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SELBERG'S CONJECTURES AND ARTIN L-FUNCTIONS, II

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Dedicated to the memory of Harish-Chandra

1. Introduction.

The construction of an automorphic L-function embodies the synthesis of a galaxy of ideas originating from the work of Riemann, Dirichlet, Ramanujan, Hecke, Siegel, Weil, Maass, Selberg and Harish-Chandra. The concept of an automorphic form for a semi-simple Lie group over $\mathbb{R}$ was first defined and studied in a fundamental paper of Harish-Chandra. (See for example, [M1].) Meanwhile, Selberg dicovered the trace formula and applied it to obtain the analytic continuation of Eisenstein series. In a unified vision, Langlands [L] generalized the themes of both Selberg and Harish-Chandra by describing the general construction of an automorphic L-function attached to any cuspidal automorphic representation of a reductive algebraic group. These L-functions have (or are expected to have) analytic continuation, a functional equation and can be written as an Euler product (in the half plane $\Re(s) > 1$). One expects that any function having similar properties must be an automorphic L-function. A special case of such an expectation is the celebrated Shimura-Taniyama conjecture.

In a previous paper [RM], we described conjectures of Selberg [S] that isolate and study Dirichlet series, which admit analytic continuations, Euler products and functional equations. We observed that these conjectures imply the Artin conjecture concerning analytic continuation of non-abelian L-series. In this paper, we prove that these conjectures are incompatible with certain conjectures of van der Waall [vW] and Vinogradov [Vi]. We then proceed to disprove van der Waall's and Vinogradov's conjectures without assuming Selberg's conjectures.

More precisely, let a be squarefree, q be a prime and ζ_q denote a primitive q-th root of unity. Set $K_q = \mathbb{Q}(\zeta_q, a^{1/q})$. Then $K_q/\mathbb{Q}$ is Galois with a metacyclic Galois group of order $q(q-1)$. This group has q irreducible characters $1 = \chi_1, \chi_2, ..., \chi_{q-1}, \chi_q$, where χ_q is a non-abelian character of degree $q-1$ and $\chi_1, ..., \chi_{q-1}$ are abelian characters corresponding to the cyclotomic extension $\mathbb{Q}(\zeta_q)/\mathbb{Q}$. In a conjectural approach to solve Artin's primitive root conjecture, Vinogradov [Vi] suggested that one should look for a factorization of

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$L(s,\chi_q)$ into L-functions of conductor of size approximately $O(q)$. In trying to determine what these factors may be, van der Waall [vW] was led to:

Conjecture:

$$L(s,\chi_q)/L(s,\chi_2)...L(s,\chi_{q-1})$$

is entire.

Using existing tables of computed zeroes of L-functions on the critical line (see [LO] and [Sp]), it is not hard to construct specific counterexamples. In fact, since χ_q is monomial, $L(s,\chi_q)$ is entire by Artin's reciprocity law. We will show that van der Waall's conjecture is false for every q. If $q=3$, we show in Theorem 3.1 and Corollary 3.3 that $L(s,\chi_q)$ is primitive so that Vinogradov's expected factorization does not exist.

In the last section, we discuss how to generalize the conjectures of Selberg into the context of a number field. The generalization is by no means obvious and a natural translation of them turns out to be false.

Most likely, the Selberg class of Dirichlet series is identical with the class of automorphic L-functions. Indeed, all known examples of the Selberg class are automorphic.

2. Preliminary results.

We begin by recalling the Selberg conjectures. The Selberg class $\mathcal{S}$ consists of functions $F(s)$ of a complex variable s satisfying the following properties:

(i) (Dirichlet series) for $\Re(s)>1$,

$$F(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s}$$

where $a_1=1$. (We will write $a_n(F)=a_n$ for the coefficients of the Dirichlet series of F.);

(ii) (Analytic continuation) for some integer $m\geq 0$, $(s-1)^m F(s)$ extends to an entire function of finite order;

