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General Section

On the moments of averages of Ramanujan sums

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A B S T R A C T

Chan and Kumchev studied averages of the first and second moments of Ramanujan sums. In this article, we extend this investigation by estimating the higher moments of averages of Ramanujan sums using a Tauberian theorem due to La Bretèche. We also give a result for the moments of averages of Cohen-Ramanujan sums.

1. Introduction and main results

In 1918, Ramanujan [23] studied the function $c_q(n)$ defined for positive integers q and n as follows:

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$$c_q(n) := \sum_{\substack{1\le j\le n\\(j,q)=1}} e\left(\frac{nj}{q}\right) = \sum_{\substack{d|n\\d|q}} d\mu\left(\frac{q}{d}\right), \qquad e(t) := e^{2\pi it}. \tag{1.1}$$

These sums are now called Ramanujan sums. The equality of the two sums in (1.1) is a simple consequence of the Möbius inversion formula. Perhaps inspired by the theory of Fourier expansions of continuous functions, Ramanujan first encountered these sums while exploring trigonometric series representations of normalized arithmetic functions of the form $\sum_q a_q c_q(n)$, now known as Ramanujan expansions. Subsequently, in 1932, Carmichael [4] established that these sums also possess an orthogonality property. Ramanujan and Carmichael's work set the stage for a general theory of Ramanujan sums and Ramanujan expansions. Ramanujan sums exhibit deep connections in number theory and arithmetic, including their role in proving Vinogradov's theorem on the ternary Goldbach problem [20, Chapter 8], Waring-type formulas [15], the distribution of rational numbers in short intervals [14], equipartition modulo odd integers [3], the large sieve inequality [24], and various other branches of mathematics. For more recent developments in the direction of Ramanujan expansions, we refer to [11,13,18,27,28,30,31].

Understanding these sums and their distribution is an essential and interesting topic. Alkan [1,2] studied the weighted average of Ramanujan sums. The question on the average order over both variables n and q of $c_q(n)$ was first considered by Chan and Kumchev [6] motivated by applications to problems on Diophantine approximations of reals by sums of rational numbers. In [6], using both elementary and analytic techniques, they found asymptotic formulas for

$$S_k(x,y) := \sum_{n\le y}\left(\sum_{q\le x} c_q(n)\right)^k \tag{1.2}$$

for $k = 1, 2$. Robles and Roy adapted their methodology to compute the averages of generalized Ramanujan sums introduced by Cohen. It is worth noting, however, that in their study presented in [25], Robles and Roy claimed a result for higher moments, which has since been determined to be incorrect. To be precise, their Proposition 1.1 implies for $k > 1$, (and $\beta = 1$ in their notation) that

$$S_k(x,y) = \frac{3yx^2}{\pi^2} + O(yx\log x + x^{2k}\log^k x), \tag{1.3}$$

for $y > x^{2k}\log^{k+1} x$. This is correct for $k = 2$, but for $k = 4$, the theorem contradicts itself as can be seen by a simple application of the Cauchy-Schwarz inequality:

$$S_2(x,y) \le y^{1/2} S_4(x,y)^{1/2}.$$

This raises the question of what exactly is the behavior of (1.2) for $k \ge 3$. This problem prompts an exploration of the theory of the arithmetical functions of several variables, a study initiated by Vaidyanathaswamy [29] in 1931. This theory is still in evolution and several recent papers [5,10,26] highlight the importance of developing such a theory. In this paper, we derive the asymptotic behavior of the moments of Ramanujan sums (1.2) for $k \geq 3$. This result is an important step in developing the theory of the arithmetical functions of several variables. More precisely, we prove:

Theorem 1.1. *For $k \geq 3$ and $y > x^k$, as $x \to \infty$, we have*

$$S_k(x, y) = yx^k Q(\log x) + \mathrm{O}\left(yx^{k-\theta}\right),$$

where $Q \in \mathbb{R}[X]$ is a polynomial of exact degree $2^k - 2k - 1$ and $\theta > 0$.

Thus, (1.3) is egregiously false in two ways: first in the dominant power of x and second, in the powers of the logarithm that must be added to the main term.

Our main tool is a Tauberian theorem due to La Bretèche [10] for non-negative arithmetical functions of several variables, which we review in the next section.

Analogous questions of moments of these sums over number fields have been studied by numerous mathematicians [7,8,12,21,22,19,32] for the cases $k = 1, 2$. Our methods should extend to handle the number field cases also for $k \geq 3$ and we relegate this to future work.

2. La Bretèche Tauberian theorem

The study of (1.2) inevitably leads one into the theory of arithmetical functions of several variables. In the one variable case, the classical Tauberian theorems provide us with asymptotic behaviors of the summatory function of the non-negative arithmetical function of a single variable by relating it with the analytic properties of the associated Dirichlet series. In the multivariable case, a similar theorem exists but it does not seem to be well-known. The extension of Cauchy's residue theorem for functions of several variables seems to have been first addressed by Leray [16] in 1959 using the language of sheaf theory. Later, in the 1980's, Cassou-Noguès [5] and Sargos [26] derived more precise results that could be applied to counting problems involving arithmetical functions of several variables. We should also mention the work of Lichtin [17] in this regard. In the early part of the 21st century, de la Bretèche [10] derived a multi-variable version of the Tauberian theorem using classical methods of analytic number theory and it is this version that we apply to our situation. His theorems in this context are as follows.

Theorem 2.1. *Let $f : \mathbb{N}^k \to \mathbb{R}$ be a non-negative function and F the associated Dirichlet series of f defined by*

$$F(\boldsymbol{s}) = F(s_1, \ldots, s_k) = \sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1, \ldots, n_k)}{n_1^{s_1} \cdots n_k^{s_k}}.$$

Denote by $\mathcal{LR}_k^+(\mathbb{C})$ the set of non-negative $\mathbb{C}$ linear forms from $\mathbb{C}^k$ to $\mathbb{C}$ on $\mathbb{R}_+^k$. Moreover, assume that there exists $(c_1, \ldots, c_k) \in \mathbb{R}_+^k$ such that:

(1) *For $\boldsymbol{s} \in \mathbb{C}^k$, $F(s_1, \ldots, s_k)$ is absolutely convergent for $\mathrm{Re}(s_i) > c_i$ for all $1 \leq i \leq k$.*
(2) *There exist a finite family $\mathcal{L} = (l^{(i)})_{1 \leq i \leq q}$ of non-zero elements of $\mathcal{LR}_k^+(\mathbb{C})$, a finite family $(h^{(i)})_{1 \leq i \leq q'}$ of elements of $\mathcal{LR}_k^+(\mathbb{C})$ and $\delta_1, \delta_2, \delta_3 > 0$ such that the function H defined by*

$$H(\boldsymbol{s}) = F(\boldsymbol{s} + \boldsymbol{c}) \prod_{i=1}^{q} l^{(i)}(\boldsymbol{s})$$

has a holomorphic continuation to the domain

$$\begin{aligned} &D(\delta_1, \delta_3) \\ &= \left\{ \boldsymbol{s} \in \mathbb{C}^k : \mathrm{Re}\left\{ l^{(i)}(\boldsymbol{s}) \right\} > -\delta_1 \text{ for all } i = 1, \ldots, q \text{ and } \mathrm{Re}\left\{ h^{(i)}(\boldsymbol{s}) \right\} > -\delta_3 \right. \\ &\qquad \left. \text{for all } i = 1, \ldots, q' \right\}, \end{aligned}$$

and verifies the estimate: for $\epsilon, \epsilon' > 0$ we have uniformly in $\boldsymbol{s} \in D(\delta_1 - \epsilon, \delta_3 - \epsilon')$

$$H(\boldsymbol{s}) \ll \prod_{i=1}^{q} \left(\left| \mathrm{Im}\left\{ l^{(i)}(\boldsymbol{s}) \right\} \right| + 1 \right)^{1 - \delta_2 \min\left(0, \mathrm{Re}\left\{ l^{(i)}(\boldsymbol{s}) \right\}\right)} (1 + (\mathrm{Im}\{s_1\} + \cdots + \mathrm{Im}\{s_k\})^{\epsilon}).$$

Set $J = J(\mathbb{C}) = \{ j \in \{1, \ldots, k\} : c_j = 0 \}$. Denote w to be the cardinality of J and by $j_1 < \ldots < j_w$ its elements in increasing order. Define the w linear forms $l^{(q+i)}$ $(1 \leq i \leq w)$ by $l^{(q+i)}(\boldsymbol{s}) = e_{j_i}^*(\boldsymbol{s}) = s_{j_i}$.

Then, for any $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k) \in (0, \infty)^k$, there exist a polynomial $Q_\beta \in \mathbb{R}[X]$ of degree at most $q + w -$ Rank$\{l^{(1)}, \ldots, l^{(q)}\}$ and $\theta > 0$ such that as $x \to \infty$

$$\sum_{n_1 \leq x^{\beta_1}} \cdots \sum_{n_k \leq x^{\beta_k}} f(n_1, \ldots, n_k) = x^{\langle \boldsymbol{c}, \boldsymbol{\beta} \rangle} Q_{\boldsymbol{\beta}}(\log x) + \mathrm{O}\left(x^{\langle \boldsymbol{c}, \boldsymbol{\beta} \rangle - \theta} \right).$$

Here, $\langle \cdot, \cdot \rangle$ denotes the usual dot product in $\mathbb{R}^k$.

The next theorem gives a determination of the precise degree of the polynomial $Q_{\boldsymbol{\beta}}$ appearing in the previous theorem. Denoting by $\mathbb{R}_*^+$ the set of strictly positive real numbers, the notation $\mathrm{con}^*(\{l^{(1)}, \ldots, l^{(q)}\})$ means $\mathbb{R}_*^+ l^{(1)} + \ldots + \mathbb{R}_*^+ l^{(q)}$.

Theorem 2.2. *Let $f : \mathbb{N}^k \to \mathbb{R}$ be a non-negative function satisfying the assumptions of Theorem 2.1. Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_k) \in (0, \infty)^k$ and set $\mathcal{B} = \sum_{i=1}^{k} \beta_i e_i^* \in \mathcal{LR}_k^+(\mathbb{C})$. If the Dirichlet series F satisfies the additional assumptions:*

(1) *There exists a function G such that* $H(\boldsymbol{s}) = G(l^{(1)}, \dots, l^{(q)})$.
(2) *There is no subfamily* $\mathcal{L}$ *of* $\mathcal{L}_0 = \{l^{(1)}, \dots, l^{(q)}\}$ *such that* $\mathcal{L}_0 \neq \mathcal{L}$, $\mathcal{B} = \mathrm{Vect}(\mathcal{L})$ *and* $\#\mathcal{L} - \mathrm{Rank}(\mathcal{L}) = \#\mathcal{L}_0 - \mathrm{Rank}(\mathcal{L}_0)$.

Then, the polynomial $Q_{\boldsymbol{\beta}}(\log x)$ *satisfies the relation*

$$Q_{\boldsymbol{\beta}}(\log x) = H(\mathbf{0})x^{\langle \mathbf{c}, \boldsymbol{\beta}\rangle} I_{\beta}(x) + O((\log x)^{\rho-1}),$$

where $\rho = q + w - Rank\{l^{(1)}, \dots, l^{(q)}\}$ *and*

$$I_{\boldsymbol{\beta}}(x) := \int\limits_{\mathcal{C}_{\boldsymbol{\beta}}(x)} \frac{dy_1 \dots dy_q}{\prod_{i=1}^{q} y_i^{1-l^{(i)}(\mathbf{c})}},$$

with

$$\mathcal{C}_{\boldsymbol{\beta}}(x) := \left\{ \boldsymbol{y} \in [1, \infty)^q : \prod_{i=1}^{q} y_i^{l^{(i)}(\mathbf{c})} \leq x^{\beta_i} \text{ for all} 1 \leq i \leq k \right\}.$$

Also, if $Rank\{l^{(1)}, \dots, l^{(q)}\} = n$, $H(\mathbf{0}) \neq 0$, *and* $\mathcal{B} \in con^*(\{l^{(1)}, \dots, l^{(q)}\})$, *then* $\deg(Q_{\boldsymbol{\beta}}) = q + w - n$.

With these theorems in place, we begin with a review of multiplicative arithmetical functions of several variables.

