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SIGN CHANGES OF FOURIER COEFFICIENTS OF HALF-INTEGRAL WEIGHT CUSP FORMS

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We prove a quantitative result for the number of sign changes of the Fourier coefficients of half-integral weight cusp forms in the Kohnen plus space, provided the Fourier coefficients are real numbers.

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1. Introduction

Let k be a positive integer. Suppose that f is a modular form of weight $k + 1/2$ on $\Gamma_0(4)$. The Shimura correspondence defined in [12] maps f to a modular form F of integral weight $2k$. In addition, if f is an eigenform of the Hecke operator $T_{k+1/2}(p^2)$, then F is also an eigenform of the Hecke operators $T_{2k}(p)$ with the same eigenvalue. For more on half-integral weight modular forms, see [12]. Let the Fourier expansion of f be given by

$$f(z) = \sum_{n=0}^{\infty} a(n) n^{\frac{k}{2}-\frac{1}{4}} e^{2\pi i n z}. \tag{1}$$

Also, assume that the Fourier coefficients $a(n)$ are real numbers. Kohnen [6] proved that for any half-integral weight cusp form, the sequence $a(tn^2)$ for a fixed t square-free has infinitely many sign changes. However, Kohnen does not prove any quantitative result on the number of sign changes in that paper. Recently, Kohnen, Lau and Wu [7] have proved some quantitative results on the number of sign changes in the sequence $a(tn^2)$, where t is a fixed square-free positive integer. If f is an eigenform, then recently, Hulse, Kiral, Kuan and Lim [3] have proved that the sequence

$a(t)$, where t is running over all square-free integers, changes sign infinitely often. If f is in the Kohnen plus space, that is, $a(n) = 0$ when $(-1)^k n \equiv 2, 3 \pmod 4$, Bruinier and Kohnen [1] make the following conjectures:

$$\lim_{x\to\infty} \frac{|\{n \le x : a(n) < 0\}|}{|\{n \le x : a(n) \ne 0\}|} = \frac{1}{2}, \tag{2}$$

and

$$\lim_{x\to\infty} \frac{|\{|d| \le x : d \text{ fundamental discriminant}, a(|d|) < 0\}|}{|\{|d| \le x : d \text{ fundamental discriminant}, a(|d|) \ne 0\}|} = \frac{1}{2}. \tag{3}$$

The proofs of the above conjectures seem to be out of reach as of now. However, in this article we prove a quantitative result on the number of sign changes in the sequence $a(n)(n \ge 1)$. In general, one may ask about a quantitative result on the number of sign changes for any real sequence. In fact, under certain conditions, we are able to prove a quantitative result on number of sign changes for any real sequence.

More precisely, we prove the following result.

Theorem 1.1. *Let $a(n)(n \ge 1)$ be a sequence of real numbers satisfying $a(n) = O(n^\alpha)$ such that*

$$\sum_{n\le x} a(n) \ll x^\beta,$$

and

$$\sum_{n\le x} a(n)^2 = cx + O(x^\gamma),$$

where α, β, γ and c are non-negative constants. If $\alpha + \beta < 1$, then for any r satisfying $\max\{\alpha + \beta, \gamma\} < r < 1$, the sequence $a(n)(n \ge 1)$ has at least one sign change for $n \in (x, x + x^r]$. In particular, the sequence $a(n)(n \ge 1)$ has infinitely many sign changes and the number of sign changes for $n \le x$ is $\gg x^{1-r}$ for sufficiently large x.

The above theorem can be applied in a very general context. Here, we focus our attention on the sequence of Fourier coefficients of cusp forms of half-integral weight on $\Gamma_0(4)$. In a later paper [9], we investigate several applications to the theory of L-functions attached to automorphic representations.

