Math 401/801 – Solutions to Sample Midterm

1. The adjacency matrix of the graph is:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The eigenvalues are roots of the determinant:

$$\Delta = \begin{vmatrix} \lambda & -1 & -1 & -1 \\ -1 & \lambda & -1 & 0 \\ -1 & -1 & \lambda & -1 \\ -1 & 0 & -1 & \lambda \end{vmatrix}$$

Adding (−1)(second row) to the third & fourth rows:

$$\Delta = \begin{vmatrix} \lambda & -1 & -1 & -1 \\ -1 & \lambda & -1 & 0 \\ 0 & -(\lambda+1) & \lambda+1 & -1 \\ 0 & -\lambda & 0 & \lambda \end{vmatrix}$$

Multiplying λ(second row) to the first row:

$$\Delta = \begin{vmatrix} 0 & \lambda^2-1 & -(\lambda+1) & -1 \\ -1 & \lambda & -1 & 0 \\ 0 & -(\lambda+1) & (\lambda+1) & -1 \\ 0 & -\lambda & 0 & \lambda \end{vmatrix}$$

Now expand (Laplace expansion) on the 1^{st} column:

$$\Delta = \begin{vmatrix} \lambda^2-1 & -(\lambda+1) & -1 \\ -(\lambda+1) & (\lambda+1) & -1 \\ -\lambda & 0 & \lambda \end{vmatrix}$$

$$= \lambda \begin{vmatrix} \lambda^2-1 & -(\lambda+1) & -1 \\ -(\lambda+1) & (\lambda+1) & -1 \\ -1 & 0 & 1 \end{vmatrix}$$

using standard rules for determinants.

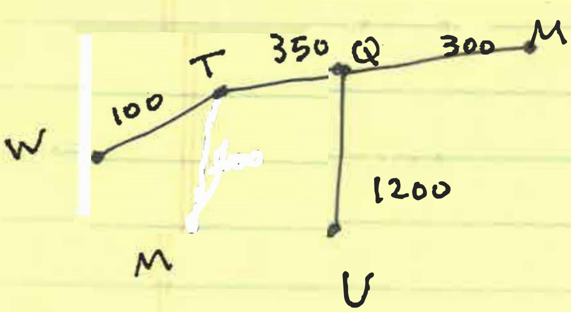
$$= \lambda \left\{ (\lambda^2-1)(\lambda+1) - (\lambda+1) - \left[(\lambda+1) + (\lambda+1)^2 \right] \right\}$$

$$= \lambda(\lambda+1)\left[\lambda^2 - \lambda - 4 \right]$$

So the eigenvalues of A are $0, -1, \dfrac{1 \pm \sqrt{17}}{2}$.

2. The number of closed walks of length n $= \operatorname{tr} A^n$ which by the spectral theorem is the sum of the n-th powers of the eigenvalues, as stated.

3. The greedy algorithm generates the spanning tree:



So the minimal cost is

$100 + 300 + 350 \quad + 1200 = 1950$