3. A quick review of arithmetical functions of several variables

The study of arithmetical functions of several variables seems to begin with the fundamental paper of Vaidyanathaswamy [29] written in 1931. There, he defines a multiplicative arithmetical function of several variables $f(n_1, \dots, n_k)$ as a map $f : \mathbb{N}^k \to \mathbb{C}$ satisfying the equation

$$f(m_1 n_1, \dots, m_k n_k) = f(m_1, \dots, m_k) f(n_1, \dots, n_k)$$

whenever $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$. The classical theory of one-variable arithmetical functions extends nicely to the multi-variable case with this definition. For example, we can define the convolution $f \star g$ of two functions f and g via

$$(f \star g)(n_1, \dots, n_k) = \sum_{d_1 | n_1,\, d_2 | n_2,\, \dots, d_k | n_k} f(d_1, \dots, d_k) g(n_1/d_1, \dots, n_k/d_k).$$

If f and g are multiplicative, then it is easy to see that $f \star g$ is also multiplicative.

For multiplicative functions f, we can introduce a formal Dirichlet series of several variables along with an Euler product:

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1,\ldots,n_k)}{n_1^{s_1}\cdots n_k^{s_k}} = \prod_p \left(\sum_{v_1,\ldots,v_k=0}^{\infty} \frac{f(p^{v_1},\ldots,p^{v_k})}{p^{v_1 s_1}\cdots p^{v_k s_k}} \right).$$

In our context, the function we will study is

$$f(n_1,\ldots,n_k) := \sum_{d_1|n_1,\, d_2|n_2,\,\ldots,d_k|n_k} \mu(n_1/d_1)\cdots\mu(n_k/d_k) g(d_1,\ldots,d_k) \tag{3.1}$$

where

$$g(n_1,\ldots,n_k) := \frac{n_1\cdots n_k}{[n_1,\ldots,n_k]},$$

and $[n_1,\ldots,n_k]$ denotes the least common multiple of $n_1,\ldots,n_k$. Since g is multiplicative, we see that f is multiplicative by our remarks above.

Theorem 3.1. *For the function $f(n_1,\ldots,n_k)$, we have*

$$\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1,\ldots,n_k)}{n_1^{s_1}\cdots n_k^{s_k}} = \left(\prod_{\substack{I\subseteq[k]\\ |I|\geq 2}} \zeta(s_I - |I| + 1) \right) E(s_1,\ldots,s_k), \tag{3.2}$$

where $[k] := \{1,\ldots,k\}$ and for any subset $I = \{l_1,\ldots,l_r\}$ of $[k]$, we have $s_I := s_{l_1} + \cdots + s_{l_r}$ and $E(s_1,\ldots,s_k)$ is a Dirichlet series absolutely convergent for $\mathrm{Re}(s_i) > 1 - 1/k$.

Proof. That a factorization of the form (3.2) exists is easily proved as follows. We first note that $f(d_1,\ldots,d_k)$ is a convolution of multiplicative functions. Therefore, from (3.1), we have

$$\begin{aligned}\sum_{n_1,\ldots,n_k=1}^{\infty} \frac{f(n_1,\ldots,n_k)}{n_1^{s_1}\cdots n_k^{s_k}} &= \sum_{d_1,\ldots,d_k=1}^{\infty} \frac{g(d_1,\ldots,d_k)}{d_1^{s_1}\cdots d_k^{s_k}} \sum_{e_1,\ldots,e_k=1}^{\infty} \frac{\mu(e_1)\cdots\mu(e_k)}{e_1^{s_1}\cdots e_k^{s_k}} \\ &= \frac{1}{\zeta(s_1)\cdots\zeta(s_k)} \sum_{d_1,\ldots,d_k=1}^{\infty} \frac{g(d_1,\ldots,d_k)}{d_1^{s_1}\cdots d_k^{s_k}} \end{aligned} \tag{3.3}$$

since

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

We examine the series on the right of (3.3) as follows.

$$\sum_{d_1,\ldots,d_k=1}^{\infty} \frac{g(d_1,\ldots,d_k)}{d_1^{s_1}\cdots d_k^{s_k}} = \prod_p \left(\sum_{v_1,\ldots,v_k=0}^{\infty} \frac{p^{v_1+\cdots+v_k-\max(v_1,\ldots,v_k)}}{p^{s_1v_1+\cdots+s_kv_k}} \right)$$

$$= \prod_p \left(\sum_{n=0}^{\infty} p^{-n} \sum_{\substack{v_1,\ldots,v_k=0 \\ \max(v_1,\ldots,v_k)=n}}^{\infty} \frac{p^{v_1+\cdots+v_k}}{p^{s_1v_1+\cdots+s_kv_k}} \right). \tag{3.4}$$

The Euler factor can be written as

$$1+p^{-1} \sum_{\substack{v_1,\ldots,v_k=0 \\ \max(v_1,\ldots,v_k)=1}}^{\infty} \frac{p^{v_1+\cdots+v_k}}{p^{s_1v_1+\cdots+s_kv_k}} + \sum_{n=2}^{\infty} p^{-n} \sum_{\substack{v_1,\ldots,v_k=0 \\ \max(v_1,\ldots,v_k)=n}}^{\infty} \frac{p^{v_1+\cdots+v_k}}{p^{s_1v_1+\cdots+s_kv_k}}. \tag{3.5}$$

The inner sum in the second summation is actually a finite sum with precisely $(n+1)^k - n^k$ terms and with $\sigma = \mathrm{Re}(s_i)$, it is easily estimated to be

$$\ll (n+1)^k p^{kn(1-\sigma)}.$$

This means that

$$\sum_p \sum_{n=2}^{\infty} p^{-n} \sum_{\substack{v_1,\ldots,v_k=0 \\ \max(v_1,\ldots,v_k)=n}}^{\infty} \frac{p^{v_1+\cdots+v_k}}{p^{s_1v_1+\cdots+s_kv_k}}$$

converges absolutely for $\mathrm{Re}(s_i) > 1 - 1/k$. We can therefore factor the Euler product in (3.4) to get

$$\sum_{d_1,\ldots,d_k=1}^{\infty} \frac{g(d_1,\ldots,d_k)}{d_1^{s_1}\cdots d_k^{s_k}} = \left(\prod_p \left(1+p^{-1} \sum_{\substack{v_1,\ldots,v_k=0 \\ \max(v_1,\ldots,v_k)=1}}^{\infty} \frac{p^{v_1+\cdots+v_k}}{p^{s_1v_1+\cdots+s_kv_k}} \right) \right) E^*(s_1,\ldots,s_k),$$

