Canadian Mathematical Society
Conference Proceedings
Volume 15, 1995

The Analytic Rank of $J_0(N)(\mathbb{Q})$

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ABSTRACT. Let $J_0(N)$ be the Jacobian of the modular curve $X_0(N)$. We will investigate the 'analytic rank' of the group of rational points of $J_0(N)$. For instance, assuming a 'Riemann hypothesis' for certain L-functions associated to $J_0(N)$, we prove that the analytic rank is bounded by (3/2)dim $S_2(N)$, where $S_2(N)$ is the space of cusp forms of weight 2 and level N. Our investigation suggests that the rank grows like (1/2)dim $S_2(N)$ as N tends to infinity.

1. Introduction

In this paper, we will study the analytic rank of $J_0(N)(\mathbb{Q})$ when N is prime by means of the explicit formula method. This method has been used by various previous authors, see Duke [Du], Goldfeld [Go], Mestre [Me], and Murty [Mu], to study similar questions. Several years ago, Brumer [Br] used the method in conjunction with the Eichler-Selberg trace formula to obtain results similar to the ones we will establish in this paper. Our approach diverges from Brumer's in the use of Poincaré series instead of the trace formula and this in turn seems to lead to better results. The restriction to N prime is made for technical simplicity. The general case is a little more delicate involving a finer analysis of the contribution from the oldforms.

Our investigation was motivated by a conjecture of Greenberg [Gr]: let Σ be a finite set of primes and let f vary over all normalized new forms of weight 2 for $\Gamma_0(M)$ where M is divisible only by primes of Σ. Then the order of vanishing of the L-function $L(f,s)$ at $s=1$ is 0 or 1 except for at most finitely many such f's.

Assuming his conjecture, Greenberg shows that for any modular elliptic curve E over $\mathbb{Q}$ with root number -1, either $E(\mathbb{Q})$ is infinite or the p-primary part of the Shafarevich-Tate group is infinite for all p where E has good, ordinary

1991 *Mathematics Subject Classification.* Primary 11G40; Secondary 14G10, 14H40.
Research supported by NSERC, FCAR and CICMA grants

reduction. Such a result would be a precursor to a generalization of the celebrated theorem of Kolyvagin.

Let

$$r_f = ord_{s=1} L(f, s).$$

We would like to study the sum

$$\sum_f{}' r_f$$

where the dash on the sum indicates that f ranges over normalized newforms of weight 2 and level N. Such a sum is in fact the "analytic rank" of $J_0(N)^{new}(\mathbb{Q})$ by virtue of Shimura's theorem concerning the Hasse-Weil L-function of $J_0(N)^{new}$. The conjectures of Birch and Swinnerton-Dyer predict that it is the "algebraic rank" of $J_0(N)^{new}(\mathbb{Q})$. We will prove

THEOREM 1. *Let N be prime. Suppose that for each newform f, $L(f, s)$ satisfies the analogue of the Riemann hypothesis. Then,*

$$\sum_f{}' \frac{r_f}{4\pi(f,f)} \leq \frac{7}{6} + o(1)$$

as $N \to \infty$, and where (f, f) denotes the Petersson inner product on the space of cusp forms $S_2(N)$ of weight 2 on $\Gamma_0(N)$.

The proof of Theorem 1 is a straightforward application of the "semi-orthogonality" of the Fourier coefficients of an orthonormal basis of Hecke eigenforms in conjunction with the explicit formula method. By invoking the theory of the L-function attached to the symmetric square L-function, one can remove the "weight" of $1/4\pi(f, f)$ from the above sum in the proof of Theorem 1 to deduce:

THEOREM 2. *Under the same conditions as in Theorem 1, and the Lindelöf hypothesis for $L(s, \mathrm{sym}^2(f))$,*

$$\sum_f{}' r_f \leq \left(\frac{3}{2} + \epsilon\right) \dim S_2(N) + o(N)$$

as $N \to \infty$.

COROLLARY. *Under the hypothesis of Theorem 2, the analytic rank of $J_0(N)^{new}(\mathbb{Q})$ satisfies*

$$\leq \left(\frac{3}{2} + \epsilon\right) \dim S_2^{new}(N) + o(N)$$

as $N \to \infty$.

