Thanks are due to Nour Fahmy and Chelsea Crocker for helpful conversations

Problem 1.

Solution.

a) We know that for any finite graph, we must have

$$\sum_x d(x) = 2|E(x)|.$$

However, we note that

$$3+3+3+3+5+6+6+6+6 = 41$$

which is not an even number. Therefore, there can be no such graph.

b) In a bipartite graph, say composed of parts A and B we must have

$$\sum_{a \in A} d(a) = \sum_{b \in B} d(b) = |E(x)|.$$

We note that

$$3+3+3+5+6+6+6+6 = 38,$$

so we would need to find two subsets which both add up to $38/2 = 19$. However, we note that both 3 and 6 have 3 as a prime factor, but 5 does not, so there can be no way to find two subsets of this set that add up to the same total, much less two subsets that add up to 19. Therefore, there exists no bipartite graph with this degree sequence.

$\square$

Problem 2.

Solution. We will prove this by induction. Start with our base case, $n = 0$, with $F_0 = 1$.

$$\sqrt{5} \cdot 1 = \left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1$$
$$\sqrt{5} = \frac{1}{2} + \frac{\sqrt{5}}{2} - \frac{1}{2} + \frac{\sqrt{5}}{2}$$
$$\sqrt{5} = \sqrt{5}$$

Now, our inductive step. We assume that

$$\sqrt{5}F_{n-1} = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$$

and we use that to show

$$\sqrt{5}F_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

We know that $F_n = F_{n-1} + F_{n-2}$, by the definition of the fibbonacci numbers.

$$\begin{aligned}\sqrt{5}F_n &= \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n + \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}+1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}+1\right) \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{3-\sqrt{5}}{2}\right)\end{aligned}$$

We note that

$$\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

and that

$$\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{6-2\sqrt{5}}{4} = \frac{3-\sqrt{5}}{2}$$

So we then have

$$\begin{aligned}\sqrt{5}F_n &= \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{3-\sqrt{5}}{2}\right) \\ \sqrt{5}F_n &= \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\left(\frac{1-\sqrt{5}}{2}\right)^2 \\ \sqrt{5}F_n &= \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\end{aligned}$$

Which is what we wanted to prove. □

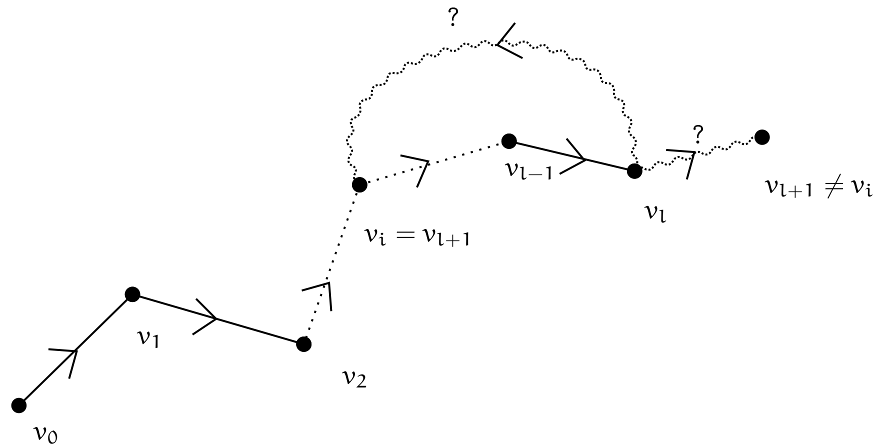
Problem 3.

Solution. In a connected simple graph with n vertices, the possible degrees of a vertex range from 1 (since the graph is simple and every vertex is connected to at least one other vertex) to $n-1$, in the case where the vertex is connected to every other vertex. So, if we have n vertices, and $n-1$ options for the degree of a vertex, by the pigeonhole principle, at least two vertices must have the same degree. There may be more, but there must be at least two.

Now, suppose the graph was not connected, and suppose that there were j vertices which do not connect to any other vertices. The possibilities for the degree of a vertex range from 0, for the disconnected vertices, to $n-j-1$. However, we have $n-j$ connected vertices, so at least two of them must have the same degree. □

Problem 5.

Solution. Suppose that X did *not* contain a cycle. If it did not, we could find a path of maximum length, say of length l. However, the last vertex in this path, say v_l, is the tail of at least one edge leading to another vertex say v_{l+1} by assumption. That means that we have two cases: either $v_{l+1} \neq v_i, i \in \{0, 1, \ldots, l\}$, in which case we did not have a path of maximum length, contradicting our assumption, or $v_{l+1} = v_i$ for some i, which means that we do have a cycle.



□

Problem 6.

Solution.

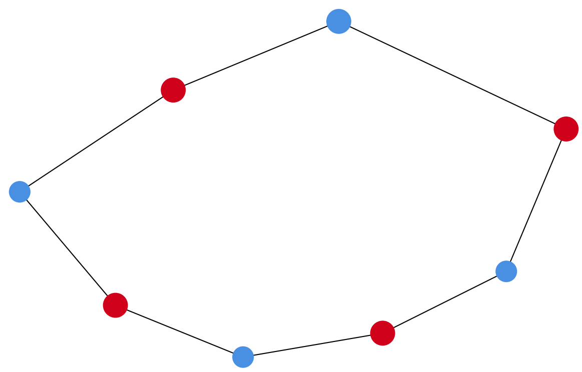
- ⇒ First, assume that we have an eulerian circuit in some graph X. We can select some vertex v as the starting point of our circuit. As we traverse a path through the graph, we must enter and exit each vertex we encounter which is not v. Each entrance and exit accounts for two edges, so $d^+(u) = d^-(u)$ for all $u \neq v$. We must also end at v to be a closed path, so $d^+(v) = d^-(v)$, and the indegree and outdegree of all vertices is the same.