where $E^*(s_1,\ldots,s_k)$ is a Dirichlet series absolutely convergent in $\mathrm{Re}(s_i) > 1 - \frac{1}{k}$. The Euler product above can be analyzed as follows. The p-Euler factor can be written as

$$1 + \frac{1}{p} \sum_{\emptyset \neq I \subseteq [k]} \prod_{i \in I} (pT_i)^{v_i}$$

where $T_i = p^{-s_i}$. This observation allows us to further factor the Euler product as

$$\left(\prod_{\emptyset \neq I \subseteq [k]} \zeta(s_I - |I| + 1) \right) E^{**}(s_1,\ldots,s_k)$$

where $E^{**}(s_1,\ldots,s_k)$ is a Dirichlet series absolutely convergent for $\mathrm{Re}(s_i) > 1/2$. Combining all these observations and setting

$$E(s_1,\dots,s_k) = E^*(s_1,\dots,s_k)E^{**}(s_1,\dots,s_k),$$

we obtain

$$\sum_{d_1,\dots,d_k=1}^{\infty} \frac{g(d_1,\dots,d_k)}{d_1^{s_1}\cdots d_k^{s_k}} = \left(\prod_{\emptyset\neq I\subseteq[k]} \zeta(s_I - |I| + 1)\right) E(s_1,\dots,s_k).$$

Taking into account (3.3) and noting that the singleton sets are removed from our product of zeta functions, we obtain (3.2), as claimed. □

Remark. We can determine $E(s_1,\dots,s_k)$ very explicitly:

$$E(s_1,\dots,s_k) = \prod_p \left(\prod_{\emptyset\neq I\subset[k]} (1-p^{|I|-1-s_I}) + \frac{\sum_{\emptyset\neq I\subseteq[k]} (-1)^{|I|} p^{2|I|-1-2s_I}(p^{|I|-s_I}-1)\prod_{\substack{\emptyset\neq J\subseteq[k]\\ I\neq J}}(1-p^{|J|-1-s_J})}{1+\sum_{\emptyset\neq I\subseteq[k]}(-1)^{|I|}p^{|I|-s_I}}\right).$$

To see this, let $p^{-s_1} = T_1,\dots,p^{-s_k} = T_k$ as before. Then, for $n\geq 1$, we have

$$\begin{aligned}
\sum_{\substack{v_1,\dots,v_k=0\\ \max(v_1,\dots,v_k)=n}}^{\infty} \frac{p^{v_1+\cdots+v_k}}{p^{s_1v_1+\cdots+s_kv_k}} &= \sum_{\substack{v_1,\dots,v_k=0\\ \max(v_1,\dots,v_k)\leq n}}^{\infty} p^{v_1+\cdots+v_k}T_1^{v_1}\cdots T_k^{v_k} \\
&\quad - \sum_{\substack{v_1,\dots,v_k=0\\ \max(v_1,\dots,v_k)\leq n-1}}^{\infty} p^{v_1+\cdots+v_k}T_1^{v_1}\cdots T_k^{v_k} \\
&= \prod_{i=1}^{k}\left(\sum_{v_i\leq n} p^{v_i}T_i^{v_i}\right) - \prod_{i=1}^{k}\left(\sum_{v_i\leq n-1} p^{v_i}T_i^{v_i}\right) \\
&= \prod_{i=1}^{k}\left(\frac{(1-(pT_i)^{n+1})}{1-pT_i}\right) - \prod_{i=1}^{k}\left(\frac{1-(pT_i)^n}{1-pT_i}\right). \qquad (3.6)
\end{aligned}$$

Inserting the above estimate into (3.5), we obtain

$$\begin{aligned}
&1+\frac{1}{\prod_{i=1}^k(1-pT_i)}\left(\sum_{n=1}^{\infty}\sum_{\emptyset\neq I\subseteq[k]}(-1)^{|I|}\left(p^{n(|I|-1)+|I|}T_I^{n+1} - p^{n(|I|-1)}T_I^n\right)\right) \\
&= 1+\frac{1}{\prod_{i=1}^k(1-pT_i)}\left(\sum_{\emptyset\neq I\subset[k]}(-1)^{|I|}p^{|I|-1}T_I\left(\frac{p^{|I|}T_I}{1-p^{|I|-1}T_I} - \frac{1}{1-p^{|I|-1}T_I}\right)\right)
\end{aligned}$$

$$= 1 + \frac{1}{\prod_{i=1}^{k}(1-pT_i)} \left(\sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|} p^{|I|-1} T_I \left(\frac{p^{|I|}T_I - 1}{1 - p^{|I|-1}T_I} \right) \right). \tag{3.7}$$

Therefore from (3.4) and (3.7), we have

$$\begin{aligned} &\sum_{d_1,\ldots,d_k=1}^{\infty} \frac{g(d_1,\ldots,d_k)}{d_1^{s_1}\cdots d_k^{s_k}} \\ &= \prod_p \left(1 + \frac{1}{\prod_{i=1}^{k}(1-pT_i)} \sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|} p^{|I|-1} T_I \left(\frac{p^{|I|}T_I - 1}{1 - p^{|I|-1}T_I} \right) \right) \\ &= \left(\prod_{\emptyset \neq I \subseteq [k]} \zeta(s_I - |I| + 1) \right) E(s_1,\ldots,s_k). \end{aligned} \tag{3.8}$$

Here,

$$\begin{aligned} &E(s_1,\ldots,s_k) \\ &= \prod_p \left(\prod_{\emptyset \neq I \subset [k]} (1-p^{|I|-1}T_I) + \frac{\sum_{\emptyset \neq I \subset [k]} (-1)^{|I|} p^{|I|-1} T_I (p^{|I|}T_I - 1) \prod_{\substack{\emptyset \neq J \subset [k] \\ I \neq J}} (1 - p^{|J|-1}T_J)}{\prod_{i=1}^{k}(1-pT_i)} \right) \\ &= \prod_p \left(\prod_{\emptyset \neq I \subset [k]} (1-p^{|I|-1}T_I) + \frac{\sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|} p^{|I|-1} T_I (p^{|I|}T_I - 1) \prod_{\substack{\emptyset \neq J \subseteq [k] \\ I \neq J}} (1 - p^{|J|-1}T_J)}{1 + \sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|} p^{|I|} T_I} \right). \end{aligned}$$

From this explicit expression, the region of absolute convergence for $E(s_1,\ldots,s_k)$ is not immediately clear and thus, we have opted for the more expedient method in the proof of our theorem.

