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Special values of derivatives of L-series and generalized Stieltjes constants

by

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To Professor Robert Tijdeman on the occasion of his 75th birthday

1. Introduction. The Riemann zeta function $\zeta(s)$ plays a crucial role in mathematics. The Laurent series expansion of $\zeta(s)$ around $s=1$ (see [3]) can be written as

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\gamma_k (s-1)^k,$$

where

$$\gamma_k := \lim_{N\to\infty}\left\{\sum_{n=1}^{N} \frac{\log^k n}{n} - \frac{\log^{k+1} N}{k+1}\right\}$$

are called the *Stieltjes constants* and γ is the well-known Euler–Mascheroni constant. Even though these constants are important ingredients of the theory of the Riemann zeta function and appear in many contexts, it is unknown whether they are rational or irrational although they are expected to be transcendental. As a generalization of this question to arithmetic progressions, Knopfmacher [10] defined

$$\gamma_k(a,q) := \lim_{x\to\infty}\left\{\sum_{\substack{n\le x\\ n\equiv a \bmod q}} \frac{\log^k n}{n} - \frac{\log^{k+1} x}{q(k+1)}\right\}$$

for natural numbers a and q. The case $k=0$ was studied earlier (1975) by D. H. Lehmer [12]. We refer to these constants as the *generalized Stieltjes constants.*

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The motivation for studying these constants comes from our desire to understand special values of L-series. More precisely, when f is an arithmetical function, with period q, the Dirichlet series

$$L(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

has been the focus of intense study (see for example the survey article by Tijdeman [17] as well as [4], [8] and [15]). However, these papers studied the special value $L(1,f)$ whenever it is defined. Interestingly, this special value can be studied using Baker's theory of linear forms in logarithms. In this paper, our focus will be on the derivative $L'(1,f)$. This problem has received scant attention. For example, there is the curious result of Murty and Murty [14] which states that if there is some squarefree $D > 0$ and χ_D is the quadratic character attached to $\mathbb{Q}(\sqrt{-D})$ such that $L'(1,\chi_D) = 0$, then e^{γ} is transcendental. An analogous question of non-vanishing seems to occur in other contexts as well (see for example [16]). These are not unrelated to the Euler–Kronecker constants studied in [9] and [13].

Many arithmetic properties and computational aspects of these constants have been studied in [7] but the only known result about their transcendental nature is a theorem due to M. Ram Murty and N. Saradha [15, Theorem 1], who tackle the case $k = 0$. In the present paper, we concentrate on the arithmetic nature of these constants when $k = 1$.

The nature of values of the gamma function at rational arguments and relations among them have been the subject of research for a long time. In light of this, a conjecture put forth by S. Gun, M. Ram Murty and P. Rath [8] will be useful towards a partial solution to our question:

Conjecture 1. *For any positive integer $q > 2$, let $\overline{V_\Gamma(q)}$ be the $\overline{\mathbb{Q}}$-vector space spanned by the real numbers*

$$\log \Gamma\left(\frac{a}{q}\right), \qquad 1 \le a \le q,\ (a,q) = 1.$$

Then the dimension of $\overline{V_\Gamma(q)}$ is $\phi(q)$.

This conjecture was inspired by a conjecture of Rohrlich (see [18]) regarding possible relations among special values of the Γ-function. We note that Conjecture 1 is equivalent to the numbers $\{\log \Gamma(a/q) \mid 1 \le a \le q,\ (a,q) = 1\}$ being $\overline{\mathbb{Q}}$-linearly independent for $q > 2$. This is a major unsolved problem in number theory and is believed to be outside the scope of current mathematical tools.

For a natural number q, a function f defined on the integers which is periodic with period q is said to be *odd* if $f(q-n) = -f(n)$ for all natural numbers n, and *even* if $f(q-n) = f(n)$ for all natural numbers n.

As noted earlier, the L-series attached to f is defined as

$$L(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

for $\Re(s) > 1$. Using the theory of the Hurwitz zeta function, one can extend $L(s,f)$ to an entire function as long as $\sum_{a=1}^{q} f(a) = 0$.