Theorem 1.2. *Let $k \ge 2$ be an integer. Assume that*

$$f(z) = \sum_{(-1)^k n \equiv 0,1 \pmod 4} a(n) n^{\frac{k}{2} - \frac{1}{4}} e^{2\pi i n z}$$

is an eigenform of weight $k + 1/2$ on $\Gamma_0(4)$ in the Kohnen's plus space $S^+_{k+1/2}(\Gamma_0(4))$ such that the cusp form F associated to f under the Shimura correspondence is an eigenform. If the coefficients of f are real numbers, then there exists $0 < \delta < 1$ such that the sequence $a(n)$ $(n \ge 1)$ has at least one sign change for $n \in (x, x + x^\delta]$

for sufficiently large x. In particular, the number of sign changes for $n \le x$ is $\gg x^{1-\delta}$. We prove that one may choose δ as any number greater than $\frac{43}{70}$. If we assume the Ramanujan conjecture, then one may choose δ as any number greater than $\frac{3}{5}$.

2. Preparatory Results

One of key ingredients in the proof is Waldspurger's result [13]. For an eigenform $f \in S^+_{k+1/2}(\Gamma_0(4))$, the Waldspurger formula is more explicit as proved by Kohnen and Zagier [8]. For a fundamental discriminant D, the L-series $L_D(s)$ is defined as

$$L_D(s) = \sum_{n\ge 1} \left(\frac{D}{n}\right) n^{-s}.$$

If $D = l^2 D_0$, where D_0 is a fundamental discrminant, then $L_D(s)$ is defined as $L_{D_0}(s)$ times a finite Euler product over the prime divisors of l. If $D \not\equiv 0, 1 \pmod 4$, then $L_D(s)$ is defined to be identically 0. Thus, we have defined $L_D(s)$ for all $D \neq 0$. For an arbitrary integer $D \neq 0$, $L(f, D, s)$ is defined as the convolution of $L(f, s)$ and $L_D(s)$, where

$$L(f, s) = \sum_{n\ge 1} \frac{a(n)}{n^s}.$$

More precisely,

$$L(f, D, s) = \sum_{n\ge 1} \left(\frac{D}{n}\right) \frac{a(n)}{n^s}.$$

We state the following result of Kohnen and Zagier [8, Corollary 4].

Lemma 2.1. *Let*

$$f(z) = \sum_{(-1)^k n \equiv 0,1 \ (\mathrm{mod}\ 4)} a(n) n^{\frac{k}{2}-\frac{1}{4}} e^{2\pi i n z}$$

is an eigenform of weight $k+1/2$ on $\Gamma_0(4)$ in the Kohnen's plus space $S^+_{k+1/2}(\Gamma_0(4))$ such that the cusp form F associated to f under the Shimura correspondence is an eigenform. Then one has

$$\frac{|a(n)|^2}{\langle f, f\rangle} = \frac{(k-1)!}{\pi^k \langle F, F\rangle} L(F, (-1)^k n, k), \tag{4}$$

where $\langle f, f\rangle$ and $\langle F, F\rangle$ are the Petersson inner products in the spaces of half-integral weight and integral weight cusp forms, respectively.

We review some more facts about half-integral weight modular forms. Assume that $h(z) = \sum_{n\ge 1} c(n) e^{2\pi i n z}$ is a cusp form of weight $k + 1/2$ on $\Gamma_0(4)$. Let

$$D_h^*(s) = \pi^{-s}\Gamma(s) \sum_{n\ge 1} c(n) n^{-s}.$$

Then $D_h^*(s)$ has an analytic continuation to the whole complex plane $\mathbb{C}$ and it satisfies the functional equation

$$D_h^*(k+1/2-s) = D_{h|W_4}^*(s),$$

where

$$h\,|\,W_4 = (-2iz)^{-k-1/2}h(-1/4z) = \sum_{n\geq 1} c^*(n)e^{2\pi inz}$$

is a cusp form of weight $k+1/2$ on $\Gamma_0(4)$ [5, p. 430]. If $c(n) = a(n)n^{k/2-1/4}$ and $c^*(n) = a^*(n)n^{k/2-1/4}$, let $M_1(s) = \sum_{n\geq 1} a(n)n^{-s}$ and $M_2(s) = \sum_{n\geq 1} a^*(n)n^{-s}$. Then the above functional equation becomes

$$\pi^{-(s+\frac{k}{2}-\frac{1}{4})}\Gamma\left(s+\frac{k}{2}-\frac{1}{4}\right)M_1(s) = \pi^{-(-s+\frac{k}{2}-\frac{3}{4})}\Gamma\left(-s+\frac{k}{2}-\frac{3}{4}\right)M_2(1-s). \quad (5)$$

We next recall a theorem due to Chandrasekharan and Narasimhan. We state a simplified version of their result for our purpose. We need to define a few things before proceeding to the theorem.