Is it true that

$$\operatorname{rank} J_0(N)^{new}(\mathbb{Q}) \sim c \dim S_2^{new}(N)$$

for some constant c ? Does $c = 1/2$? There are good reasons to suspect this. One can establish a lower bound with $c = 1/2$.

It is interesting to note that Brumer [Br] also proves assuming the Riemann hypothesis for each $L(s,f)$ that the analytic rank of $J_0(N)^{new}(\mathbb{Q})$ is

$$\leq \left(\frac{3}{2}+\epsilon\right)\dim S_2^{new}(N) + o(N).$$

Our method needs the extra assumption on $L(s, \mathrm{sym}^2(f))$. Nevertheless, the method has its advantages and a few more related results can be deduced without this extra hypothesis (see Theorem 4 below).

As noted earlier, our methods are different. Brumer uses the Eichler-Selberg trace formula as generalized by Skoruppa and Zagier [SZ]. We will use the theory of Poincaré series. The common feature of both the methods is the use of Weil's explicit formula which necessitates the invoking of the Riemann hypothesis so that we may obtain optimal results.

The method allows for interesting unconditional results. Namely, we can prove:

THEOREM 3.

$$\sum_f{}'(f,f) = \frac{\pi}{24}(\dim S_2(N))^2 + O(N^{1.3}\log^2 N).$$

This will lead to the following modification of Theorems 1 and 2:

THEOREM 4. *Under the same conditions as Theorem 1,*

$$\sum_f{}' r_f^{1/2} \leq \frac{\sqrt{7}\pi}{6}\dim S_2(N) + o(N).$$

2. Poincaré series

We will recall (see Iwaniec [Iw]) the basic facts concerning Poincaré series.

Let Γ_∞ be the stabilizer of $i\infty$ in $\Gamma_0(N)$. The space $S_k(N)$ of cusp forms of weight k and level N is a finite dimensional vector space over $\mathbb{C}$, spanned by the Poincaré series: for $m \geq 1$,

$$P_m(z,k,N) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} j(\gamma,z)^{-k} e(m\gamma z)$$

where

$$e(z) = e^{2\pi i z}, \quad j(\gamma,z) = (\det\gamma)^{-1/2}(cz+d)$$

and

$$\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}.$$

In the case of $k = 2$, one does not have absolute convergence of the Poincaré series. Nevertheless, what we state below is valid in this case as well. If $f \in S_k(N)$, we write the Fourier expansion of f as:

$$f(z) = \sum_{n=1}^{\infty} \hat{f}(n) e(nz)$$

at $i\infty$. The space $S_k(N)$ has an inner product (Petersson inner product):

$$(f, g) = \int_{\Gamma_0(N)\backslash\mathfrak{h}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}$$

where $\mathfrak{h}$ denotes the upper half plane. Petersson proved:

$$\hat{f}(n) = \frac{(4\pi n)^{k-1}}{\Gamma(k-1)} (f, P_n(\cdot, k, N)) \tag{1}$$

so that if $f_1, ..., f_r$ is an orthonormal basis for $S_k(N)$, and

$$P_n(\cdot, k, N) = \sum_i c_i f_i$$

we deduce

$$c_i = (f_i, P_n(\cdot, k, N)) = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \hat{f}_i(n).$$

Thus,

$$\frac{(4\pi n)^{k-1}}{\Gamma(k-1)} P_n(\cdot, k, N) = \sum_i \hat{f}_i(n) f_i.$$

Comparing the m-th coefficients on both sides gives

$$\frac{(4\pi n)^{k-1}}{\Gamma(k-1)} \hat{P}_n(m, k, N) = \sum_i \hat{f}_i(n) \hat{f}_i(m). \tag{2}$$

The m-th Fourier coefficient of the n-th Poincaré series $P_n(z, k, N)$ can be computed explicitly as:

$$\hat{P}_n(m, k, N) =$$

$$(m/n)^{(k-1)/2} \left\{ \delta_{mn} + 2\pi i^{-k} \sum_{c \equiv 0 \pmod{N}} c^{-1} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) S(m, n, c) \right\}$$

where $\delta_{mn} = 0$ unless $m = n$ in which case it is 1, $J_{k-1}(x)$ is the Bessel function of order $k-1$, $S(m, n, c)$ is the Kloosterman sum:

$$S(m, n, c) = \sum_{d \pmod{c}} e\left(\frac{md + n\bar{d}}{c}\right)$$

where $d\bar{d} \equiv 1 \pmod{c}$. Again this formula is due to Petersson.