(iii) (Functional equation) there are numbers $Q>0$, $\alpha_i>0$ and $r_i\in\mathbf{C}$ with $\Re(r_i)\geq 0$ so that

$$\Phi(s)=Q^s\prod_{i=1}^{d}\Gamma(\alpha_i s+r_i)F(s)$$

satisfies

$$\Phi(s)=w\overline{\Phi}(1-s)$$

where w is a complex number with $|w| = 1$ and $\overline{\Phi}(s) = \overline{\Phi(\overline{s})}$;

(iv) (Euler product)

$$F(s) = \prod_p F_p(s)$$

where

$$F_p(s) = \exp\left(\sum_{k=1}^{\infty} \frac{b_{p^k}}{p^{ks}}\right)$$

where $b_{p^k} = O(p^{k\theta})$ for some $\theta < 1/2$ and p denotes a prime number here and throughout the paper. (We shall write $b_p(F) = b_p$.);

(v) (Ramanujan hypothesis) For any fixed $\epsilon > 0$, $a_n = O(n^\epsilon)$.

A function $F \in \mathcal{S}$ is called **primitive** if it cannot be written as a non-trivial product of two elements in $\mathcal{S}$. It can be shown that every element in $\mathcal{S}$ can be written as a product of primitive elements. (See [RM].) Selberg [S, (1.12) and (1.13)] conjectures:

(a) for any primitive function F,

$$\sum_{p \le x} \frac{|a_p(F)|^2}{p} = \log\log x + O(1);$$

(b) for two distinct primitive functions F and G,

$$\sum_{p \le x} \frac{a_p(F)\overline{a_p(G)}}{p} = O(1).$$

We define the dimension of F as

$$\dim F = 2\alpha_F$$

where

$$\alpha_F = \sum_{i=1}^{d} \alpha_i.$$

As was shown in our previous paper [RM], this concept is well-defined. Selberg conjectures that dim F is always a non-negative integer.

Here are the fundamental results of the theory.

Proposition 2.1

(i) if $\dim F = 0$, then $F = 1$.

(ii) if $F \neq 1$, then $\dim F \geq 1$.

Proof. For (ii), see Bochner [B]. For (i), see Conrey-Ghosh [CG].

Proposition 2.2 Every $F \in \mathcal{S}$ has a factorization into primitive elements.

Proof. This follows easily by induction and proposition 2.1.

Proposition 2.3 Conjectures (a) and (b) imply factorization into primitive elements is unique.

Proof. This is straightforward (see [RM]).

Proposition 2.4 If F has dimension 1 and all the α_i are rational numbers, then $d = 1$ and $\alpha_1 = 1/2$. The only elements of $\mathcal{S}$ with $\alpha_1 = 1/2$ and $d = 1$ are $\zeta(s)$ and $L(s + it, \chi)$ for some Dirichlet character χ mod N and a real number t.

Remark. This is the quintessence of a sequence of results. First, Bochner [B] studied series with analytic continuation and a given functional equation. Vignéras [V] reconsidered Bochner's work and formulated his results in terms of the classical Dirichlet L-functions. However, corollary 1 of [V] contained a gap which was fixed by Gérardin-Li [GL]. Conrey-Ghosh [CG] give an independent proof of this result.

Proposition 2.5 Assume Selberg's conjectures. If K/k is a finite Galois extension of algebraic number fields, and χ is an irreducible character $\neq 1$ of $\mathrm{Gal}(K/k)$, then $L(s, \chi, K/k)$ is entire and hence in $\mathcal{S}$. Moreover, if $k = \mathbb{Q}$, then $L(s, \chi, K/\mathbb{Q})$ is primitive.

Proof. This was proved in our previous paper [RM].

§3. Conjectures of van der Waall and Vinogradov.

As described in section 1, let a be squarefree and q be prime. If ζ_q denotes a primitive q-th root of unity and K_q is the field $\mathbb{Q}(\zeta_q, a^{1/q})$, then $K_q/\mathbb{Q}$ is Galois with a group G_q that is naturally isomorphic to the group of matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with $a \in \mathbb{F}_q^\times$, $b \in \mathbb{F}_q$. The irreducible characters of G_q are q in number: $1 = \chi_1, ..., \chi_q$ with the first $q-1$ being abelian and χ_q being of degree $q-1$. Motivated by earlier work of A. I. Vinogradov [Vi], van der Waall [vW] conjectured that

$$\Phi(s) = L(s, \chi_q)/L(s, \chi_2)...L(s, \chi_{q-1})$$

is entire.

