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A function related to the Mordell–Weil rank of elliptic curves

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Let p be a prime number. For each natural number n, we study the behavior of the function $f_p(n)$ which enumerates the number of factorizations $ab = n$ with $a + b$ a perfect square (mod p). The study of this function is inspired by the cognate function $f(n)$ which enumerates the number of factorizations $ab = n$ with $a + b$ a perfect square. The descent theory of elliptic curves would show that if $f(n)$ is unbounded for squarefree values of n, then there are elliptic curves over the rational number field with arbitrarily large rank. In this note, we show for every prime p, $f_p(n)$ is unbounded as n ranges over squarefree values, thus providing some evidence for the conjecture that $f(n)$ is unbounded for squarefree n.

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1. Introduction

For a natural number n, let

$$f(n) := \#\{1 \leq a,\, b \leq n \,:\, ab = n,\, a + b \text{ is a perfect square}\}.$$

The unboundedness of $f(n)$ for n squarefree has a connection to the unbounded rank conjecture of elliptic curves which we will describe in Sec. 2.

For a fixed prime p, let

$$f_p(n) := \#\left\{1 \leq a,\, b \leq n \,:\, ab = n,\, \left(\frac{a+b}{p}\right) = 1\right\},$$

where $(\frac{a}{p})$ denotes the Legendre symbol. In this paper, we show that for any fixed prime number p, the function $f_p(n)$ is unbounded as n ranges over squarefree numbers. The study of this function is inspired by the cognate function $f(n)$ defined above. The descent theory of elliptic curves (see Sec. 2) would show that if $f(n)$ is unbounded as a function of n, then there are elliptic curves over the rational number field with arbitrarily large rank. If $f(n)$ is unbounded, then so is $f_p(n)$ for every prime p. We show the following.

Theorem 1.1. *Let p be a prime number. Then,*

$$\sum_{\substack{n \leq x \\ n \text{ squarefree}}} f_p(n) \gg x \log x,$$

with the implied constant dependent on the prime p. Consequently, $f_p(n)$ is unbounded as n varies over squarefree positive integers.

2. Two Descent Via a 2-Isogeny

In his famous 1961 Haverford lectures, Tate [5] (see also the appendix in [2]) described a simple algorithm for determining the Mordell–Weil rank of elliptic curves of the form

$$E: \quad y^2 = x^3 + ax^2 + bx, \quad a, b \in \mathbb{Z}.$$

We let $W = (0, 0)$ and observe that it is a rational point on $E(\mathbb{Q})$ of order 2. Now define the curve E' as

$$E': \quad y^2 = x^3 + a'x^2 + b'x$$

with $a' = -2a$ and $b' = a^2 - 4b$. Denoting by $\mathcal{O}$ the identity element of $E(\mathbb{Q})$, we define the map

$$\alpha_E : E(\mathbb{Q}) \to \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$$

by $\alpha(\mathcal{O}) = 1 \mod \mathbb{Q}^{\times 2}$, $\alpha_E(W) = b \mod \mathbb{Q}^{\times 2}$ and for $x \neq 0$,

$$\alpha_E(x, y) = x \mod \mathbb{Q}^{\times 2}.$$

The definition of $\alpha_{E'}$ is analogous. The image of α_E and $\alpha_{E'}$ are then shown to be finite. If r denotes the rank of $E(\mathbb{Q})$, Tate [5] proves that

$$2^{r+2} = |\mathrm{Im}(\alpha_E)|\, |\mathrm{Im}(\alpha_{E'})|\,.$$

Thus, to determine the rank r, one needs to determine the size of the images of α_E and $\alpha_{E'}$.

To this end, we consider every possible factorization $b = b_1 b_2$ with $b_1, b_2 \in \mathbb{Z}$. For each such factorization, we examine the Diophantine equation involving the variables M, N and e:

$$N^2 = b_1 M^4 + a M^2 e^2 + b_2 e^4. \tag{2.1}$$

If (2.1) has a solution in nonzero integers M, N, e with $e > 0$, then a routine verification shows that

$$x = \frac{b_1 M^2}{e^2}, \quad y = \frac{b_1 M N}{e^3}$$

gives a rational point $P = (x, y)$ on E so that $\alpha_E(P) = b_1 \mod \mathbb{Q}^{\times 2}$.

For curves with $a = 0$, we now see the connection to the function $f(n)$. For n squarefree, consider the family of curves

$$E_n : \quad y^2 = x^3 + nx.$$

The algorithm for the rank of this curve derived by Tate would imply $2^{r+2} \geq f(n)$. Thus, if $f(n)$ is unbounded, then the Mordell–Weil ranks of $E_n(\mathbb{Q})$ would be unbounded. This is the motivation for studying $f(n)$ and $f_p(n)$.