Given a q-periodic function f, we define the Fourier transform of f as

$$\hat{f}(b) := \frac{1}{q} \sum_{a=1}^{q} f(a)\zeta_q^{-ab}, \tag{1}$$

where $\zeta_q = e^{2\pi i/q}$. This can be inverted using the identity

$$f(n) = \sum_{b=1}^{q} \hat{f}(b)\zeta_q^{bn}.$$

Thus, the condition for convergence of $L(1,f)$, i.e. $\sum_{a=1}^{q} f(a) = 0$, can be interpreted as $\hat{f}(q) = 0$.

A q-periodic arithmetical function is said to be *of Dirichlet type* if

$$f(n) = 0 \quad \text{whenever} \quad (n,q) > 1.$$

Another important notion is that of linear independence of arithmetical functions. A set $\{f_1, \ldots, f_m\}$ of arithmetical functions is said to be *linearly independent over* $\overline{\mathbb{Q}}$ if

$$\sum_{j=1}^{m} \alpha_j f_j = 0 \text{ with } \alpha_j \in \overline{\mathbb{Q}} \;\Rightarrow\; \alpha_j = 0 \text{ for all } 1 \le j \le m.$$

Again, using the theory of the Hurwitz zeta function, one can derive (as we will see below) formulas for $L'(1,f)$ in terms of the Stieltjes constants.

We can now state the main theorems of this paper.

Theorem 1.1. *Let p be a prime greater than 7. Define*

$$\mathfrak{F}_p := \{f : \mathbb{Z} \to \overline{\mathbb{Q}} \mid f \text{ is } p\text{-periodic and odd}, \widehat{f}(p) = 0, L(1,f) \neq 0\}.$$

For $r > 2$, let $f_1, \ldots, f_r$ be $\overline{\mathbb{Q}}$-linearly independent elements of $\mathfrak{F}_p$. Then Conjecture 1 implies that at most three of the numbers

$$\left\{ L'(1,f_j) = -\sum_{a=1}^{p} f_j(a)\gamma_1(a,p) \;\middle|\; 1 \le j \le r \right\}$$

are algebraic.

We also handle the case of p-periodic arithmetical functions satisfying $L(1,f) = 0$:

Theorem 1.2. *Let p be a prime number greater than 5. Define*

$$\mathfrak{G}_p := \{f : \mathbb{Z} \to \overline{\mathbb{Q}} \mid f \text{ is } p\text{-periodic and odd}, \widehat{f}(p) = 0, L(1,f) = 0\}.$$

For $r \geq 2$, let $f_1, \dots, f_r$ be $\overline{\mathbb{Q}}$-linearly independent elements of $\mathfrak{G}_p$. Then under Conjecture 1, at most one of the numbers

$$\Big\{ L'(1, f_j) = -\sum_{a=1}^{p} f_j(a)\gamma_1(a,p) \;\Big|\; 1 \leq j \leq r \Big\}$$

is algebraic.

Remark. Both the above theorems also hold when the period of the functions under consideration is a composite number q, provided that the Fourier transforms of the functions are of Dirichlet type. This restriction comes from the nature of Conjecture 1. Indeed, we will prove that Theorem 1.1 holds for the general set of functions

$$\mathfrak{F}_q := \{ f : \mathbb{Z} \to \overline{\mathbb{Q}} \mid f \text{ is } q\text{-periodic and odd}, \\ \widehat{f} \text{ is of Dirichlet type}, L(1,f) \neq 0 \},$$

and Theorem 1.2 holds for

$$\mathfrak{G}_q := \{ f : \mathbb{Z} \to \overline{\mathbb{Q}} \mid f \text{ is } q\text{-periodic and odd}, \\ \widehat{f} \text{ is of Dirichlet type}, L(1,f) = 0 \},$$

where q is not necessarily prime.

We note that the above defined set $\mathfrak{F}_q$ is non-empty since odd primitive Dirichlet characters modulo q are in $\mathfrak{F}_q$. We will see this in the course of proof of the following corollary.