In the simplest case of $q = 3$ and $a = 2$, it is possible to disprove this conjecture merely by examining existing computations of zeroes of Artin L-functions of Lagarias and Odlyzko [LO] and Dirichlet L-functions of Spira [Sp]. It is easily checked that the zeroes of $L(s, \chi_2)$ are not among the zeroes of $L(s, \chi_3)$ for $|\operatorname{Im}(s)| \leq 15$ as tabulated in [Sp].

However, this does not explain why the conjecture is false. Since χ_q is monomial, $L(s, \chi_q)$ is entire and so belongs to $\mathcal{S}$. By proposition 2.5, $L(s, \chi_q)$ is conjecturally primitive and so should not have any factorization whatsoever.

Nevertheless, the conjecture is not so outlandish from a group-theoretic point of view. There is a theorem of Yoshida [Y] which states that given any group G and irreducible characters ϕ_1 and ϕ_2 such that $\phi_1(1) > \phi_2(1)$, one can construct a group $\mathcal{G} \supseteq G$ such that

$$\operatorname{Ind}_G^{\mathcal{G}} (\phi_1 - \phi_2) = \sum_i m_i \operatorname{Ind}_{H_i}^{\mathcal{G}} \psi_i$$

with **positive** integers m_i, and ψ_i abelian characters of $\mathcal{G}$. Thus, if K/k is an extension with group $\mathcal{G}$, then $\operatorname{Gal}(K/K^G) \simeq G$. The corresponding Artin L-function

$$L(s, \phi_1 - \phi_2, K/K^G) = \prod_i L(s, \psi_i, K/K^{H_i})^{m_i}$$

is entire and therefore

$$L(s, \phi_1, K/K^G)/L(s, \phi_2, K/K^G)$$

is entire. When applied to our context above, all we can conclude from this is that there is a Galois extension K/k with group isomorphic to G_q such that

$$L(s, \chi_q, K/k)/L(s, \chi_2, K/k)...L(s, \chi_{q-1}, K/k)$$

is entire. However, things over $\mathbb{Q}$ seem not to be so simple.

Theorem 3.1 *For $q \geq 3$, $\Phi(s)$ is not entire.*

Proof. Recall that the Artin L-function satisfies a functional equation given as follows. Put

$$\gamma_\chi(s) = [\pi^{-(s+1)/2}\Gamma(\frac{s+1}{2})]^d[\pi^{-s/2}\Gamma(\frac{s}{2})]^c$$

where

$$c = \frac{1}{2}(\chi(1) + \chi(g_0)) \qquad d = \frac{1}{2}(\chi(1) - \chi(g_0))$$

where g_0 is an element of $\mathrm{Gal}\,(E/\mathbb{Q})$ that corresponds to complex conjugation. Let

$$\xi(s, \chi) = [s(s-1)]^{\delta(\chi)} A(\chi)^{s/2}\gamma_\chi(s)L(s, \chi)$$

with $\delta(\chi) = 1$ if $\chi = \chi_1$ and zero otherwise. Then,

$$\xi(1-s, \overline{\chi}) = w(\chi)\xi(s, \chi),$$

with $|w(\chi)| = 1$. In our situation, $c = (q-1)/2$ and $d = (q-1)/2$ and thus

$$\Lambda(s, \chi_q) = A(\chi_q)^{s/2}[\pi^{-(s+1)/2}\Gamma(\frac{s+1}{2})]^{(q-1)/2}[\pi^{-s/2}\Gamma(s/2)]^{(q-1)/2}L(s, \chi_q)$$

satisfies a functional equation relating s with $1-s$. (Note that χ_q is a real character.) Thus, $\Phi(s)$ satisfies the functional equation

$$A^s[\pi^{-s/2}\Gamma(s/2)]\Phi(s) = A^{1-s}[\pi^{-(1-s)/2}\Gamma(\frac{1-s}{2})]\Phi(1-s).$$