In his MSc thesis (written under the direction of the senior author), David Clark [1] proved that $f(n)$ is unbounded if we remove the restriction that n is squarefree. In this paper, we study $f_p(n)$ for n squarefree so as to elucidate the more difficult study of $f(n)$ when n is squarefree.

3. Preliminary Lemmas

In the proof of our theorem, we need various results that we collect in this section for ease of reference.

The first is a Tauberian theorem. We use the classical version as stated below. See [4, Exercise 4.4.17] for a reference.

Lemma 3.1. *Let $f(s) = \sum_{n=1}^{\infty} a_n/n^s$ with $a_n = O(n^\epsilon)$. Suppose that*

$$f(s) = \zeta(s)^k g(s),$$

where k is a natural number and $g(s)$ is a Dirichlet series absolutely convergent in $\Re(s) > 1 - \delta$ for some $0 < \delta < 1$. Then we have

$$\sum_{n \leq x} a_n \sim \frac{g(1)}{(k-1)!}\, x\, (\log x)^{k-1}$$

as $x \to \infty$.

The following result is due to Hooley [3].

Lemma 3.2 ([3]). *Let $R(x;a,q)$ be the number of squarefree numbers in the arithmetic progression $a \bmod q$ with $(a,q)=1$. For any $\epsilon > 0$, we have*

$$R(x;a,q) = \frac{1}{\zeta(2)} \prod_{\substack{p|q \\ p \text{ prime}}} \left(1 - \frac{1}{p^2}\right)^{-1} \frac{x}{q} + O_\epsilon\left(\left(\frac{x}{q}\right)^{1/2} + q^{1/2+\epsilon}\right). \tag{3.1}$$

Lemma 3.3.

$$\sum_{n \le x} \mu^2(n)\, d(n) = Cx \log x + o(x \log x),$$

where

$$C = \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3}\right) = 0.33\ldots$$

Proof. This is a simple application of Lemma 3.1. Here are the relevant details. We have

$$\sum_{n=1}^{\infty} \frac{\mu^2(n)d(n)}{n^s} = \prod_p \left(1 + \frac{2}{p^s}\right).$$

The infinite product can be re-written as

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-2} \prod_p \left(1 - \frac{1}{p^s}\right)^2 \left(1 + \frac{2}{p^s}\right) = \zeta(s)^2 g(s) \quad \text{(say)}.$$

An application of the Tauberian theorem gives

$$\sum_{n \le x} \mu^2(n)d(n) = Cx \log x + o(x \log x)$$

with

$$C = \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3}\right) = \prod_p \left(\frac{p^3 - 3p + 2}{p^3}\right).$$

Since each factor in the absolutely convergent product is nonzero, we deduce $C \neq 0$, as desired. □

Remark 3.4. We remark that it is possible to refine the lemma to give constants C, D such that

$$\sum_{n \le x} \mu^2(n)d(n) = Cx \log x + Dx + O(x^{1/2}),$$

using the technique of contour integration as discussed in [4, Chap. 4].

We also use the following estimates. Let $R(x)$ denote the number of squarefree numbers $n \le x$. It is well known that (see for example, [4, Exercise 1.4.4])

$$R(x) = \frac{x}{\zeta(2)} + O(\sqrt{x}).$$

By partial summation,

$$\sum_{\substack{n\le x\\ n\text{ squarefree}}} \frac{1}{n} = \sum_{n\le x}\frac{\mu^2(n)}{n} = \int_1^x \frac{R(t)}{t^2}\,dt + O(1) = \frac{\log x}{\zeta(2)} + O(1). \tag{3.2}$$

Similarly, we can deduce

$$\sum_{\substack{n\le x\\ n\text{ squarefree}}} \frac{1}{\sqrt{n}} = \frac{2\sqrt{x}}{\zeta(2)} + O(\log x). \tag{3.3}$$

Using these, we prove the crucial lemma below.