We now specialize these formulas for $k = 2$. Using well-known estimates for the Bessel function and Weil's estimate for the Kloosterman sum

$$|S(m, n, c)| \leq (m, n, c)^{1/2} d(c) c^{1/2}$$

where $d(c)$ denotes the number of positive divisors of c, we deduce (see Duke [Du]) from (2) and (3):

PROPOSITION 1.

$$\left|\sum_f{}' \frac{\hat{f}(m)\hat{f}(n)}{4\pi(f,f)} - \delta_{mn}\sqrt{m}\sqrt{n}\right| = O(N^{-3/2}(m,n)^{1/2}mn).$$

Putting $m = 1$, we deduce that

PROPOSITION 2.

$$\sum_f{}' \frac{\hat{f}(n)}{4\pi(f,f)} = \delta_{1n}\sqrt{n} + O(N^{-3/2}n),$$

where the sum is over normalised eigenforms of $S_2(N)$.

Remark. One could improve this further on the assumption of the Selberg-Linnik conjecture which asserts that

$$\sum_{c\leq x} \frac{S(m,n,c)}{c} = O(x^{1/2+\epsilon}).$$

In fact, the factor of $N^{-3/2}$ can be replaced by N^{-2} on this conjecture. We make further remarks concerning this at the end of the paper.

3. The explicit formula method

Let us write

$$-\frac{L'}{L}(f,s) = \sum_{n=1}^{\infty} \frac{c_n(f)}{n^s}\Lambda(n)$$

where

$$c_n(f) = \alpha_p^a + \overline{\alpha}_p^a$$

whenever $n = p^a$ and is zero otherwise. Here, $\alpha_p + \overline{\alpha}_p = \hat{f}(p)$, with $|\alpha_p| = \sqrt{p}$.

LEMMA 1 (WEIL'S EXPLICIT FORMULA). *Let $F : \mathbb{R}\to\mathbb{R}$ satisfy the following conditions:*

(a) *there is an $\epsilon > 0$ such that $F(x)\exp((1+\epsilon)x)$ is integrable and of bounded variation,*
(b) *the function $(F(x) - F(0))/x$ is of bounded variation.*

Define

$$\phi(\gamma) = \int_{-\infty}^{\infty} F(x)e^{i\gamma x}dx.$$

Then,

$$\sum_{L(f,1+i\gamma)=0} \phi(\gamma) = 2F(0)\log\frac{\sqrt{N}}{2\pi}$$

$$-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}(1+it)\phi(t)dt - 2\sum_{n=1}^{\infty}\frac{c_n(f)}{n}\Lambda(n)F(\log n)$$

where the sum on the left hand side is over γ *such that* $L(f,1+i\gamma)=0$, $1 \le \Re(1+i\gamma) \le 3/2$.

PROOF. See Mestre [Me].

We choose $T > 0$ and define

$$F(x) = \begin{cases} 2T - |x| & if\ |x| \le 2T, \\ 0 & otherwise. \end{cases}$$

Then F satisfies the conditions of Lemma 1 and

$$\phi(\gamma) = \left(\frac{2\sin\gamma T}{\gamma}\right)^2.$$

Moreover, the corresponding integral involving the logarithmic derivative of the gamma function is easily estimated to be $O(T)$. Let us observe that

$$\phi(0) = T^2\lim_{\gamma\to 0}\left(\frac{2\sin\gamma T}{\gamma T}\right)^2 = 4T^2$$

so that choosing $T = (\log x)/2$ gives $\phi(0) = \log^2 x$. On the GRH, $\phi(\gamma) \ge 0$ for all γ and so from the explicit formula we get that

$$r_f(\log x)^2 \le 2(\log x)\log\frac{\sqrt{N}}{2\pi} - 2\sum_{n\le x}\frac{c_n(f)}{n}\Lambda(n)\log\frac{x}{n} + O(\log x).$$