Corollary 1. *Let q be a natural number greater than 7. Then under Conjecture 1, at most three of the following numbers are algebraic:*

$$\Big\{ L'(1, \chi) = -\sum_{a=1}^{q} \chi(a)\gamma_1(a,q) \;\Big|\; \chi \textit{ is an odd primitive} \\ \textit{Dirichlet character modulo } q \Big\}.$$

Applying Theorem 1.1 to the scenario when $q = p$, an odd prime greater than 7, and

$$f_j(n) := \begin{cases} 1 & \text{if } n \equiv j \bmod p, \\ -1 & \text{if } n \equiv -j \bmod p, \\ 0 & \text{otherwise}, \end{cases}$$

for $1 \leq j \leq (p-1)/2$, we infer the following:

Corollary 2. *For an odd prime p greater than 7, Conjecture 1 implies that at least $(p-7)/2$ of the numbers*

$$\{ \gamma_1(a,p) \mid 1 \leq a \leq p-1 \}$$

are transcendental.

2. Preliminaries. The aim of this section is to introduce notation and some fundamental results that will be used later. Let q be a fixed positive integer. Let $f : \mathbb{N} \to \overline{\mathbb{Q}}$ be q-periodic. Define

$$L(s,f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Observe that $L(s,f)$ converges absolutely for $\Re(s) > 1$. Since f is periodic,

$$L(s,f) = \sum_{a=1}^{q} f(a) \sum_{k=0}^{\infty} \frac{1}{(a+kq)^s} = \frac{1}{q^s} \sum_{a=1}^{q} f(a)\zeta(s,a/q), \tag{2}$$

where $\zeta(s,x)$ is the Hurwitz zeta function. For $\Re(s) > 1$ and $0 < x \leq 1$, the Hurwitz zeta function is defined as

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}.$$

In 1882, Hurwitz [1, Chapter 12, Section 5] proved that $\zeta(s,x)$ has an analytic continuation to the entire complex plane except for a simple pole at $s = 1$ with residue 1. Using this, we conclude that $L(s,f)$ can be extended analytically to $\mathbb{C}$ except for a simple pole at $s = 1$ with residue $\frac{1}{q}\sum_{a=1}^{q} f(a)$. Thus, it is easy to deduce that $\sum_{n=1}^{\infty} \frac{f(n)}{n}$ converges whenever $\sum_{a=1}^{q} f(a) = 0$. Hence, $f(q) = \hat{f}(q) = 0$ implies that both $L(s,f)$ and $L(s,\hat{f})$ are entire.

Before proceeding, we prove a few lemmas for q-periodic arithmetical functions.

LEMMA 2.1. *Let f be a q-periodic arithmetical function. Then*

$$L(1-s,f) = 2\Gamma(s)\left(\frac{q}{2\pi}\right)^s \cos\left(\frac{s\pi}{2}\right) L(s,\hat{f})$$

when f is even (i.e., $f(q-n) = f(n)$ for all n), and

$$L(1-s,f) = 2i\Gamma(s)\left(\frac{q}{2\pi}\right)^s \sin\left(\frac{s\pi}{2}\right) L(s,\hat{f})$$

when f is odd (i.e., $f(q-n) = -f(n)$ for all n).

Proof. We refer the reader to [11, Chapter XIV, Theorem 2.1]. ∎

In analogy with the notation of generalized Bernoulli numbers associated to Dirichlet characters, we define

$$B_{1,f} := \sum_{a=1}^{q} a f(a),$$

where f is an odd q-periodic arithmetical function. We make another important observation.