Lemma 3.5. *For a prime* p,

$$\left|\sum_{\substack{ab\le x,\\ a,b\ \text{squarefree}}} \left(\frac{a+b}{p}\right)\right| \le \frac{1}{\sqrt{p}\,(p+1)\,\zeta(2)^2}\, x\log x + O\left(x\right).$$

Proof. Recall that the Legendre symbol can be written using the Gauss sum as

$$\left(\frac{a}{p}\right) = \frac{1}{\tau}\sum_{c\ne 0}\left(\frac{c}{p}\right) e\left(\frac{ca}{p}\right),$$

where $e(t) = e^{2\pi i t}$ and

$$\tau = \sum_{b=1}^{p-1}\left(\frac{b}{p}\right) e\left(\frac{b}{p}\right)$$

is the Gauss sum. Hence, we have

$$\left(\frac{a+b}{p}\right) = \frac{1}{\tau}\sum_{c\ne 0}\left(\frac{c}{p}\right) e\left(\frac{c(a+b)}{p}\right).$$

Therefore,

$$\sum_{\substack{ab\le x,\\ a,b\text{ squarefree}}} \left(\frac{a+b}{p}\right) = \frac{1}{\tau}\sum_{c\ne 0}\left(\frac{c}{p}\right) \sum_{\substack{ab\le x,\\ a,b\text{ squarefree}}} e\left(\frac{c(a+b)}{p}\right),$$

and the innermost sum can be written as

$$\sum_{\substack{a\le x,\\ a,\text{ squarefree}}} e\left(\frac{ca}{p}\right) \sum_{\substack{b\le x/a,\\ b\text{ squarefree}}} e\left(\frac{cb}{p}\right). \tag{3.4}$$

This motivates us to consider

$$\sum_{\substack{b\le Y,\\ b\text{ squarefree}}} e\left(\frac{cb}{p}\right).$$

Again, using the Möbius function to sift out non-squarefree numbers, we have

$$\sum_{\substack{b\le Y,\\ b \text{ squarefree}}} e\left(\frac{cb}{p}\right) = \sum_{b\le Y} e\left(\frac{cb}{p}\right)\sum_{t^2|b}\mu(t) = \sum_{t\le\sqrt{Y}}\mu(t)\sum_{s\le Y/t^2} e\left(\frac{ct^2s}{p}\right).$$

If t is not divisible by p, the inner sum is bounded giving a final contribution of $O(\sqrt{Y})$ in this case. Inserting this into (3.4) gives an estimate of $O(x)$, where the constant depends on p. If t is divisible by p, the contribution is

$$\sum_{\substack{t\le\sqrt{Y}\\ p|t}}\mu(t)\left[\frac{Y}{t^2}\right].$$

Note that

$$\sum_{\substack{t=1\\ p|t}}^{\infty}\frac{\mu(t)}{t^2} = \frac{-1}{p^2}\prod_{\substack{l \text{ prime}\\ l\neq p}}\left(1-\frac{1}{l^2}\right)$$
$$= \frac{-1}{p^2}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{l \text{ prime}}\left(1-\frac{1}{l^2}\right) = \frac{-1}{(p^2-1)\zeta(2)}.$$

Thus, we have

$$\left|\sum_{\substack{t\le\sqrt{Y}\\ p|t}}\mu(t)\left[\frac{Y}{t^2}\right]\right| \le \frac{Y}{(p^2-1)\,\zeta(2)} + O\left(\sqrt{Y}\right).$$

Putting everything together along with the fact that $|\tau| = \sqrt{p}$, we get the lemma. □

4. Proof of the Main Theorem

When $p = 2$, note that $f_2(n) = \#\{1 \le a, b \le n : ab = n\} = d(n)$, the divisor function. It has already been established in Lemma 3.3 that

$$\sum_{\substack{n\le x\\ n \text{ squarefree}}} d(n) = Cx\log x + o(x\log x).$$

This proves the theorem for $p = 2$.

Henceforth, let $p \ge 3$ be a fixed prime and $f_p(n)$ be as above. Note that

$$2f_p(n) = \sum_{\substack{1\le a,b\le n\\ ab=n}}\left(\left(\frac{a+b}{p}\right)+1\right) - \sum_{\substack{1\le a,b\le n,\\ ab=n,\ p|a+b}} 1.$$

Let

$$S(x) = 2\sum_{\substack{n\le x\\ n \text{ squarefree}}} f_p(n).$$

Then

$$S(x) = \sum_{\substack{n\le x\\ n \text{ squarefree}}} \sum_{ab=n} \left(\frac{a+b}{p}\right) + \sum_{\substack{n\le x\\ n \text{ squarefree}}} \sum_{ab=n} 1 - \sum_{\substack{n\le x\\ n \text{ squarefree}}} \sum_{ab=n,\, p|a+b} 1.$$

Let us denote the three summations over $n \le x$ on the right-hand side as S_1, S_2 and S_3 respectively.