4. Proof of Theorem 1

We consider

$$\sum_f{}' \frac{r_f}{4\pi(f,f)}.$$

We have

$$(\log x)^2\sum_f{}'\frac{r_f}{4\pi(f,f)} \le 2(\log x)(\log\frac{\sqrt{N}}{2\pi})\sum_f{}'\frac{1}{4\pi(f,f)}$$

$$-2\sum_{n\le x}\frac{\Lambda(n)}{n}\log\frac{x}{n}\left(\sum_f{}'\frac{c_n(f)}{4\pi(f,f)}\right) + O\left((\log x)\sum_f{}'\frac{1}{4\pi(f,f)}\right).$$

Note that Proposition 2 with $n = 1$ gives

$$\sum_f{}'\frac{1}{4\pi(f,f)} = 1 + O(N^{-3/2}).$$

Therefore

$$(\log x)^2 {\sum_f}' \frac{r_f}{4\pi(f,f)} \le 2(\log x)\log\frac{\sqrt{N}}{2\pi} + O(\log x) \tag{4}$$
$$- 2\sum_{n\le x}\frac{\Lambda(n)}{n}\log\frac{x}{n}\left({\sum_f}'\frac{c_n(f)}{4\pi(f,f)}\right).$$

We now analyze

$${\sum_f}'\frac{c_n(f)}{4\pi(f,f)}.$$

In case $n = p$ is prime, $c_p(f) = \hat{f}(p)$ and therefore by Proposition 2,

$${\sum_f}'\frac{\hat{f}(p)}{4\pi(f,f)} = O(N^{-3/2}p)$$

which contributes

$$O\left(\sum_{p\le x}\log p\log\frac{x}{p}N^{-3/2}\right) = O(N^{-3/2}x)$$

to the second sum in (4). The contribution for $n = p^a$ with $a \ge 3$ is at most

$$\sum_{n=p^a\le x, a\ge 3}\frac{\Lambda(n)}{\sqrt{n}}\log\frac{x}{n} \ll (\log x)\sum_{a\ge 3,p}\frac{\log p}{p^{a/2}} \ll \log x.$$

We must still deal with $n = p^2$. Note that

$$c_{p^2}(f) = \alpha_p^2 + \overline{\alpha}_p^2 = (\alpha_p + \overline{\alpha}_p)^2 - 2\alpha_p\overline{\alpha}_p = a_p(f)^2 - 2p.$$

But $a_p(f)^2 = a_{p^2}(f) + p$. By the Rankin-Selberg method,

$$\sum_{p\le x}\frac{\log p}{p}a_p(f)^2 \sim x$$

so that, by partial summation, we deduce that

$$\sum_{p\le x}\frac{\log p}{p^2}a_p(f)^2 \sim \log x.$$

Therefore,

$$\sum_{p^2\le x}\frac{\log p}{p^2}a_p(f)^2 \sim \frac{1}{2}\log x.$$

Again, by partial summation,

$$\sum_{p^2\le x}\frac{\log p}{p^2}\left(\log\frac{x}{p^2}\right)a_p(f)^2 \sim \frac{1}{4}\log^2 x.$$

In addition,

$$-2\sum_{p^2\le x}\frac{\log p}{p}\left(\log\frac{x}{p^2}\right)\sim -\frac{1}{2}\log^2 x$$

so that

$$\sum_{p^2\le x}\frac{\log p}{p^2}\left(\log\frac{x}{p^2}\right)(a_p(f)^2-2p)\sim -\frac{1}{4}\log^2 x$$

as $x\to\infty$. Summing over f with weights $1/4\pi(f,f)$, we obtain a contribution of

$$\frac{1}{2}(\log x)^2+\mathrm{O}(N^{-3/2}(\log x)^2).$$

Finally, we get

$$\sum_f{}' \frac{r_f}{4\pi(f,f)}\le\frac{2\log\frac{\sqrt{N}}{2\pi}}{\log x}+\frac{1}{2}+\mathrm{O}(N^{-3/2}+N^{-3/2}x(\log x)^{-2}+(\log x)^{-1})$$

which simplifies to

$$\frac{1}{2}+\frac{\log N}{\log x}+\mathrm{O}(N^{-3/2}x(\log x)^{-2}).$$

Choose $x=N^{3/2}$ to get

$$\sum_f{}' \frac{r_f}{4\pi(f,f)}\le\left(\frac{7}{6}+\mathrm{O}((\log N)^{-2})\right).$$

This completes the proof of Theorem 1.