LEMMA 2.2. *For any q-periodic arithmetical function f,*

$$L'(0,f) = \frac{\log q}{q} B_{1,f} + \sum_{b=1}^{q} f(b) \log \Gamma\left(\frac{b}{q}\right).$$

Proof. By differentiating (2) with respect to s, we have

$$L'(s,f) = \frac{-\log q}{q^s}\left[\sum_{a=1}^{q} f(a)\,\zeta\left(s,\frac{a}{q}\right)\right] + \left[\frac{1}{q^s}\sum_{a=1}^{q} f(a)\zeta'\left(s,\frac{a}{q}\right)\right].$$

Substituting $s=0$, we have

$$L'(0,f) = -\log q\left[\sum_{a=1}^{q} f(a)\,\zeta\left(0,\frac{a}{q}\right)\right] + \left[\sum_{a=1}^{q} f(a)\zeta'\left(0,\frac{a}{q}\right)\right].$$

The values of the Hurwitz zeta function and its derivative at $s=0$ are given by

$$\zeta(0,x) = 1+\zeta(0)-x, \qquad \zeta'(0,x) = \log\Gamma(x) + \zeta'(0),$$

where $\zeta(s)$ is the Riemann zeta function (for the proof see [6]). Substituting these values in the expression obtained earlier, we get

$$\begin{aligned} L'(0,f) &= -(\log q)(1+\zeta(0))\Big[\sum_{a=1}^{q} f(a)\Big] + \frac{\log q}{q}\sum_{a=1}^{q} f(a)a \\ &\quad + \zeta'(0)\Big[\sum_{a=1}^{q} f(a)\Big] + \sum_{a=1}^{q} f(a)\log\Gamma\left(\frac{a}{q}\right) \\ &= \frac{\log q}{q}\sum_{a=1}^{q} f(a)a + \sum_{a=1}^{q} f(a)\log\Gamma\left(\frac{a}{q}\right), \end{aligned}$$

since $\sum_{a=1}^{q} f(a) = 0$. ∎

The functional equation obtained in Lemma 2.1 gives an expression for $L'(1,f)$ when f is an odd periodic function.

LEMMA 2.3. *Let f be an odd q-periodic arithmetical function satisfying $f(q)=\hat{f}(q)=0$. Then*

$$L'(1,f) = \frac{i\pi}{q}\left\{\left(\left(1+\frac{1}{q}\right)\log q - \log 2\pi - \gamma\right)B_{1,\hat{f}} + \sum_{b=1}^{q}\hat{f}(b)\log\Gamma\left(\frac{b}{q}\right)\right\},$$

where $B_{1,g} := \sum_{a=1}^{q} ag(a)$ for any odd q-periodic arithmetical function g.

Proof. Note that if f is an odd periodic arithmetical function, then so is $\hat{f}$. Thus, differentiating the functional equation for $L(s,\hat{f})$ from Lemma 2.1 gives

$$
\begin{aligned}
-L'(1-s,\hat{f}) = {} & 2\Gamma'(s)\left(\frac{q}{2\pi}\right)^s \sin\left(\frac{s\pi}{2}\right)L(s,f)\\
& + 2\Gamma(s)\left(\frac{q}{2\pi}\right)^s \log\left(\frac{q}{2\pi}\right)\sin\left(\frac{s\pi}{2}\right)L(s,f)\\
& + 2\Gamma(s)\left(\frac{q}{2\pi}\right)^s \frac{\pi}{2}\cos\left(\frac{s\pi}{2}\right)L(s,f)\\
& + 2\Gamma(s)\left(\frac{q}{2\pi}\right)^s \sin\left(\frac{s\pi}{2}\right)L'(s,f).
\end{aligned}
$$

Since $f(q)=\hat{f}(q)=0$, both $L(s,f)$ and $L(s,\hat{f})$ are entire. Taking the limit as s tends to 1 in the above expression, we obtain

$$
\begin{aligned}
-L'(0,\hat{f}) &= 2i\Gamma(1)\frac{q}{2\pi}\sin\left(\frac{\pi}{2}\right)\left\{\left(\frac{\Gamma'}{\Gamma}(1)+\log\left(\frac{q}{2\pi}\right)\right)L(1,f)+L'(1,f)\right\}\\
&= \frac{iq}{\pi}\left\{L'(1,f)+L(1,f)\left(\log\left(\frac{q}{2\pi}\right)-\gamma\right)\right\},
\end{aligned}
$$

as $\Gamma'(1)/\Gamma(1)=-\gamma$. By rearrangement, we get

$$
L'(1,f)=\frac{i\pi}{q}L'(0,\hat{f})-\left(\log\left(\frac{q}{2\pi}\right)-\gamma\right)L(1,f).
$$