4. Non-negativity of $f(n_1,\ldots,n_k)$

In this section, we will show that $f(n_1,\ldots,n_k)$ is a non-negative function, thus paving the way for an application of the Bretèche Tauberian theorem.

Theorem 4.1. *$f(n_1,\ldots,n_k) \geq 0$ for all $(n_1,\ldots,n_k) \in \mathbb{N}^k$.*

Proof. Since f is a convolution of two multiplicative functions, it is also multiplicative. Therefore, to prove the lemma, it suffices to show for each prime p,

$$f(p^{v_1}, p^{v_2}, \ldots, p^{v_k}) \geq 0, \qquad v_1,\ldots,v_k \geq 0.$$

Since f is symmetric, we can suppose without any loss of generality that $v_1 \geq v_2 \geq \ldots \geq v_k$. We proceed by induction on k. For $k = 1$, the result is clear. We may also suppose that all $v_i \geq 1$ for otherwise, we are again done by induction. If $v_1 > v_2$, then noting that

$$f(p^{v_1}, p^{v_2}, \ldots, p^{v_k}) = \tag{4.1}$$
$$\sum_{d_2|p^{v_2},\ldots,d_k|p^{v_k}} \mu(d_2)\cdots\mu(d_k)\left\{g\left(p^{v_1}, \frac{p^{v_2}}{d_2}, \ldots, \frac{p^{v_k}}{d_k}\right) - g\left(p^{v_1-1}, \frac{p^{v_2}}{d_2}, \ldots, \frac{p^{v_k}}{d_k}\right)\right\},$$

we have

$$g\left(p^{v_1}, \frac{p^{v_2}}{d_2}, \ldots, \frac{p^{v_k}}{d_k}\right) = g\left(1, \frac{p^{v_2}}{d_2}, \ldots, \frac{p^{v_k}}{d_k}\right)\left(p^{v_1}, \left[\frac{p^{v_2}}{d_2}, \ldots, \frac{p^{v_k}}{d_k}\right]\right)$$

and

$$g\left(p^{v_1-1}, \frac{p^{v_2}}{d_2}, \ldots, \frac{p^{v_k}}{d_k}\right) = g\left(1, \frac{p^{v_2}}{d_2}, \ldots, \frac{p^{v_k}}{d_k}\right)\left(p^{v_1-1}, \left[\frac{p^{v_2}}{d_2}, \ldots, \frac{p^{v_k}}{d_k}\right]\right).$$

We see in this case that the gcd in both cases is the same and so the term in braces in (4.1) above is zero. Now suppose that $v_1 = v_2 = \cdots v_\ell > v_{\ell+1} \geq \cdots \geq v_k$. We have

$$f(p^{v_1}, \ldots, p^{v_k}) = \sum_{d_1|p,\ldots,d_k|p} \mu(d_1)\cdots\mu(d_k) g(p^{v_1}/d_1, \ldots, p^{v_k}/d_k).$$

Noting that in the sum over divisors that each d_i can only be 1 or p, we arrange the sum as follows. We write $d_i = p^{e_i}$ where $e_i = 0$ or 1. We can then identify each tuple $(d_1, \ldots, d_k)$ with a subset $I \subseteq [k]$ where $I = \{i : e_i = 1\}$. Our sum becomes

$$f(p^{v_1}, \ldots, p^{v_k}) = \sum_{I\subseteq[k]} (-1)^{|I|} p^{s-|I|-\max(v_i-e_i:1\leq i\leq k)},$$

where

$$s = v_1 + \cdots + v_k.$$

We let $I_0 = \{1, 2, \ldots, \ell\}$ and set $s' = v_2 + \cdots + v_k$. We split the sum on the right into three parts:

$$\sum_{I:I\cap I_0=\emptyset} + \sum_{I:\emptyset\neq I\cap I_0\neq I_0} + \sum_{I:I\supseteq I_0}.$$

Letting $J = \{\ell+1, \ldots, k\}$, the first part is equal to

$$\sum_{I\subseteq J} (-1)^{|I|} p^{s'-|I|} = p^{s'}\left(1 - \frac{1}{p}\right)^{k-\ell},$$

because in this case $\max(v_i - e_i : 1 \leq i \leq k) = v_1$. In the second part, we again have $\max(v_i - e_i : 1 \leq i \leq k) = v_1$ so that the second part is equal to

$$\sum_{I:\emptyset \neq I \cap I_0 \neq I_0} (-1)^{|I|} p^{s'-|I|} = p^{s'} \sum_{j=1}^{\ell-1} \binom{\ell}{j} (-1)^j p^{-j} \left(1 - \frac{1}{p}\right)^{k-\ell}.$$

Finally, in the third part, $\max(v_i - e_i : 1 \leq i \leq k) = v_1 - 1$ so that the third part equals

$$\sum_{I:I \supseteq I_0} (-1)^{|I|} p^{s'+1-|I|} = p^{s'+1-\ell} (-1)^\ell \left(1 - \frac{1}{p}\right)^{k-\ell}.$$

Notice that the first part and the second part combine to give

$$p^{s'} \sum_{j=0}^{\ell-1} \binom{\ell}{j} (-1)^j p^{-j} \left(1 - \frac{1}{p}\right)^{k-\ell},$$

since the term corresponding to $j = 0$ is the contribution from the first part. Putting everything together gives

$$p^{s'} \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j p^{-j} \left(1 - \frac{1}{p}\right)^{k-\ell} - (-1)^\ell p^{s'-\ell} \left(1 - \frac{1}{p}\right)^{k-\ell} + p^{s'+1-\ell} (-1)^\ell \left(1 - \frac{1}{p}\right)^{k-\ell}.$$

This simplifies to

$$p^{s'} \left(1 - \frac{1}{p}\right)^k + (-1)^\ell \left(1 - \frac{1}{p}\right)^{k-\ell} \left(p^{s'+1-\ell} - p^{s'-\ell}\right).$$

Noting that $\ell \geq 2$, we see that this is certainly positive if ℓ is even. If ℓ is odd, the term is equal to

$$p^{s'} \left(1 - \frac{1}{p}\right)^k - \left(1 - \frac{1}{p}\right)^{k-\ell} \left(p^{s'+1-\ell} - p^{s'-\ell}\right).$$

We easily see that this reduces to checking that

$$p^{s'} (p-1)^\ell \geq p^\ell \left(p^{s'+1-\ell} - p^{s'-\ell}\right) = p^{s'}(p-1),$$

which is evidently true. This completes the proof of non-negativity. □

5. Average order of $f(n_1, \ldots, n_k)$

The average order of the function $f(n_1, \ldots, n_k)$ is the most essential step in the proof of Theorem 1.1. In this section, we estimate an average of $f(n_1, \ldots, n_k)$ as an application of Theorems 2.1 and 2.2.