We remark that the Selberg-Linnik conjecture would lead to an upper bound of $1+\mathrm{o}(1)$ for the same sum.

5. An approximate trace formula

In this section, we will derive a formula for

$$\sum_f{}' \hat{f}(n).$$

This will enable us to prove Theorems 2 and 3. We make use of the fact (see Stark [St, p. 90])

$$L(2,\mathrm{sym}^2(f))=8\pi^3\phi(N)\frac{(f,f)}{N^2}$$

which in the case of N prime simplifies to

(5) $$L(2,\mathrm{sym}^2(f))=8\pi^3(N-1)(f,f)/N^2.$$

(We take this opportunity to point out a numerical error in the formula for Coates-Schmidt [CS, p. 123].) Moreover, by a well-known elementary identity (see Coates-Schmidt [CS, p. 124]),

$$L(s,\mathrm{sym}^2(f))=\zeta_N(2s-2)\sum_{n=1}^{\infty}\frac{\hat{f}(n^2)}{n^s}=\sum_{n=1}^{\infty}\frac{g_f(n)}{n^s}$$

where $\zeta_N(s)$ is the Riemann zeta function with the Euler factors corresponding to $p|N$ removed. Consider the integral

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} L(2+s,\mathrm{sym}^2(f))T^s\Gamma(s)ds = \sum_{n=1}^{\infty}\frac{g_f(n)}{n^2}\exp(-n/T)$$

and this is

$$= L(2,\mathrm{sym}^2(f)) + \frac{1}{2\pi i}\int_{(-1/2)} L(2+s,\mathrm{sym}^2(f))T^s\Gamma(s)ds.$$

By the Phragmén - Lindelöf principle, the integral is easily estimated as

$$O(N^{\epsilon}T^{-1/2})$$

assuming the Lindelöf hypothesis (which is a consequence of the generalised Riemann hypothesis for $L(s,\mathrm{sym}^2(f))$. Unconditionally, we have the estimate (see [MM, p. 336]):

$$O(N^{3/4}T^{-1/2}).$$

Therefore,

$$L(2,\mathrm{sym}^2(f)) = \sum_{n=1}^{\infty}\frac{g_f(n)}{n^2}\exp(-n/T) + O(N^{\theta}T^{-1/2}) \tag{6}$$

for a certain $\theta > 0$. Substituting the expression for $g_f(n)$ gives:

PROPOSITION 3.

$$L(2,\mathrm{sym}^2(f)) = \sum_{\substack{d,e\\(d,N)=1}}\frac{\hat{f}(e^2)}{d^2e^2}\exp(-d^2e/T) + O(N^{\theta}T^{-1/2}),$$

where $\theta = 3/4$ unconditionally. If we assume the Lindelöf hypothesis for $L(s,\mathrm{sym}^2(f))$, we can take any $\theta > 0$.

We now utilise this proposition to prove:

PROPOSITION 4. *Let θ be as above. If n is not a square, we have for any $T > 0$,*

$$\sum_f{}' \hat{f}(n) = O(N^{-1/2}nTd(n) + \sqrt{n}d(n)N^{1+\theta}T^{-1/2})$$

where $d(n)$ is the number of divisors of n. If n is a square, we have

$$\sum_f{}' \hat{f}(n) = \frac{N}{2\pi^2}\left\{\zeta_N(2) + O(T^{-1/2+\epsilon}n^{1/4+\epsilon})\right\}$$
$$+ O(N^{-1/2}nT + \sqrt{n}d(n)N^{1+\theta}T^{-1/2}).$$