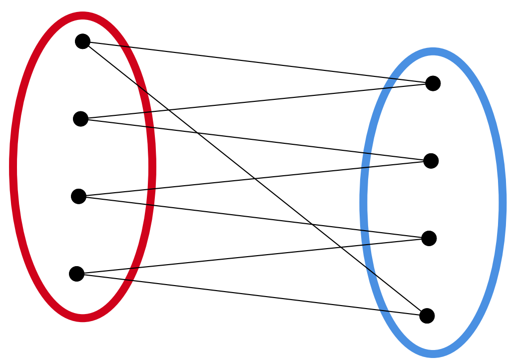
- ⇐ Now, suppose that we have $d^+(v) = d^-(v)$ for all $v \in X$. We prove that X contains an Eulerian circuit, by induction. The simplest case where we have $d^+(v) = d^-(v)$ is when we have 3 edges, a simple graph in the shape of a triangle. Now, we assume that the statement is true for a graph with $m - 1$ edges, and use that to show that it is true for graphs with m edges. Since the graph is connected, the indegree and outdegree of every vertex is at least 1, and therefore the graph must contain a cycle by question 5. We can remove the edges of the cycle to obtain a graph with $|E(x)| = \ell < m$. Since we have removed an even number of edges from each vertex (one in, one out), the remaining graph still has that $d^+(v) = d^-(v)\ \forall\ v$. It is composed of smaller circuits, each of which have Eulerian circuits by our inductive hypothesis. Traversing the edges of the circuit we removed, and then going through the Eulerian circuits as we encounter them gives us an Eulerian circuit on the whole graph, and the statement is proved.

□

Problem 7.

Solution. If n is even, then we can separate the vertices into the even-numbered ones, c_{2j}, and the odd-numbered ones, c_{2j+1}. Clearly, each even vertex will touch two odd vertices, and each odd vertex will touch two even vertices. We can colour the even vertices blue, and the odd vertices red, to make the graph bipartite. For example, in the case $n = 8$:





$\square$

Problem 8. a) We know by the binomial theorem that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

We can derive this with respect to x:

$$\frac{d}{dx}(1+x)^n = \frac{d}{dx}\sum_{k=0}^{n} \binom{n}{k} x^k$$
$$n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} kx^{k-1}$$

Which is what we were trying to prove.

b) First, we note that the quantity we are trying to evaluate is

$$\sum_{A\subseteq[n]} |A|^2 = \sum_{k=0}^{n} \binom{n}{k} k^2$$

Since $\sum_{k=0}^{n}\binom{n}{k}$ counts the number of subsets of a set of size n, so to count the sizes of subsets squared, we multiply by k^2. Now, recall that we proved

$$n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} kx^{k-1} \tag{1}$$

. We can derive again to obtain

$$\frac{d}{dx}n(1+x)^{n-1} = \frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}kx^{k-1}$$
$$n(n-1)(1+x)^{n-2} = \sum_{k=0}^{n}\binom{n}{k}k(k-1)x^{k-2}$$

Now, we can set $x = 1$:

$$(n^2-n)(2^{n-2}) = \sum_{k=0}^{n}\binom{n}{k}(k^2-k)$$
$$(n^2-n)(2^{n-2}) = \sum_{k=0}^{n}\binom{n}{k}k^2 - \sum_{k=0}^{n}\binom{n}{k}k$$

And we can note that, by setting $x = 1$ in (1), we obtain

$$n2^{n-1} = \sum_{k=0}^{n}\binom{n}{k}k$$

So we can write

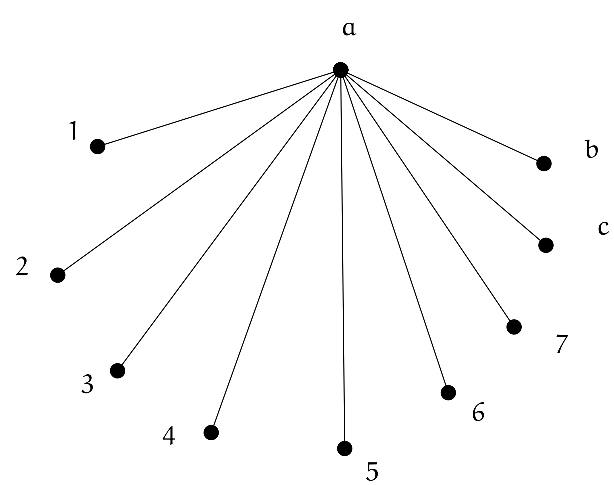
$$(n^2-n)2^{n-2} = \sum_{k=0}^{n}\binom{n}{k}k^2 - n2^{n-1}$$
$$n^2 2^{n-2} - n2^{n-2} + n2^{n-1} = \sum_{k=0}^{n}\binom{n}{k}k^2$$
$$n^2 2^{n-2} + n2^{n-2} = \sum_{k=0}^{n}\binom{n}{k}k^2$$
$$(n^2+n)2^{n-2} = \sum_{k=0}^{n}\binom{n}{k}k^2$$

So therefore,

$$\sum_{A\subseteq[n]}|A|^2 = (n^2+n)2^{n-2}$$

Problem 9.

Solution. There is no such simple graph. Since one vertex has degree 9, it is connected to every other vertex, in this graph called vertex a.



Two of the other vertices have degree 1, so no more edges can connect to them (vertices b, c). However, there are only 7 more free vertices, and one of the vertices must have degree 8, so there can be not such graph.

$\square$