The value $L(1,f)$ for periodic arithmetical functions is well-understood (for example, see [4, Theorem 3.1]). In particular, when f is odd,

$$
L(1,f)=\frac{-i\pi}{q}\sum_{a=1}^{q}\hat{f}(a)a=\frac{-i\pi}{q}B_{1,\hat{f}}. \tag{3}
$$

This evaluation, together with Lemma 2.2, gives

$$
\begin{aligned}
L'(1,f) &= \frac{i\pi}{q}\left\{\frac{\log q}{q}B_{1,\hat{f}}+\sum_{b=1}^{q}\hat{f}(b)\log\Gamma\left(\frac{b}{q}\right)+\left(\log\left(\frac{q}{2\pi}\right)-\gamma\right)B_{1,\hat{f}}\right\},\\
&= \frac{i\pi}{q}\left\{\left(\frac{\log q}{q}+\log q-\log 2\pi-\gamma\right)B_{1,\hat{f}}+\sum_{b=1}^{q}\hat{f}(b)\log\Gamma\left(\frac{b}{q}\right)\right\},
\end{aligned}
$$

from which the assertion is immediate. ∎

A useful connection between values of derivatives of L-functions, attached to periodic functions at $s=1$, and generalized Stieltjes constants is given in the following lemma. We include its proof for completeness (see [10, Proposition 3.2]).

Lemma 2.4. *For a q-periodic arithmetical function f with $\hat{f}(q)=0$,*

$$
L^{(k)}(1,f)=(-1)^k\sum_{a=1}^{q}f(a)\gamma_k(a,q),
$$

where $\gamma_k(a,q)$ are the generalized Stieltjes constants as defined earlier.

Proof. For brevity, let

$$H_k(x,a,q) := \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \frac{\log^k n}{n}$$

for any positive real number x. Observe that

$$\sum_{n \le x} f(n) \frac{\log^k n}{n} = \sum_{a=1}^{q} f(a) H_k(x,a,q) = \sum_{a=1}^{q} f(a) \left(H_k(x,a,q) - \frac{\log^{k+1} x}{q(k+1)} \right),$$

since $q\hat{f}(q) = \sum_{a=1}^{q} f(a) = 0$. Letting $x \to \infty$ gives the result. ∎

As mentioned earlier, the special value $L(1,f)$ has been extensively studied and is important in the context of our theorems. The following result of Baker, Birch and Wirsing [2] will be particularly useful.

THEOREM 2.5. *If f is a non-vanishing function defined on the integers with algebraic values and period q such that (i) $f(n) = 0$ whenever $1 < (n,q) < q$, and (ii) the qth cyclotomic polynomial Φ_q is irreducible over $\mathbb{Q}(f(1), \dots, f(q))$, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

We also observe that if $f_1, \dots, f_r$ are q-periodic arithmetical functions periodic with period q, then

$$\sum_{j=1}^{r} \alpha_j f_j = 0 \ \Leftrightarrow\ \sum_{j=1}^{r} \alpha_j \widehat{f_j} = 0, \tag{4}$$

for any complex numbers α_j, $1 \le j \le r$. This is immediate from the fact that the Fourier transform is a linear automorphism of the $\mathbb{C}$-vector space of q-periodic arithmetical functions.

3. Proofs

3.1. Proof of Theorem 1.1. For convenience, let

$$\mathscr{C} := \left(1 + \frac{1}{q}\right) \log q - \log 2\pi - \gamma.$$

Thus, Lemma 2.3 gives

$$L'(1,f_j) = \frac{i\pi}{q} \left\{ \mathscr{C} B_{1,\hat{f}_j} + \sum_{b=1}^{q} \hat{f}_j(b) \log \Gamma\left(\frac{b}{q}\right) \right\}$$

for all $1 \le j \le r$. By (3), the hypothesis $L(1,f_j) \neq 0$ implies that $B_{1,\hat{f}_j} \neq 0$.