PROOF. We begin by writing,

$$\sum_f{}' \hat{f}(n) = N\sum_f{}' \frac{\hat{f}(n)}{8\pi^3(f,f)}L(2,\mathrm{sym}^2(f))$$

by virtue of (5). By Proposition 3, we can rewrite this as

$$\frac{2\pi^2}{N}\sum_f{}' \hat{f}(n) = \sum_{\substack{d,e\\(d,N)=1}} \frac{\exp(-d^2e/T)}{d^2e^2}\sum_f{}' \frac{\hat{f}(n)\hat{f}(e^2)}{4\pi(f,f)} + \mathrm{O}(\sqrt{n}d(n)N^\theta T^{-1/2}),$$

where we have used Proposition 2 with $n = 1$ to estimate the error term. By Proposition 1, the inner sum over f is easily evaluated to give

$$\frac{2\pi^2}{N}\sum_f{}' \hat{f}(n) = \sum_{\substack{d,e\\(d,N)=1}} \frac{\exp(-d^2e/T)}{d^2e^2}\left(\delta_{n,e^2}\sqrt{n}e + \mathrm{O}(N^{-3/2}ne^2(n,e^2)^{1/2})\right)$$
$$+ \mathrm{O}(\sqrt{n}d(n)N^\theta T^{-1/2}).$$

The error term arising from the sum is

$$\ll N^{-3/2}n\sum_{d,e}\frac{\exp(-d^2e/T)}{d^2}(n,e^2)^{1/2}.$$

If we define a function $g(m)$ by

$$m^{1/2} = \sum_{d|m} g(d)$$

then we can rewrite the above sum in the error as

$$N^{-3/2}n\sum_{\delta|n} g(\delta)\sum_{\substack{d,e\\ \delta|e^2}}\frac{\exp(-d^2e/T)}{d^2} = N^{-3/2}n\sum_{\delta|n} g(\delta)\sum_d \frac{1}{d^2}\sum_{\substack{e\\ \delta|e^2}}\exp(-d^2e/T)$$

which is easily seen to be

$$\ll N^{-3/2}nT\sum_{\delta|n}\frac{g(\delta)}{\sqrt{\delta}}$$

which is

$$\ll N^{-3/2}nTd(n).$$

This proves the Proposition when n is not a square. When $n = m^2$, we need to analyze the main term arising from

$$\sum_{(d,N)=1}\frac{\exp(-d^2m/T)}{d^2}.$$

But again from the integral formula

$$\sum_{(d,N)=1}\frac{\exp(-d^2/T)}{d^2} = \frac{1}{2\pi i}\int_{(\sigma)}\zeta_N(2+2s)T^s\Gamma(s)ds$$
$$= \zeta_N(2) + \frac{1}{2\pi i}\int_{(-1/2+\epsilon)}\zeta_N(2+2s)T^s\Gamma(s)ds$$

we infer

$$\sum_{(d,N)=1} \frac{\exp(-d^2/T)}{d^2} = \zeta_N(2) + O(T^{-1/2+\epsilon}) \tag{7}$$

which when substituted into the main term gives the desired result. This completes the proof of the Proposition.

6. Proof of Theorem 2

We will assume that $L(s, \mathrm{sym}^2(f))$ satisfies the Lindelöf hypothesis so that in Proposition 3, we can take any $\theta > 0$. From the last displayed formula of Section 3, we deduce that

$$(\log x)^2 \sum_f{}' r_f = S_1 + S_2 + S_3$$

where

$$S_1 = 2(\log x)(\log \frac{\sqrt{N}}{2\pi}) \dim S_2(N),$$

$$S_2 = -2\sum_{n\leq x} \left(\sum_f{}' c_n(f)\right) \frac{\Lambda(n)}{n} \log \frac{x}{n},$$

$$S_3 = O(N \log x).$$

We now estimate S_2. We decompose this sum further into

$$S_2 = S_{21} + S_{22} + S_{23}$$

where S_{21} is the contribution from the terms $n = p$, a prime; S_{22} is the contribution from the terms $n = p^2$, the square of a prime and S_{23} is composed of terms $n = p^a$ with $a \geq 3$. Let us first consider S_{23}. The inequality

$$|c_n(f)| \leq 2\sqrt{n}$$

leads to

$$S_{23} \ll N \log x \sum_{p^a,\, a\geq 3} \frac{\log p}{p^{a/2}} \ll N \log x.$$