Let

$$\phi(s) = \sum_{n\geq 1}\frac{a(n)}{n^s},$$

and

$$\psi(s) = \sum_{n\geq 1}\frac{b(n)}{n^s}$$

be two Dirichlet series. Let $\Delta(s) = \prod_{i=1}^{l}\Gamma(\alpha_i s+\beta_i)$ and $A = \sum_{i=1}^{l}\alpha_i$. Assume that

$$Q(x) = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\phi(s)}{s}x^s\,ds,$$

where $\mathcal{C}$ encloses all the singularities of the integrand. With these definitions, we now state the theorem [2, Theorem 4.1].

Theorem 2.2. *Suppose that the functional equation*

$$\Delta(s)\phi(s) = \Delta(\delta-s)\psi(\delta-s)$$

is satisfied with $\delta > 0$, *and that the only singularities of the function* ϕ *are poles. Then, we have*

$$A(x)-Q(x) = O(x^{\frac{\delta}{2}-\frac{1}{4A}+2A\eta u}) + O(x^{q-\frac{1}{2A}-\eta}(\log x)^{r-1}) + O\left(\sum_{x<n\leq x'}|a(n)|\right),$$

for every $\eta \geq 0$, *where* $A(x) = \sum_{n\leq x} a(n)$, $x' = x + O(x^{1-\frac{1}{2A}-\eta})$, q *is the maximum of the real parts of the singularities of* ϕ, r *is the maximum order of a pole with real* *part* q, *and* $u = \beta - \frac{\delta}{2} - \frac{1}{4A}$, *where* β *is such that* $\sum_{n=1}^{\infty} |b(n)|n^{-\beta}$ *is finite. If, in addition,* $a_n \geq 0$, *then we have*

$$A(x) - Q(x) = O(x^{\frac{\delta}{2} - \frac{1}{4A} + 2A\eta u}) + O(x^{q - \frac{1}{2A} - \eta}(\log x)^{r-1}).$$

1 Next, we recall the following result by Kohnen and Zagier [8, Corollary 5].

Lemma 2.3. *Let* F *be a normalized eigenform of weight* $2k$ *on* $\mathrm{SL}_2(\mathbb{Z})$. *Then the Dirichlet series*

$$\mathcal{L}_F(s) = \sum_{n=1}^{\infty} L(F, (-1)^k n, k) n^{-s}$$

is absolutely convergent for $\Re(s) > 1$, *has a meromorphic continuation to the entire complex plane with the only singularity a simple pole at* $s = 1$, *and satisfies the functional equation*

$$\begin{aligned}&\pi^{-2s}\Gamma(s)\Gamma\left(s + k - \frac{1}{2}\right)\zeta(2s)\mathcal{L}_F(s)\\ &\quad = \pi^{-2(1-s)}\Gamma(1-s)\Gamma\left(-s + k + \frac{1}{2}\right)\zeta(2-2s)\mathcal{L}_F(1-s).\end{aligned}$$

2 *The residue of* $\mathcal{L}_F(s)$ *at* $s = 1$ *is* $\frac{3(4\pi)^{2k}}{\pi(2k-1)!}\langle F, F\rangle$.