Hence, if $\Phi(s)$ were entire, then it would belong to $\mathcal{S}$ because it can be written as a Dirichlet series with analytic continuation and Euler product. By proposition 2.4, $\Phi(s)$ must be a Dirichlet series $L(s + i\alpha, \chi)$ for some primitive Dirichlet character χ mod N. Since the coefficents of $\Phi(s)$ are real, $\alpha = 0$. Thus $\Phi(s)$ is a classical Dirichlet series for some primitive Dirichlet character χ mod N. A simple calculation shows that

$$\chi_q(p) = \begin{cases} q - 2 + \chi(p) & \text{if } p \equiv 1(\bmod q) \\ -1 + \chi(p) & \text{if } p \not\equiv 1(\bmod q). \end{cases}$$

On the other hand, the conjugacy classes of $\mathrm{Gal}\,(K_q/\mathbb{Q})$ are represented by

$$1 = \{1\},\ C_a = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : b(\bmod q)\},\ 2 \le a \le q-1, \text{ and } D = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : 0 \ne b(\bmod q)\}.$$

The character values are

$$\chi_q(1) = q-1, \quad \chi_q(C_a) = 0, \quad \chi_q(D) = -1.$$

In particular $\chi_q(p) = 0$ for $p \not\equiv 1(\mod q)$, so that $\chi(p) = 1$ for all $p \not\equiv 1(\mod q)$. If $p \equiv 1(\mod q)$ and p splits in K_q, then $\chi(p) = 1$ in this case also. If $p \equiv 1(\mod q)$ and p does not split in K_q, then $\chi_q(p) = -1$ and so $\chi(p) = 1 - q$. Since χ is a Dirichlet character, it has absolute value at most 1. Thus, $\chi(p) = 1 - q$ is impossible if $q \geq 3$. This contradiction implies that $\Phi(s)$ is not entire.

In fact, it is possible to prove more. In the special case $q = 3$, one can show that $L(s, \chi_q)$ is primitive. More generally, let f be a normalized Hecke eigenform. Then, its associated L-series $L(s, f)$ is primitive. We will need:

Theorem 3.2 *Let π be an irreducible cuspidal automorphic representation of $GL_2(\mathbb{A}_{\mathbb{Q}})$. Let $L(s,\pi)$ be its associated L-series. Suppose that the Ramanujan conjecture is true for π. Then, $L(s,\pi) \in \mathcal{S}$ and is primitive.*

Remark. If π corresponds to a holomorphic modular form, then the Ramanujan-Petersson conjecture is known to be true.

Proof. The fact that $L(s,\pi)$ belongs to $\mathcal{S}$ is in fact classical (emanating from the work of Hecke in the holomorphic case and Maass in the non-holomorphic case) and is subsumed in the work of Jacquet and Langlands [JL]. Moreover, the dimension of $L(s,\pi)$ is two so that if it is not primitive, it must factor as a product of two elements of dimension 1. First suppose that π corresponds to a holomorphic modular form. The gamma factor appearing in the functional equation tells us that $L(s,\pi)$ has trivial zeroes at $s = -n-(k-1)/2$ where k is the weight of f and n ranges over all the non-negative integers. If F_1 is any primitive factor and $\Gamma(\alpha_1 s + r_1)$ is a typical gamma factor appearing in the functional equation of F_1, then F_1 has trivial zeroes at $s = (-n - r_1)/\alpha_1$ as n ranges over non-negative integers. Since the latter set of zeroes must be a subset of the set $s = -n - (k-1)/2$ for varying n, we deduce that α_1 and r_1 must be rational numbers. Hence by proposition 2.4, we deduce that $\alpha_1 = 1/2$ and, all the factors (if any) must be classical Dirichlet L-functions. In the non-holomorphic case, we observe that the difference of any two trivial zeroes of $L(s,\pi)$ is an integer which again implies by a similar argument that α_1 must be rational. In either case, we are forced to the conclusion that

$$L(s,\pi) = L(s,\chi_1)L(s,\chi_2)$$

for two Dirichlet characters. However, the Rankin-Selberg L-function of the left hand side, namely $L(s,\pi\times\tilde{\pi})$ has a simple pole at $s=1$, whereas the right hand side has a pole of order at least 2. This contradiction proves the theorem.