Now we estimate S_{21} and S_{22}. First, let us deal with S_{21} and note that $c_p(f) = \hat{f}(p)$ so that by Proposition 4, and familiar estimates of analytic number theory,

$$S_{21} = O(N^{-1/2}Tx + N^{1+\theta}T^{-1/2}x^{1/2}).$$

Now let us look at S_{22}. If $n = p^2$, note that

$$c_{p^2}(f) = \hat{f}(p^2) - p$$

so that we derive in a straightforward manner,

$$S_{22} \leq \frac{1}{2} \dim S_2(N)(\log x)^2 + O(N^{-1/2}T\sqrt{x}\log x + N(\log x)),$$

as $T \to \infty$. The final result is

$$\sum_f{}' r_f \leq \frac{\log N}{\log x} \dim S_2(N) + \frac{1}{2} \dim S_2(N)$$
$$+ \mathrm{O}(\frac{N^{-1/2}T\sqrt{x}}{\log x} + \frac{N}{\log x}) + \mathrm{O}(\frac{x^{1/2}N^{1+\theta}T^{-1/2}}{(\log x)^2}).$$

If we take θ arbitrary and positive, then we can proceed quickly to make the optimal choices. We choose $T = x^{1+\eta}$ where $\eta > 0$ and arbitrary. The optimal choice for x is $N^{1-2\eta}$ which gives for any $\epsilon > 0$

$$\sum_f{}' r_f \leq \left(\frac{3}{2} + \epsilon\right) \dim S_2(N) + \mathrm{o}(N)$$

as $N \to \infty$. This completes the proof of Theorem 2.

The Selberg-Linnik conjecture, as has been previously remarked, leads to N^{-2} instead of $N^{-3/2}$ in Proposition 2. It is then not difficult to see that in the above derivation, this would lead to an upper bound of

$$\left(\frac{5}{4} + \epsilon\right) \dim S_2(N) + \mathrm{o}(N)$$

as $N \to \infty$.

7. Proof of Theorem 3

We have by Proposition 3,

$$\sum_f{}' L(2, \mathrm{sym}^2(f)) = \sum_{\substack{d,e \\ (d,N)=1}} \frac{\exp(-d^2e/T)}{d^2e^2} \sum_f{}' \hat{f}(e^2) + \mathrm{O}(N^{1+\theta}T^{-1/2}).$$

By Proposition 4,

$$\sum_f{}' \hat{f}(e^2) = \frac{N}{2\pi^2}\zeta_N(2) + \mathrm{O}(NT^{-\frac{1}{2}+\epsilon}e^{1-2\epsilon}) + \mathrm{O}(N^{-1/2}e^2T + ed(e^2)N^{1+\theta}T^{-1/2})$$

so that

$$\sum_f{}' L(2, \mathrm{sym}^2(f)) = \frac{N\zeta_N(2)}{2\pi^2} \sum_{\substack{d,e \\ (d,N)=1}} \frac{\exp(-d^2e/T)}{d^2e^2} + \mathcal{E}$$

where

$$\mathcal{E} \ll \sum_{d,e} \frac{\exp(-d^2e/T)}{d^2e^2} \left(NT^{-\frac{1}{2}+\epsilon}e^{1-2\epsilon} + N^{-\frac{1}{2}}Te^2 + ed(e^2)N^{1+\theta}T^{-\frac{1}{2}}\right).$$

Using the elementary estimate

$$\sum_n e^{-n/T} \ll \int_1^\infty e^{-x/T}dx \ll T$$

and

$$\sum_e \frac{d(e^2)}{e} \exp(-e/T) \ll \log^2 T$$

we deduce

$$\mathcal{E} \ll NT^{-\frac{1}{2}+\epsilon} + N^{-\frac{1}{2}}T^2 + N^{1+\theta}T^{-\frac{1}{2}} \log^2 T.$$