Theorem 5.1. *Let $k \geq 3$. There exists a $\theta > 0$, such that as $x \to \infty$, we have*

$$\sum_{n_1,\ldots,n_k \leq x} f(n_1, \ldots, n_k) = x^k Q(\log x) + \mathrm{O}\left(x^{k-\theta}\right),$$

where $Q \in \mathbb{R}[X]$ is a polynomial of exact degree $2^k - 2k - 1$.

Proof. In Theorem 4.1, we proved that $f(n_1, \ldots, n_k)$ is non-negative and in Theorem 3.1, we proved $f(n_1, \ldots, n_k)$ has an absolutely convergent series $F(\boldsymbol{s})$ for $\mathrm{Re}(s_i) > 1$ for all $1 \leq i \leq k$. This shows $f(n_1, \ldots, n_k)$ satisfies (1) of Theorem 2.1. Next, we show that $f(n_1, \ldots, n_k)$ also satisfies (2) of Theorem 2.1. Write $\mathbf{1} = (1, \ldots, 1)$ then, $F(\boldsymbol{s} + \mathbf{1})$ is an absolutely convergent series for $\mathrm{Re}(s_i) > 0$. Therefore, for the linear forms s_I for $I \subseteq [k]$ and $|I| \geq 2$, define the function

$$H(\boldsymbol{s}) := F(\boldsymbol{s} + \mathbf{1}) \prod_{\substack{I \subseteq [k] \\ |I| \geq 2}} s_I.$$

Since $c_i = 1$ for all $1 \leq i \leq k$, we take $q' = 0$ in the notation of Theorem 2.1. Furthermore, for any $\zeta(s_1 + \cdots + s_\ell + 1)$ in $F(\boldsymbol{s} + \mathbf{1})$, there is a linear form $(s_1 + \cdots + s_\ell)$ such that

$$(s_1 + \cdots + s_\ell)\zeta(s_1 + \cdots + s_\ell + 1)$$

has analytic continuation on the plane $\mathrm{Re}\left\{(s_1 + \cdots + s_\ell)\right\} > -\epsilon$, where $\epsilon > 0$ and for $I \subseteq K$, we have $|I| = \ell \geq 2$. Therefore, $H(\boldsymbol{s})$ also has analytic continuation on the plane $\mathrm{Re}\left\{(s_1 + \cdots + s_\ell)\right\} > -\epsilon$. Consider $h^i(\boldsymbol{s}) = s_i$, set $\delta_1 = \delta_3 = \epsilon$. Moreover, from Lemma 3.1, $E(\boldsymbol{s} + \mathbf{1})$ has analytic continuation on the plane $\mathrm{Re}\left\{(s_1 + \cdots + s_\ell)\right\} > -\epsilon$. We know that for $\mathrm{Re}\left\{s_i\right\} > -1$ and for all $\epsilon_0 > 0$

$$s_I \zeta(1 + s_I) \ll_{\epsilon_0} (1 + |s_I|)^{1 - \frac{1}{2}\min(0, \mathrm{Re}\{s_I\}) + \epsilon_0}.$$

The above argument shows that $H(\boldsymbol{s})$ satisfies (2) of Theorem 2.1 with $\delta_2 = 1/2$. Therefore, we have as $x \to \infty$,

$$\sum_{n_1,\ldots,n_k \leq x} f(n_1, \ldots, n_k) = x^k Q(\log x) + \mathrm{O}\left(x^{k-\theta}\right),$$

where $Q(\log x)$ is a polynomial of degree at most $2^k - 2k - 1$.

Next, $c_i > 0$ for all $1 \le i \le k$, this implies $w = 0$. Again, it is easy to see that for $k \ge 3$ the rank of the collection of linear forms s_I is k and the interior of the cone generated by linear forms is the set $\mathcal{B} = \sum_{i=1}^{k} \beta_i \mathbf{e}_i^*$ for $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k) \in (0, \infty)^k$ and $\mathbf{e}_i^*(\boldsymbol{s}) = s_i$. Also, as $s_I \to 0$, $s_I \zeta(s_I + 1) \to 1$ and hence $H(\mathbf{0}) \neq 0$. More precisely, we have

$$H(\mathbf{0}) = \lim_{\substack{s_i \to 0 \\ 1 \le i \le k}} \prod_p \Bigg(\prod_{\emptyset \neq I \subseteq [k]} (1 - p^{-1-s_I}) + \frac{\sum_{\emptyset \neq I \subseteq [k]} (-1)^{|I|} p^{-1-s_I} (p^{-s_I} - 1) \prod_{\substack{\emptyset \neq J \subseteq [k] \\ I \neq J}} \left(1 - p^{-1-s_J}\right)}{1 + \sum_{\emptyset \neq L \subseteq [k]} (-1)^{|L|} p^{-s_L}} \Bigg)$$

$$= \prod_p \left(\left(1 - \frac{1}{p}\right)^{2^k - 1} + \frac{1}{p}\left(1 - \frac{1}{p}\right)^{2^k - 2} \lim_{\substack{s_i \to 0 \\ 1 \le i \le k}} \sum_{\emptyset \neq I \subseteq [k]} \frac{(-1)^{|I|} p^{-s_I} (p^{-s_I} - 1)}{1 + \sum_{\emptyset \neq L \subseteq [k]} (-1)^{|L|} p^{-s_L}} \right). \tag{5.1}$$