Corollary 3.3 If $q=3$, $L(s,\chi_q)$ is primitive.

Proof. $L(s,\chi_q)$ is a Hecke L-function belonging to the cubic extension $\mathbb{Q}(a^{1/3})$. Hence, it is entire. Moreover, for $q=3$, χ_q is a faithful irreducible two-dimensional representation which is not of icosahedral type and so by Langlands [L2], $L(s,\chi_3)$ is the Mellin transform of a holomorphic modular form. The result now follows from the theorem.

4. Selberg conjectures over a number field. Let k be a number field. The Dedekind zeta function of k is the simplest example of a function in the family $\mathcal{S}_k$ of functions $F(s)$ of a complex variable s satisfying the following properties:

(i) (Dirichlet series) For $\mathrm{Re}\,(s)>1$,

$$F(s)=\sum_{\mathfrak{n}}\frac{a_{\mathfrak{n}}}{\mathbb{N}(\mathfrak{n})^s},$$

where the sum is over integral ideals $\mathfrak{n}$ of k, $a_1=1$ and we will write $a_{\mathfrak{n}}(F)=a_{\mathfrak{n}}$ for the coefficients of the Dirichlet series;

(ii) (Analytic continuation) $F(s)$ extends to a meromorphic function so that for some integer $m\geq 0$, $(s-1)^mF(s)$ is an entire function of finite order;

(iii) (Functional equation) There are numbers $Q>0$, $\alpha_i>0$, $\mathrm{Re}\,(r_i)\geq 0$, so that

$$\Phi(s)=Q^s\prod_{i=1}^{d}\Gamma(\alpha_i s+r_i)F(s)$$

satisfies

$$\Phi(s)=w\overline{\Phi(1-\bar{s})}$$

for some complex number w with $|w|=1$;

(iv) (Euler product)

$$F(s)=\prod_{\mathfrak{p}}F_{\mathfrak{p}}(s)$$

where

$$F_p(s)=\exp\left(\sum_{k=1}^{\infty}\frac{b_{\mathfrak{p}^k}}{\mathfrak{p}^{ks}}\right)$$

where $b_{\mathfrak{p}^k}=O(\mathbb{N}(\mathfrak{p})^{k\theta})$ for some $\theta<1/2$, where $\mathfrak{p}$ denotes a prime ideal;

(v) (Ramanujan hypothesis) $a_{\mathfrak{n}} = O(\mathrm{N}(\mathfrak{n})^{\epsilon})$ for any fixed $\epsilon > 0$.

As a class of functions of a complex variable, $\mathcal{S}_k$ is contained in $\mathcal{S}_{\mathbb{Q}}$. However, it should be stressed that the map $\mathfrak{n} \rightarrow a_{\mathfrak{n}}$ determines more than a Dirichlet series. This is an important distinction.

However, the natural generalization of the Selberg conjectures fails to be true. For instance, one would expect

$$\sum_{\mathrm{N}(\mathfrak{p}) \leq x} \frac{|a_{\mathfrak{p}}(F)|^2}{\mathrm{N}(\mathfrak{p})} = n \log\log x + O(1),$$

for some integer n. This need not be the case.

Consider a Galois extension $K/\mathbb{Q}$ with

$$G = \mathrm{Gal}\,(K/\mathbb{Q}) \simeq SL_2(\mathbb{F}_3).$$

This can be realized, for instance, (see [HK]) as the splitting field of

$$x^8 + 9x^6 + 23x^4 + 14x^2 + 1.$$

The group has three abelian characters, one irreducible character of degree 3, denoted χ_3, and three irreducible characters of degree 2, denoted $\chi_2^{(1)}, \chi_2^{(2)}, \chi_2^{(3)}$. There is a subgroup C_2 of order 2 and a non-trivial character α of C_2 such that

$$\mathrm{Ind}\,_{C_2}^{G} \alpha = 2(\chi_2^{(1)} + \chi_2^{(2)} + \chi_2^{(3)}).$$