The main term is deduced by an application of (7),

$$\frac{N\zeta_N(2)}{2\pi^2} \sum_{\substack{d,e \\ (d,N)=1}} \frac{\exp(-d^2e/T)}{d^2e^2} = \frac{N\zeta_N(2)}{2\pi^2} \sum_e \frac{1}{e^2} \left(\zeta_N(2) + \mathrm{O}((T/e)^{-\frac{1}{2}+\epsilon})\right).$$

This is easily seen to be equal to

$$= \frac{N\zeta(2)^3}{2\pi^2} + \mathrm{O}(NT^{-\frac{1}{2}+\epsilon}).$$

Hence, for any $T > 0$,

$$\frac{8\pi^3}{N} \sum_f{}' (f,f) = \frac{N\zeta(2)^3}{2\pi^2} + \mathrm{O}(NT^{-\frac{1}{2}+\epsilon} + N^{-\frac{1}{2}}T^2 + N^{1+\theta}T^{-\frac{1}{2}} \log^2 T).$$

Therefore,

$$\sum_f{}' (f,f) = \frac{\pi}{24} \dim S_2(N)^2 + \mathrm{O}(N^{.7+4\theta/5})$$

where we have chosen $T = N^{.6+.4\theta}$. By our previous remark, $\theta = 3/4$ is permissible by the Phragmén-Lindelöf theorem, and so this completes the proof.

8. Proof of Theorem 4

We now apply Theorem 3 in conjunction with the Cauchy - Schwarz inequality to deduce Theorem 4. We have

$$\sum_f{}' r_f^{\frac{1}{2}} \le \left(\sum_f{}' \frac{r_f}{4\pi(f,f)}\right)^{\frac{1}{2}} \left(\sum_f{}' 4\pi(f,f)\right)^{\frac{1}{2}}.$$

By Theorems 1 and 3, this is

$$\le 2\sqrt{\pi}\left(\frac{7}{6} + \mathrm{o}(1)\right)^{\frac{1}{2}} \left(\frac{\pi}{24} \dim S_2(N)^2 + \mathrm{O}(N^{1.3}\log^2 N)\right)^{\frac{1}{2}}$$

which is

$$\le \frac{\sqrt{7}\pi}{6} \dim S_2(N) + \mathrm{o}(N)$$

as $N \to \infty$.

We remark that the Selberg-Linnik conjecture would lead to an upper bound of

$$\le \frac{\pi}{\sqrt{6}} \dim S_2(N) + \mathrm{o}(N).$$

9. A lower bound

One can establish a lower bound for the rank which is unconditional by the following argument. Consider the Atkin-Lehner involution $w : X_0(N) \to X_0(N)$ and the natural quotient map of degree 2,

$$\phi : X_0(N) \to X_0(N)^+ = X_0(N)/w.$$

The genus of $X_0(N)^+$ is the dimension of the space $S_2^+(N)$ consisting of forms whose corresponding L-series has root number equal to -1. By the Riemann-Hurwitz formula, we then find

$$\dim S_2(N) = 2 \dim S_2^+(N) + s$$

where s is the number of fixed points of w. But such a point is represented by an isogeny

$$E \xrightarrow{\pi} E'$$

which is isomorphic to its dual

$$E' \xrightarrow{\overline{\pi}} E.$$

It follows that $E \simeq E'$ and so the isogeny must be a complex multiplication $\pi^2 = \epsilon N$ where ϵ is a unit in the order of the imaginary quadratic field $\mathbb{Q}(-\sqrt{-N})$ corresponding to E. But the number of such E's is at most $O(\sqrt{N})$ by familiar estimates for the class number of $\mathbb{Q}(\sqrt{N})$. Thus,

$$\dim S_2(N)^+ = \frac{1}{2} \dim S_2(N) + O(\sqrt{N})$$

which gives some evidence for the conjecture we had made in Section 1. This proves that the analytic rank of $J_0(N)^{new}(\mathbb{Q})$ is

$$\geq \frac{1}{2} \dim S_2(N)^{new} + O(\sqrt{N}).$$

Acknowledgments. I would like to thank C.S Rajan, H. Kisilevsky and H. Darmon for their suggestions and discussions.

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