We first obtain an upper bound on S_3. Observe that in S_3, we have that a and b are coprime for otherwise, n would not be squarefree. Therefore,

$$\begin{aligned} S_3 = \sum_{\substack{n\le x\\ n \text{ squarefree}}} \sum_{ab=n,\, p|a+b} 1 &= \sum_{\substack{a\le x\\ a \text{ squarefree}}} \sum_{\substack{b\le \frac{x}{a},\, p|a+b\\ (a,b)=1\\ b \text{ squarefree}}} 1 \\ &\le \sum_{\substack{a\le x\\ a \text{ squarefree}}} \sum_{\substack{b\le \frac{x}{a},\, p|a+b\\ b \text{ squarefree}}} 1. \end{aligned}$$

The inner sum above is counting squarefree $b \le x/a$ which are congruent to $-a$ (mod p). Therefore, using (3.1) in Lemma 3.2 with $\epsilon = 1/4$, we get

$$\sum_{\substack{b\le \frac{x}{a},\, p|a+b\\ b \text{ squarefree}}} 1 = \frac{1}{\zeta(2)} \frac{p}{(p^2-1)} \frac{x}{a} + O\left(\left(\frac{x}{ap}\right)^{1/2} + p^{3/4}\right).$$

Inserting this estimate in the upper bound for S_3, together with (3.2) and (3.3), gives

$$\begin{aligned} S_3 &\le \frac{1}{\zeta(2)} \frac{p}{(p^2-1)} \sum_{\substack{a\le x\\ a \text{ squarefree}}} \frac{x}{a} + \sqrt{x}\, O\left(\sum_{\substack{a\le x\\ a \text{ squarefree}}} \frac{1}{\sqrt{a}}\right) + O(x) \\ &= \frac{1}{\zeta(2)^2} \frac{p}{(p^2-1)} x \log x + O(x), \end{aligned} \tag{4.1}$$

where the implied constant in the O-term depends on p.

We estimate S_2 using Lemma 3.3:

$$S_2 = Cx\log x + o(x \log x). \tag{4.2}$$

Finally, we estimate S_1 as follows. The condition that $n = ab$ is squarefree can be re-written using the Möbius function

$$\begin{aligned} S_1 = \sum_{\substack{n\le x\\ n \text{ squarefree}}} \sum_{ab=n} \left(\frac{a+b}{p}\right) &= \sum_{\substack{ab\le x\\ a,b \text{ squarefree}}} \left(\frac{a+b}{p}\right) \sum_{\substack{d|a\\ d|b}} \mu(d) \\ &= \sum_{d\le x} \mu(d) \left(\frac{d}{p}\right) \sum_{\substack{ab\le x/d^2\\ a,b \text{ squarefree}}} \left(\frac{a+b}{p}\right). \end{aligned}$$

By Lemma 3.5, we deduce

$$\begin{aligned} |S_1| &\leq \frac{1}{\sqrt{p}(p+1)\zeta(2)^2}\left(\sum_{d\leq x}\frac{x}{d^2}\log\left(\frac{x}{d^2}\right)\right)+O(x) \\ &= \frac{1}{\sqrt{p}\,(p+1)\,\zeta(2)}\,x\log x+O(x). \end{aligned} \tag{4.3}$$

Putting everything together, for a fixed prime p, we have

$$S(x)=S_1+S_2-S_3,$$

Now by (4.1)–(4.3), we get that

$$S(x)\geq\left[C-\frac{1}{\sqrt{p}(p+1)\zeta(2)}-\frac{p}{(p^2-1)\zeta(2)^2}\right]x\log x+o_p(x\log x).$$

Since the constant in brackets above is minimized when $p=3$,

$$C-\frac{1}{\sqrt{p}\,(p+1)\,\zeta(2)}-\frac{p}{(p^2-1)\,\zeta(2)^2}>C-\frac{1}{4\sqrt{3}\,\zeta(2)}-\frac{3}{8\,\zeta(2)^2}=0.10\ldots>0.$$

The above inequality implies that $S(x)\gg x\log x$, thus establishing that $f_p(n)$ is unbounded as n ranges over squarefree numbers.

5. Concluding Remarks

An examination of the algorithm described in Sec. 2 shows that we can consider the more general function $f(n;A)$ for squarefree n which counts the number of factorizations $ab=n$ such that $a+b+A$ is a perfect square. The function $f(n)$ corresponds to the case $A=0$. Our analysis can be extended to study $f_p(n;A)$ which counts the number of factorizations such that $a+b+A$ is a square mod p. This may be of some help in our search for elliptic curves of unbounded Mordell–Weil rank.

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