There is also a cyclic subgroup C_4 of order 4 and a non-trivial character γ^2 of C_4 such that

$$\mathrm{Ind}\,_{C_4}^{G} \gamma^2 = 2\chi_3,$$

(see [Mu2, p. 63]). Let F_1 and F_2 be the fixed fields of C_2 and C_4 respectively. Then,

$$L(s, \alpha, K/F_1) = \prod_{i=1}^{3} L(s, \chi_2^{(i)}, K/\mathbb{Q})^2$$

and

$$L(s, \gamma^2, K/F_2) = L(s, \chi_3, K/\mathbb{Q})^2.$$

It is easy to see that if F is either

$$\prod_{i=1}^{3} L(s, \chi_2^{(i)}, K/\mathbb{Q})$$

or

$$L(s, \chi_3, K/\mathbb{Q})$$

then it is in $\mathcal{S}_{F_1}$ or $\mathcal{S}_{F_2}$ respectively.

However, in either case we have

$$\sum_{\mathbb{N}(\mathfrak{p}) \le x} \frac{|a_{\mathfrak{p}}(F)|^2}{\mathbb{N}(\mathfrak{p})} = \frac{1}{4} \log \log x + O(1)$$

as is easily seen by the Chebotarev density theorem applied to the extensions K/F_1 and K/F_2 respectively.

We therefore need to impose some additional condition to ensure a satisfactory theory. We propose

(vi) (Rankin-Selberg property)

$$\sum_{\mathbb{N}(\mathfrak{p}) \le x} \frac{|a_{\mathfrak{p}}(F)|^2}{\mathbb{N}(\mathfrak{p})} = n \log \log x + O(1)$$

for some integer n.

The class of functions satisfying (i) to (vi) we will call the Selberg class $\tilde{S}_k$ over k.

We can then introduce the notion of k-primitivity: $F \in \tilde{S}_k$ is called k-primitive if $F = F_1 F_2$ with $F_1, F_2 \in \tilde{S}_k$ implies $F = F_1$ or $F = F_2$.

Again by Bochner's theorem we have

Proposition 4.1 Every $f \in \tilde{S}_k$ has a factorization into k-primitive functions.

It is now reasonable to consider

Conjecture:

(a_k) if F_1, F_2 are distinct primitive elements of $\tilde{S}_k$, then

$$\sum_{\mathbb{N}(\mathfrak{p}) \le x} \frac{a_{\mathfrak{p}}(F_1)\overline{a_{\mathfrak{p}}(F_2)}}{\mathbb{N}(\mathfrak{p})} = O(1).$$

(b_k) if F is k-primitive, then

$$\sum_{\mathbb{N}(\mathfrak{p})\le x} \frac{|a_\mathfrak{p}(F)|^2}{\mathbb{N}(\mathfrak{p})} = \log\log x + O(1)$$

as $x \to \infty$.

Assuming this conjecture, we can show every element of $\tilde{S}_k$ has unique factorization into k-primitive elements. Moreover, if K/k is a finite Galois extension of algebraic number fields, then the Artin L-function

$$L(s, \chi, K/k)$$

attached to an irreducible character of Gal (K/k) is k-primitive. As in [RM], we find that if K/k is solvable, the above conjecture implies the Langlands reciprocity conjecture. Namely, there exists an automorphic representation π of $GL_n(\mathbb{A}_k)$ where $n = \chi(1)$, such that

$$L(s, \pi) = L(s, \chi, K/k).$$

Although conjectures a_k and b_k imply $L(s, \chi, K/k)$ is k-primitive when χ is irreducible, one can in fact show:

Proposition 4.2 *Let $k/\mathbb{Q}$ be Galois and assume Selberg's conjectures over $\mathbb{Q}$. Then, $L(s, \chi, K/k)$ is k-primitive whenever χ is irreducible.*

Proof.