Indeed, noting the denominator in the summation in (5.1) is

$$\prod_{j=1}^{k} \left(1 - \frac{1}{p^{s_j}}\right),$$

and the summation of the numerators is

$$\prod_{j=1}^{k} \left(1 - \frac{1}{p^{2s_j}}\right) - \prod_{j=1}^{k} \left(1 - \frac{1}{p^{s_j}}\right) = \prod_{j=1}^{k} \left(1 - \frac{1}{p^{s_j}}\right) \left[\prod_{j=1}^{k} \left(1 + \frac{1}{p^{s_j}}\right) - 1 \right],$$

we deduce that the limit in (5.1) is

$$H(\mathbf{0}) = \prod_p \left[\left(1 - \frac{1}{p}\right)^{2^k - 1} + \frac{2^k - 1}{p} \left(1 - \frac{1}{p}\right)^{2^k - 2} \right].$$

Now, from Theorem 2.2, $\deg(Q) = 2^k - 2k - 1$. Next, we evaluate the polynomial $Q(\log x)$ using Theorem 2.2. From the definition of $H(\boldsymbol{s})$, there exists a function G which satisfies the first assumption of Theorem 2.2 and there is no subfamily $\mathcal{L}$ of the set of linear equations $\mathcal{L}_0$ such that $\mathcal{B} \in \mathrm{Vect}(\mathcal{L})$ and $\#\mathcal{L} - \mathrm{Rank}(\mathcal{L}) = \#\mathcal{L}_0 - \mathrm{Rank}(\mathcal{L}_0)$. Therefore, we have

$$Q(\log x) = H(\mathbf{0}) x^k I(x) + O((\log x)^{\rho - 1}),$$

where $\rho = 2^k - 2k - 1$ and

$$I(x) := \int_{\mathcal{C}(x)} \frac{dy_1 \dots dy_q}{\prod_{i=1}^{q} y_i^{1 - l^{(i)}(\mathbf{c})}},$$

with

$$\mathcal{C}(x) := \left\{ \boldsymbol{y} \in [1, \infty)^q : \prod_{i=1}^{q} y_i^{l^{(i)}(\mathbf{c})} \leq x \text{ for all } 1 \leq j \leq k \right\},$$

where $l^{(i)}(\boldsymbol{c}) \leq k$ for all $1 \leq i \leq q$. We therefore see that $H(\mathbf{0})$ is positive and as $I(x)$ is positive, the lead term in the polynomial Q is also positive. This completes our proof. □

6. Higher moments of Ramanujan sums

In this section, we provide a proof of Theorem 1.1 using the average order of $f(n_1, \ldots, n_k)$ obtained in the previous section.

Proof of Theorem 1.1. From the definition of Ramanujan sums, we have

$$\begin{aligned} S_k(x,y) &= \sum_{n \leq y} \left(\sum_{q \leq x} c_q(n) \right)^k = \sum_{n \leq y} \sum_{q_1, \ldots, q_k \leq x} \sum_{\substack{d_1 | q_1 \\ d_1 | n}} d_1 \mu\left(\frac{q_1}{d_1}\right) \cdots \sum_{\substack{d_k | q_k \\ d_k | n}} d_k \mu\left(\frac{q_k}{d_k}\right) \\ &= \sum_{q_1, \ldots, q_k \leq x} \sum_{d_1 | q_1, \ldots, d_k | q_k} d_1 \cdots d_k \mu\left(\frac{q_1}{d_1}\right) \cdots \mu\left(\frac{q_k}{d_k}\right) \sum_{\substack{n \leq y \\ [d_1, \ldots, d_k] | n}} 1 \\ &= \sum_{q_1, \ldots, q_k \leq x} \sum_{d_1 | q_1, \ldots, d_k | q_k} d_1 \cdots d_k \mu\left(\frac{q_1}{d_1}\right) \cdots \mu\left(\frac{q_k}{d_k}\right) \left(\frac{y}{[d_1, \ldots, d_k]} + \mathrm{O}(1) \right) \\ &= y \sum_{q_1, \ldots, q_k \leq x} \sum_{d_1 | q_1, \ldots, d_k | q_k} \frac{d_1 \cdots d_k}{[d_1, \ldots, d_k]} \mu\left(\frac{q_1}{d_1}\right) \cdots \mu\left(\frac{q_k}{d_k}\right) + \mathrm{O}\left(x^{2k}\right). \end{aligned}$$

The inner sum is precisely $f(q_1, \ldots, q_k)$ in our notation by virtue of (3.1). Thus, from Theorem 5.1, for $0 < \theta < 1$, we have as $x \to \infty$

$$S_k(x,y) = yx^k Q(\log x) + \mathrm{O}\left(yx^{k-\theta} + x^{2k}\right),$$

where $Q(\log x)$ is a polynomial of degree $2^k - 2k - 1$. This completes the proof. □

7. Moments of Cohen Ramanujan sums

In [9], Cohen generalized the Ramanujan sums in the following way:

$$c_q^{\beta}(n) := \sum_{\substack{1 \leq j \leq q^{\beta} \\ (j, q^{\beta})_{\beta} = 1}} e\left(\frac{jn}{q^{\beta}}\right) = \sum_{\substack{d | q \\ d^{\beta} | n}} d^{\beta} \mu\left(\frac{q}{d}\right).$$

We refer to these sums as Cohen-Ramanujan sums. Here, $(m,n)_\beta$ denotes the generalized gcd which is the largest l^β dividing both m and n. In [25], Robles and Roy estimated the averages of moments of Cohen-Ramanujan sums but their result is correct only for the first and second moments. For higher moments, it is false. Our analysis of the previous sections is amenable to treat this general case too. Therefore, applying our method to obtain the higher moments of these sums, we get:

Theorem 7.1. *For $k \geq 3$ and $y > x^{k(\beta+1)/2}$, as $x \to \infty$, we have*

$$\sum_{n \leq y} \left(\sum_{q \leq x} c_q^\beta(n) \right)^k = y x^{k(\beta+1)/2} Q(\log x) + \mathrm{O}\left(y x^{k(\beta+1)/2-\theta} \right),$$

where $Q(\log x)$ is a polynomial of exact degree $2^k - 2k - 1$ and $0 \leq \theta \leq 1$.

One can prove Theorem 7.1 by slightly modifying the proof of Theorem 1.1, taking into account the (minor) differences in the definitions between Ramanujan sums and the Cohen-Ramanujan sums.