3 3. Preliminary Results

4 We now prove several results which are of independent interest.

Proposition 3.1. *If the* $a(n)$*s are as in Theorem* 1.2, *then*

$$\sum_{n \leq x} a(n)^2 = Bx + O(x^{\frac{3}{5} + \epsilon}),$$

5 *where* B *is some positive constant.*

Proof. For $\Re(s) > 1$, let

$$\zeta(2s)\mathcal{L}_F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}. \tag{6}$$

Since $b(n)(n \geq 1)$ are positive, applying Theorem 2.2 to the sequence $b(n)$, and using Lemma 2.3, we get

$$\sum_{n \leq x} b(n) - Q(x) = O(x^{\frac{1}{2} - \frac{1}{8} + 4\eta u}) + O(x^{1 - \frac{1}{4} - \eta}), \tag{7}$$

for all $\eta \geq 0$, where for $\epsilon > 0$,

$$u = 1 + \epsilon - \frac{1}{2} - \frac{1}{8} = \frac{3}{8} + \epsilon.$$

Since both $\zeta(s)$ and $\mathcal{L}_F(s)$ have only simple poles at $s = 1$, the poles of $\zeta(2s)\mathcal{L}_F(s)$ are at $s = \frac{1}{2}$ and $s = 1$. Thus

$$Q(x) = c_1 x + c_2 x^{\frac{1}{2}}$$

for some constants c_1 and c_2. Choosing $\eta = \frac{3}{8(4u+1)}$ in (7), we get

$$\sum_{n \le x} b(n) = c_1 x + O(x^{\frac{3}{5}+\epsilon}). \tag{8}$$

From (6), we see by Möbius inversion that

$$L(F, (-1)^k n, k) = \sum_{d^2 | n} \mu(d) b(n/d^2).$$

Now applying (8), we get

$$\sum_{n \le x} \sum_{d^2 | n} \mu(d) b(n/d^2) = \sum_{d^2 \le x} \mu(d) \sum_{e \le x/d^2} b(e) = \sum_{d^2 \le x} \mu(d) \left\{ c_1 \frac{x}{d^2} + O((x/d^2)^{\frac{3}{5}+\epsilon}) \right\}.$$

Thus

$$\sum_{n \le x} L(F, (-1)^k n, k) = x \left\{ \frac{6c_1}{\pi^2} + O(1/\sqrt{x}) \right\} + O(x^{\frac{3}{5}}) = \frac{6c_1}{\pi^2} x + O(x^{\frac{3}{5}+\epsilon}).$$

In the above expression, we have used the well-known fact that

$$\sum_{d^2 \le x} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} + O(1/\sqrt{x}).$$

Now applying (4), we get

$$\sum_{n \le x} a(n)^2 = Bx + O(x^{\frac{3}{5}+\epsilon}),$$

1 where B is a positive constant. This proves the required result. □

Proposition 3.2. *Let $a(n)$ be as in Theorem* 1.2. *Then for any $\epsilon > 0$, we have*

$$\sum_{n \le x} a(n) \ll x^{\frac{2}{5}+\epsilon}.$$

Proof. We apply Theorem 2.2 to prove this result. From (5), we see that $\delta = 1$, $A = 1$. By Cauchy's theorem, $Q(x) = \phi(0)$ is a constant. Applying Theorem 2.2 to the sequence $a(n)$ $(n \ge 1)$, we get

$$\sum_{n \le x} a(n) = O(x^{\frac{1}{4}+2\eta u}) + O\left(\sum_{x < n \le x'} |a(n)| \right) \tag{9}$$

2 for every $\eta \ge 0$. Here $x' = x + O(x^{\frac{1}{2}-\eta})$ and $u = 1 + \epsilon - \frac{1}{2} - \frac{1}{4} = \frac{1}{4} + \epsilon$.

Now by Cauchy–Schwarz, we have

$$\sum_{x<n\le x'} |a(n)| \ll x^{\frac{1}{4}-\frac{\eta}{2}} \left(\sum_{x<n\le x'} |a(n)|^2\right)^{\frac{1}{2}}.$$

Now applying Proposition 3.1 to the above expression, we get

$$\sum_{x<n\le x'} |a(n)| \ll x^{\frac{1}{4}-\frac{\eta}{2}} x^{\frac{3}{10}} \ll x^{\frac{11}{20}-\frac{\eta}{2}}.$$

Substituting the above estimate in (9), we get that

$$\sum_{n\le x} a(n) = O(x^{\frac{1}{4}+2\eta u}) + O(x^{\frac{11}{20}-\frac{\eta}{2}}).$$

