use the results of the previous section to deduce lower bounds for the a_p, valid for almost all p.

THEOREM 6.1. *Suppose f satisfies the hypotheses of Proposition 5.5. Then, for any $\epsilon > 0$, we have*

$$\log|a_p| \geqslant we^{-3w}\log p \geqslant (\log p)^{1-\epsilon}$$

for almost all primes p, where $w = (\log\log p)^{1/2+\epsilon}$.

Proof. Let h be a monotone increasing function such that $h(x) = O(\log x)$ and $h(x) \geqslant 1$. Let $y = x^{1/h(x)}$. From Theorem 3.1, we find that

$$\sum_{(1/2)x \leqslant p \leqslant x}{}' (\nu_y(a_p) - \log\log y)^2 \ll \pi(x)\log\log y.$$

Let $z = p^{1/h(p)}$. Then, for almost all p in between $\frac{1}{2}x$ and x,

$$\nu_y(a_p) < \log\log y + (\log\log y)^{1/2+\epsilon} \leqslant \log\log z + (\log\log z)^{1/2+\epsilon} \leqslant \log\log z + w.$$

On the other hand, Proposition 5.5 implies that

$$\nu(a_p) > \log\log p - w$$

for almost all p. Hence, for almost all p, a_p has at least

$$(\log\log p - w) - (\log\log z + w) = \log h(p) - 2w$$

distinct prime divisors larger than z. Now if we choose h so that $\log h = 3w$ (say), we find

$$|a_p| \geqslant z^w$$

and the result follows on taking logarithms.

Remark. From a different point of view, if we assume the Sato–Tate conjecture for f, then it is easy to see that

$$|a_p| \geqslant p^{(1/2)(k-1)-\epsilon}$$

for almost all p.

Finally, we remark that Theorems 3.1 and 4.1 also apply to some classical arithmetical functions. For example, we have the following.

THEOREM 6.2. *$\nu(p \pm 1)$ has normal order $\log\log p$. Also, $\nu(\phi(n))$ and $\nu(\sigma(n))$ have normal order $\frac{1}{2}(\log\log n)^2$.*

Proof. This follows on taking $f(n)$ (now reverting to the notation of Sections 2–4) to be $\phi(n)$ or $\sigma(n)$. If $\pi(x,d,a)$ denotes the number of primes $p \leqslant x$ such that $p \equiv a \pmod d$, then $\pi_f(x,d) = \pi(x,d,1)$ in the first case and $\pi(x,d,-1)$ in the second. We may take $\beta = 2$, any $\gamma > 0$ and $y = x^{1/2-\epsilon}$. Then (C1) is the prime number theorem, (C3′) is Bombieri's theorem, and (C2), (C4′) are trivial. This proves the result. The first assertion of the theorem is an old result of Erdös [3]. It is also possible to treat

$$f(n) = \sigma_k(n) = \sum_{\substack{d \mid n \\ d \geqslant 1}} d^k \qquad \text{for} \quad k \in \mathbf{N}.$$

Let

$$\alpha(k,p) = \begin{cases} 0 & \text{if} \quad (k, p-1) \nmid \frac{1}{2}(p-1) \\ (k, p-1) & \text{otherwise.} \end{cases}$$

Then, in this case,

$$\delta(p) = \frac{\alpha(k,p)}{p} + O\left(\frac{1}{p^2}\right)$$

and so our assumption on δ is not satisfied. However, as mentioned at the end of §3 and §4, the methods still work, and we find that $\nu(\sigma_k(n))$ has normal order $\frac{1}{2}\beta(k)(\log\log n)^2$ where $\beta(k)$ is the number of odd positive divisors of k.

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