For $1 \le k < l \le r$, define

$$d_{k,l} := B_{1,\hat{f}_l} L'(1,f_k) - B_{1,\hat{f}_k} L'(1,f_l).$$

We claim that $d_{k,l} \neq 0$. Indeed, if $d_{k,l} = 0$, then

$$\begin{aligned}
0 &= B_{1,\hat{f}_l} L'(1, f_k) - B_{1,\hat{f}_k} L'(1, f_l) \\
&= \frac{i\pi}{q}\left\{\mathscr{C}(B_{1,\hat{f}_l}B_{1,\hat{f}_k} - B_{1,\hat{f}_k}B_{1,\hat{f}_l}) + \sum_{b=1}^{q}[B_{1,\hat{f}_l}\hat{f}_k(b) - B_{1,\hat{f}_K}\hat{f}_l(b)]\log\Gamma\left(\frac{b}{q}\right)\right\} \\
&= \sum_{b=1}^{q}[B_{1,\hat{f}_l}\hat{f}_k(b) - B_{1,\hat{f}_K}\hat{f}_l(b)]\log\Gamma\left(\frac{b}{q}\right),
\end{aligned}$$

which is a $\overline{\mathbb{Q}}$-linear relation among values of the log gamma function as $B_{1,\hat{f}_j} \in \overline{\mathbb{Q}}$ for all $1 \leq j \leq r$. Therefore, Conjecture 1 gives

$$B_{1,\hat{f}_l}\hat{f}_k - B_{1,\hat{f}_k}\hat{f}_l = 0$$

on all natural numbers. This implies $\overline{\mathbb{Q}}$-linear dependence of $\hat{f}_k$ and $\hat{f}_l$, and thus contradicts the $\overline{\mathbb{Q}}$-linear independence of f_k and f_l by (4). Hence, $d_{k,l}$ is not zero.

We now consider the ratio $d_{k,l}/d_{u,v}$ for $1 \leq k, u < l, v \leq r$ and $(k,l) \neq (u,v)$. If this ratio is algebraic, i.e.,

$$\frac{d_{k,l}}{d_{u,v}} = \eta \in \overline{\mathbb{Q}},$$

then

$$\begin{aligned}
0 &= d_{k,l} - \eta d_{u,v} \\
&= \sum_{b=1}^{q}[B_{1,\hat{f}_l}\hat{f}_k(b) - B_{1,\hat{f}_k}\hat{f}_l(b) - \eta B_{1,\hat{f}_w}\hat{f}_u(b) + \eta B_{1,\hat{f}_u}\hat{f}_w(b)]\log\Gamma\left(\frac{b}{q}\right),
\end{aligned}$$

which is a $\overline{\mathbb{Q}}$-linear relation among log gamma values. Hence, by Conjecture 1, we have

$$B_{1,\hat{f}_l}\hat{f}_k - B_{1,\hat{f}_k}\hat{f}_l - \eta B_{1,\hat{f}_w}\hat{f}_u + \eta B_{1,\hat{f}_u}\hat{f}_w = 0$$

on all natural numbers. Since $B_{1,\hat{f}_j}$ are non-zero algebraic numbers, we obtain a non-trivial $\overline{\mathbb{Q}}$-linear relation among $\hat{f}_k$, $\hat{f}_l$, $\hat{f}_u$ and $\hat{f}_w$. Then (4) transports this to $\overline{\mathbb{Q}}$-linear dependence of f_k, f_l, f_u and f_w, which contradicts our hypothesis. Thus, at most one of the $d_{k,l}$'s can be algebraic for $1 \leq k < l \leq r$.