Now choosing $\eta = \frac{3}{5(4u+1)}$, we obtain

$$\sum_{n\le x} a(n) = O(x^{\frac{2}{5}+\epsilon}). \qquad \square$$

4. The Ramanujan Conjecture for Half-Integral Weight Forms

The Ramanujan conjecture for half-integral weight cusp forms is the assertion that for any $\epsilon > 0$, $a(n) = O(n^{\epsilon})$. This would follow from the Lindelöf hypothesis for $L(f, D, s)$ in the D-aspect, which is an unsolved problem. However, using the estimate given in the above proposition, we deduce that $a(n) = O(n^{\frac{3}{10}+\epsilon})$. This is weaker than a result by Iwaniec [4], who showed that we can take $\epsilon = \frac{3}{14}$.

Using Proposition 3.1, we prove the following theorem, which is of independent interest.

Theorem 4.1. *Let $a(n)$ be as in Theorem 1.2. Fix $\epsilon > 0$. Then for any $\theta > 3/5$,*

$$\#\{x < n \le x + x^{\theta} : |a(n)| > n^{\epsilon}\} \ll x^{\theta-2\epsilon}.$$

In particular, there are infinitely many square-free n with $(-1)^k n \equiv 0, 1 \pmod 4$ for which the sequence $a(n)$ satisfies Ramanujan conjecture.

Proof. From the above proposition, for any $\theta > 0$,

$$\sum_{x<n\le x+x^{\theta}} a(n)^2 = Bx^{\theta} + O(x^{\frac{3}{5}+\epsilon}) \ll x^{\max\{\theta, \frac{3}{5}+\epsilon\}}.$$

Thus

$$x^{2\epsilon} \cdot \#\{x < n \le x + x^{\theta} : |a(n)| > n^{\epsilon}\} \ll \sum_{x<n\le x+x^{\theta}} a(n)^2 \ll x^{\max\{\theta, \frac{3}{5}+\epsilon\}},$$

which gives

$$\#\{x < n \le x + x^{\theta} : |a(n)| > n^{\epsilon}\} \ll x^{\max\{\theta, \frac{3}{5}+\epsilon\}-2\epsilon}.$$

In particular, if $\theta > 3/5$,

$$\#\{x < n \leq x + x^\theta : |a(n)| \leq n^\epsilon\} = x^\theta + O(x^{\theta - 2\epsilon}). \tag{10}$$

We know the well-known fact that for $\theta > 1/2$, we have (see [11, Exercise 1.4.4])

$$\#\{x < n \leq x + x^\theta : n \text{ is square-free}\} = \frac{6}{\pi^2} x^\theta + O(\sqrt{x}).$$

Thus

$$\#\{x < n \leq x + x^\theta : n \text{ is not square-free}\} = \left(1 - \frac{6}{\pi^2}\right) x^\theta + O(\sqrt{x}). \tag{11}$$

Comparing (10) and (11), we conclude that there is a square-free integer n in the interval $(x, x + x^\theta]$ such that $|a(n)| \leq n^\epsilon$. This proves the second assertion. □

As Kohnen remarks (in a private communication), for any fixed t, we have

$$a(tn^2) = O(n^\epsilon |a(t)|)$$

by virtue of the Shimura correspondence and the Ramanujan estimate for integral weight forms. However, the O-estimate depends on t. And so, if $t \ll n^\epsilon$, the sequence $a(tn^2)$ satisfies the Ramanujan estimate. Now, the number of such numbers tn^2 in the interval is easily seen to be

$$\ll \sum_{t \leq x^\epsilon} \sqrt{x/t} \ll x^{\frac{1}{2} + \epsilon} = o(x^\theta).$$

In other words, such numbers form a sparse subset of the set $[x, x + x^\theta]$.