8. Concluding remarks

There is undoubtedly a deeper significance of our main theorem to our current understanding of the Riemann hypothesis. Indeed, Ramanujan (see formula (7.2) in [23]) showed that

$$\frac{\sigma_{1-s}(n)}{\zeta(s)} = \sum_{q=1}^{\infty} \frac{c_q(n)}{q^s}, \qquad \sigma_w(n) := \sum_{d|n} d^w,$$

valid for $\mathrm{Re}(s) > 1$. As the left hand side admits a meromorphic continuation to the entire complex plane with poles located at the zeros of the zeta function, we can see from standard analytic number theory that the Riemann hypothesis is equivalent to the estimate

$$\sum_{q \leq x} c_q(n) = O(x^{\frac{1}{2}+\epsilon}),$$

for any $\epsilon > 0$ and for any fixed n. The implied constant in the estimate depends on n and a careful analysis gives $O((xn)^{\frac{1}{2}+\epsilon})$ as the final estimate for the sum. We therefore can expect $O(y^{1+k/2+\epsilon} x^{k/2+\epsilon})$ as a final estimate for $S_k(x,y)$ assuming the Riemann hypothesis. What our result shows is that this is unconditionally true if y is around x.

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Data availability

No data was used for the research described in the article.

References

[1] E. Alkan, Distribution of averages of Ramanujan sums, Ramanujan J. 29 (1–3) (2012) 385–408.
[2] E. Alkan, Ramanujan sums and the Burgess zeta function, Int. J. Number Theory 8 (8) (2012) 2069–2092.
[3] É. Balandraud, An application of Ramanujan sums to equirepartition modulo an odd integer, Unif. Distrib. Theory 2 (2) (2007) 1–17.
[4] R.D. Carmichael, Expansions of arithmetical functions in infinite series, Proc. Lond. Math. Soc. (2) 34 (1) (1932) 1–26.
[5] P. Cassou-Noguès, Séries de Dirichlet et intégrales associées à un polynôme à deux indéterminées, J. Number Theory 23 (1) (1986) 1–54.
[6] T.H. Chan, A.V. Kumchev, On sums of Ramanujan sums, Acta Arith. 152 (1) (2012) 1–10.
[7] S. Chaubey, S. Goel, On the distribution of Ramanujan sums over number fields, Ramanujan J. 61 (3) (2023) 813–837.
[8] S. Chaubey, S. Goel, Moments of averages of Ramanujan sums over number fields, Funct. Approx. Comment. Math. 71 (1) (2024) 137–155.
[9] E. Cohen, An extension of Ramanujan's sum, Duke Math. J. 16 (1949) 85–90.
[10] R. de la Bretèche, Estimation de sommes multiples de fonctions arithmétiques, Compos. Math. 128 (3) (2001) 261–298.
[11] H. Delange, On Ramanujan expansions of certain arithmetical functions, Acta Arith. 31 (3) (1976) 259–270.
[12] Y. Fujisawa, On sums of generalized Ramanujan sums, Indian J. Pure Appl. Math. 46 (1) (2015) 1–10.
[13] A. Hildebrand, Über die punktweise Konvergenz von Ramanujan-Entwicklungen zahlentheoretischer Funktionen, Acta Arith. 44 (2) (1984) 109–140.
[14] M. Jutila, Distribution of rational numbers in short intervals, Ramanujan J. 14 (2) (2007) 321–327.
[15] J. Konvalina, A generalization of Waring's formula, J. Comb. Theory, Ser. A 75 (2) (1996) 281–294.
[16] J. Leray, Le calcul différentiel et intégral sur une variété analytique complexe. (Problème de Cauchy. III), Bull. Soc. Math. Fr. 87 (1959) 81–180.
[17] B. Lichtin, The asymptotics of a lattice point problem associated to a finite number of polynomials. I, Duke Math. J. 63 (1) (1991) 139–192.
[18] L.G. Lucht, K. Reifenrath, Mean-value theorems in arithmetic semigroups, Acta Math. Hung. 93 (1–2) (2001) 27–57.
[19] J. Ma, H. Sun, W. Zhai, The average size of Ramanujan sums over cubic number fields, arXiv preprint arXiv:2105.11699, 2021.
[20] M.B. Nathanson, Additive number theory, in: Inverse Problems and the Geometry of Sumsets, in: Graduate Texts in Mathematics, vol. 165, Springer-Verlag, New York, 1996.
[21] W.G. Nowak, The average size of Ramanujan sums over quadratic number fields, Arch. Math. (Basel) 99 (5) (2012) 433–442.
[22] W.G. Nowak, On Ramanujan sums over the Gaussian integers, Math. Slovaca 63 (4) (2013) 725–732.
[23] S. Ramanujan, On certain trigonometrical sums and their applications in the theory of numbers [Trans. Cambridge Philos. Soc. 22 (1918), no. 13, 259–276], in: Collected Papers of Srinivasa Ramanujan, AMS Chelsea Publ., Providence, RI, 2000, pp. 179–199.
[24] O. Ramaré, Eigenvalues in the large sieve inequality, Funct. Approx. Comment. Math. 37 (part 2) (2007) 399–427.

[25] N. Robles, A. Roy, Moments of averages of generalized Ramanujan sums, Monatshefte Math. 182 (2) (2017) 433–461.
[26] P. Sargos, Prolongement méromorphe des séries de Dirichlet associées à des fractions rationnelles de plusieurs variables, Ann. Inst. Fourier (Grenoble) 34 (3) (1984) 83–123.
[27] W. Schwarz, Ramanujan expansions of arithmetical functions, in: Ramanujan Revisited, Urbana-Champaign, Ill., 1987, 1988, pp. 187–214.
[28] W. Schwarz, J. Spilker, An introduction to elementary and analytic properties of arithmetic functions and to some of their almost-periodic properties, in: Arithmetical Functions, in: London Mathematical Society Lecture Note Series, vol. 184, Cambridge University Press, Cambridge, 1994.
[29] R. Vaidyanathaswamy, The theory of multiplicative arithmetic functions, Trans. Am. Math. Soc. 33 (2) (1931) 579–662.
[30] A. Wintner, Eratosthenian Averages, Publisher Unknown, Baltimore, Md., 1943.
[31] E. Wirsing, Das asymptotische Verhalten von Summen über multiplikative Funktionen, Math. Ann. 143 (1961) 75–102.
[32] W. Zhai, The average size of Ramanujan sums over quadratic number fields, Ramanujan J. (2021) 1–17.