Let $\tilde{K}$ be the normal closure of K over $\mathbb{Q}$. Then, $\tilde{K}/k$ is Galois, as well as $\tilde{K}/\mathbb{Q}$, and χ can be thought of as a character $\tilde{\chi}$ of Gal $(\tilde{K}/k)$. By the property of Artin L-functions,

$$L(s, \tilde{\chi}, \tilde{K}/k) = L(s, \chi, K/k).$$

Moreover, if Ind $\tilde{\chi}$ denotes the induction of $\tilde{\chi}$ from Gal $(\tilde{K}/k)$ to $G =$ Gal $(\tilde{K}/\mathbb{Q})$, then

$$L(s, \tilde{\chi}, \tilde{K}/k) = L(s, \operatorname{Ind}\tilde{\chi}, \tilde{K}/\mathbb{Q}),$$

by the invariance of Artin L-functions under induction. Hence, we can write

$$L(s, \chi, K/k) = \prod_{\phi} L(s, \phi, \tilde{K}/\mathbb{Q})^{m(\phi)}$$

where the product is over irreducible characters ϕ of $\mathrm{Gal}(\tilde{K}/\mathbb{Q})$ and $m(\phi)$ are positive integers. Now suppose

$$L(s,\chi,K/k)=A(s)B(s)$$

with $A,B\in\mathcal{S}_k$. By unique factorization in $\mathcal{S}_{\mathbb{Q}}$, we must have

$$A(s)=\prod_{\phi}L(s,\phi,\tilde{K}/\mathbb{Q})^{n(\phi)},$$

for some positive integers $n(\phi)$. But then,

$$\psi=\sum_{\phi}n(\phi)\phi$$

vanishes unless the Artin symbol of p, $\sigma_p\in H=\mathrm{Gal}(\tilde{K}/k)$. Thus,

$$\psi=\frac{1}{[G:H]}\left(\mathrm{Ind}_H^G 1\right)\sum_{\phi}n(\phi)\phi=\frac{1}{[G:H]}\sum_{\phi}n(\phi)\,\mathrm{Ind}_H^G\phi_H.$$

Hence,

$$A(s)=L(s,\psi)=\prod_{\phi}L(s,\phi_H,\tilde{K}/k)^{n(\phi)/[G:H]}.$$

Since χ is irreducible and H is normal, ϕ_H is $m(\phi)$ times the sum of the conjugates of χ (see [Is, p. 84]). Since, H is normal, the L-functions of conjugate characters are equal so that,

$$L(s,\phi_H,\tilde{K}/k)=L(s,\tilde{\chi},\tilde{K}/k)^{m(\phi)v}.$$

Hence,

$$A(s)=L(s,\tilde{\chi},\tilde{K}/k)^{\frac{v}{[G:H]}\sum_{\phi}m(\phi)n(\phi)}.$$

Similarly,

$$B(s)=L(s,\tilde{\chi},\tilde{K}/k)^{\frac{v}{[G:H]}\sum_{\phi}m(\phi)(m(\phi)-n(\phi))}$$

so that

$$1=\frac{v}{[G:H]}\sum_{\phi}m(\phi)^2.$$

Since

$$\sum_{\phi}m(\phi)^2=[G:H]$$

(see [Is, p. 84]), we must have $v = 1$. Thus, $\tilde{\chi}$ is an invariant character. But

$$\sum_{\phi} m(\phi)n(\phi) \leq [G : H]$$

and so by (vi), we are forced to have $n(\phi) = m(\phi)$ for all ϕ. Therefore,

$$A(s) = L(s, \tilde{\chi}, \tilde{K}/k)$$

is k-primitive.

Remark. It will be noted that axiom (vi) was used only at the last step in the proof. If $\chi = 1$ then $L(s, \tilde{\chi}, \tilde{K}/k)$ is just $\zeta_k(s)$. In this case, without invoking (vi), $A(s)$ would have a pole at $s = 1$ of order less than 1, which contradicts axiom (ii). Thus, $\zeta_k(s)$ cannot be written as a product of two elements of $\mathcal{S}_k$. If $\tilde{\chi}$ is not invariant, then a similar remark applies. There is one more case in which a similar result holds. If $\chi \neq 1$ and $L(s, \chi, K/k)$ has at least one simple zero, the same argument shows that it cannot be factored in $\mathcal{S}_k$.

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