5. Proof of Theorem 1.1

Assume on the contrary that $a(n)$ are of same sign for all integers n in the interval $(x, x + x^r]$. Without any loss of generality we may assume that $a(n)$ are positive for $n \in (x, x + x^r]$. Then we have

AQ: Please check the sentence "...a(n) are ......" for clarity.

$$\sum_{x < n \leq x + x^r} a(n)^2 \ll x^\alpha \sum_{x < n \leq x + x^r} a(n) \ll x^{\alpha + \beta}. \tag{12}$$

Since $r > \gamma$, we see from the hypothesis that

$$x^r \ll \sum_{x < n \leq x + x^r} a(n)^2. \tag{13}$$

Since $r > \alpha + \beta$, (12) and (13) contradict each other. Thus, there is at least one sign change of the sequence $a(n)$ $(n \geq 1)$ for $n \in (x, x + x^r]$.

6. Proof of Theorem 1.2

We have the Iwaniec's bound [4] for the Fourier coefficients of half-integral weight cusp forms

$$a(n) = O(n^{\frac{3}{14}}).$$

From Propositions 3.1 and 3.2, we have

$$\sum_{n\le x} a(n)^2 = Bx + O(x^{\frac{3}{5}+\epsilon}),$$

and

$$\sum_{n\le x} a(n) = O(x^{\frac{2}{5}+\epsilon}).$$

Thus all the required conditions of Theorem 1.1 are satisfied. Hence by Theorem 1.1, there is at least one sign change of the sequence $a(n)(n \ge 1)$ for $n \in (x, x + x^{\delta}]$, where

$$\delta > \max\left\{\frac{3}{14} + \frac{2}{5} + \epsilon, \frac{3}{5} + \epsilon\right\} = \frac{43}{70} + \epsilon,$$

for any $\epsilon > 0$. This proves that $\delta > \frac{43}{70}$. If we assume the Ramanujan conjecture, that is,

$$a(n) = O(n^{\epsilon}),$$

for any $\epsilon > 0$, from (9), we have

$$\sum_{n\le x} a(n) = O(x^{\frac{1}{4}+2\eta u}) + O(x^{\frac{1}{2}-\eta+\epsilon}),$$

for every $\eta > 0$, where $u = \frac{1}{4} + \epsilon$. Choosing $\eta = \frac{1}{2u+1}(\frac{1}{4} + \epsilon)$ in the above expression, we deduce that

$$\sum_{n\le x} a(n) = O(x^{\frac{1}{3}+\epsilon}),$$

for any $\epsilon > 0$. Then as above,

$$\delta > \max\left\{\frac{1}{3} + \epsilon + \epsilon, \frac{3}{5} + \epsilon\right\} = \frac{3}{5} + \epsilon.$$

Since the above inequality is true for any $\epsilon > 0$, this proves the last assertion.

7. Concluding Remarks

The method applied here also applies to sequences of complex-valued coefficients. Indeed, in Theorem 1.1, we can deduce that in the interval stated, there is at least one "angular change". From this, one can deduce a lower bound for the number of "angular changes" in the sequence of complex numbers $a(n)$, $n \le x$ by a suitable modification of our proof.

One can further investigate if our results here are optimal. This is unlikely. Even assuming Ramanujan conjecture, we deduce only $\gg x^{\frac{2}{5}}$ sign changes for $n \le x$. This falls short of the Bruinier–Kohnen conjecture. However, based on analogies with the classical Dirichlet divisor problem and the circle problem, it seems reasonable to

AQ: Please check the sentence "This is ...." for clarity.

conjecture that

$$\sum_{n\leq x} a(n) = O(x^{\frac{1}{4}+\epsilon}),$$

and

$$\sum_{n\leq x} a(n)^2 = Ax + O(x^{\frac{3}{8}+\epsilon}).$$

This is consistent with the philosophy outlined in [10]. Our conjectures lead to $\gg x^{\frac{5}{8}-\epsilon}$ sign changes of $a(n)$ with $n \leq x$. (We take this opportunity to correct two typos in [10]. On p. 525, line 3, $(2A-1)(2A+1)$ should be $(2A-1)/(2A+1)$ and on p. 532, line 1, the formula for p should be $p = 2\sum_{j=1}^{N} \alpha_j$.)

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