As a result, if four numbers, $L'(1, f_k)$, $L'(1, f_l)$, $L'(1, f_u)$ and $L'(1, f_w)$, were algebraic for $(k,l) \neq (u,w)$, then $d_{k,l}/d_{u,w}$ would be algebraic, leading to a contradiction. ∎

3.2. Proof of Theorem 1.2. Using the hypothesis that $L(1, f) = 0$ for all $f \in \mathfrak{G}_{\mathfrak{q}}$ and (3), we find that

$$B_{1,\hat{f}_j} = 0$$

for all $1 \le j \le r$. Hence, Lemma 2.3 gives

$$L'(1, f_j) = \frac{i\pi}{q}\left\{\sum_{b=1}^{q} \hat{f}_j(b) \log \Gamma\left(\frac{b}{q}\right)\right\}$$

for all $1 \le j \le r$. Suppose that for $1 \le k < l \le r$,

$$\frac{L'(1, f_k)}{L'(1, f_l)} = \xi \in \overline{\mathbb{Q}}.$$

Then simplifying the above expression gives

$$\sum_{b=1}^{q} [\hat{f}_k(b) - \xi \hat{f}_l(b)] \log \Gamma\left(\frac{b}{q}\right) = 0,$$

which is an algebraic linear relation among log gamma values. Therefore, by Conjecture 1,

$$\hat{f}_k - \xi \hat{f}_l = 0$$

on all natural numbers. This implies the $\overline{\mathbb{Q}}$-linear dependence of the functions $\hat{f}_k$ and $\hat{f}_l$, and thus contradicts the $\overline{\mathbb{Q}}$-linear independence of f_k and f_l by (4). Hence, $L'(1, f_k)/L'(1, f_l)$ is transcendental for all $1 \le k < l \le r$, which implies that at most one of the numbers under consideration is algebraic. ∎

3.3. Proof of Corollary 1. Let q be any natural number greater than 7 and let χ be an odd primitive Dirichlet character modulo q. It suffices to show that $\chi \in \mathfrak{F}_{\mathfrak{q}}$, i.e., $\hat{\chi}$ is of Dirichlet type and $L(1, \chi) \neq 0$. The latter follows from the famous theorem of Dirichlet [1, Theorem 6.20 and Section 7.3], while the former comes from [1, Chapter 8, Theorem 8.19] since

$$\hat{\chi}(n) = \frac{1}{q} \sum_{a=1}^{q} \chi(a) \zeta_q^{-an}. \ ∎$$

3.4. Proof of Corollary 2. We begin by observing that the functions f_j defined below are in $\mathfrak{F}_{\mathfrak{p}}$. For $1 \le j \le (p-1)/2$,

$$f_j(n) := \begin{cases} 1 & \text{if } n \equiv j \bmod p, \\ -1 & \text{if } n \equiv -j \bmod p, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, each f_j is periodic with odd prime period p, f_j is odd and $f_j(p) = 0$. Moreover, $\sum_{a=1}^{p} f_j(a) = 0$ and by Theorem 2.5, $L(1, f_j) \neq 0$ for all $1 \le j \le r$. Thus, $f_j \in \mathfrak{F}_{\mathfrak{p}}$ for all $1 \le j \le r$. Also note that the functions $\{f_j \mid 1 \le j \le (p-1)/2\}$ are $\overline{\mathbb{Q}}$-linearly independent. Therefore, Theorem 1.1 implies that at least $(p-1)/2 - 3$ of the numbers

$$\{\gamma_1(a, p) - \gamma_1(p-a, p) \mid 1 \le a \le (p-1)/2\}$$

are transcendental. Since the difference of two numbers being transcendental implies that at least one of them is transcendental, the result follows. ∎

4. Concluding remarks. Our work here represents a modest beginning of research into the arithmetic nature of generalized Stieltjes constants. These constants have emerged in other contexts. Most notably, they appear in Li's criterion for the Riemann hypothesis (see for example [5]). It is quite possible that the study of these constants can lead us to the holy grail of mathematics.

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Abstract (will appear on the journal's web site only)

The connection between derivatives of $L(s, f)$ for periodic arithmetical functions f at $s = 1$ and generalized Stieltjes constants has been noted earlier. In this paper, we utilize this link to throw light on the arithmetic nature of $L'(1, f)$ and certain Stieltjes constants. In particular, if p is an odd prime greater than 7, then we deduce the transcendence of at least $(p-7)/2$ of the generalized Stieltjes constants, $\{\gamma_1(a, p) : 1 \leq a < p\}$, conditional on a conjecture of S. Gun, M. Ram Murty and P. Rath (2009).