critical line \( \text{Re} \zeta = \frac{1}{2} \) does not contain an arithmetical progression. It follows that \( f_{a,b} \) is not constant.

Case 4. \( a = \frac{1}{2} \). In this case we invoke a theorem of Putnam [2] saying that the set of zeros of \( \zeta(s) \) on the critical line \( \text{Re} \zeta = \frac{1}{2} \) does not contain an arithmetical progression. It follows that also in this case \( f_{a,b} \) cannot be constant.

Case 5. \( 0 < a < \frac{1}{2} \). Because of the functional equation for \( \zeta(s) \) this case may be reduced to Case 3.

Summarizing, we have the following

Theorem. If \( a \) and \( b \) are positive constants and \( f_{a,b} : \mathbb{R} \to \mathbb{R} \) is defined by (3) then \( f_{a,b} \) is a constant function only in case \( a = b = \frac{\log 2}{k} \), where \( k \) is any positive integer.

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References


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Some results in number theory, I

by

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Dedicated to the memory of Professor Paul Turán

Let \( \varphi(n) \) denote Euler's totient function and \( V(n) \) the number of distinct prime factors of \( n \). In this paper, we shall study the quantity \( V(n, \varphi(n)) \) which arises naturally in group theory. For example, letting \( G(n) \) denote the number of non-isomorphic groups of order \( n \), we have a classical result of Burnside that \( G(n) = 1 \) if and only if \( V(n, \varphi(n)) = 0 \) (i.e. \( n, \varphi(n) = 1 \)). Erdős [1] showed that the number \( F_k(x) \) of \( n \leq x \) satisfying the latter condition is

\[
F_k(x) = (1 + o(1)) \frac{x^{1-\gamma}}{k \log x}
\]

where \( \gamma \) is Euler's constant and we write \( \log x = \log x \), \( \log x = \log (\log x) \). More generally, we can define \( F_k(x) \) to be the number of \( n \leq x \) for which \( G(n) = k \). The authors [2] have shown that for each \( k \),

\[
F_k(x) \ll x/ \log x.
\]

The proof depended essentially on a weak form of the following result stated by Erdős in [1]: for each \( \varepsilon > 0 \), the number of \( n \leq x \) that fail to satisfy

\[
(1 - \varepsilon) \log n < V(n, \varphi(n)) < (1 + \varepsilon) \log n
\]

is \( o(n) \). (A proof of this was supplied by the authors in [2].)

It is an interesting number-theoretic problem to estimate the number \( A_k(x) \) of \( n \leq x \) for which \( V(n, \varphi(n)) = k \). Our main result here is the following theorem.

Theorem. For each \( k \geq 0 \), we have

\[
A_k(x) = \frac{(1 + o(1)) x^{1-\gamma} (\log x)^k}{k \log x}.
\]

The proof will require several lemmas and intermediate results. The first two lemmas are due to Erdős [1].

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LEMMA 1. Let \( p \) be a fixed prime. Then
\[
\sum_{s \leq 2} \frac{1}{s} \ll \frac{1}{p} (\log p + \log \log p),
\]
where the asterisk indicates that the sum is over primes \( s \leq x \), \( s = 1 \pmod{p} \).
We remark here that unless otherwise stated, \( p, q \) and \( s \) will denote primes.

LEMMA 2. Let \( p < (\log_4 x)^{1-t} \). Then the number of \( pm \leq x \) such that \( m \)
has no prime divisor \( \equiv 1 \pmod{p} \) is \( o(\sqrt{\log x}) \), uniformly in \( p \).

LEMMA 3. Let \( H_k(x) \) be the number of \( n \leq x \) of the form \( n = p_1 p_2 \ldots p_k m \),
where
(i) \( p_i < (\log_4 x)^{1-t} \), \( i = 1, 2, \ldots, k \),
(ii) all the prime divisors of \( m \) are \( \geq (\log_4 x)^{1-t} \),
(iii) \( (m, \varphi(m)) = 1 \).
Then for each fixed \( k \),
\[
H_k(x) = \frac{[1 + o(1)] x e^{-\gamma} (\log_4 x)^k}{k \log_4 x}.
\]

Proof. By definition,
\[
H_k(x) = \sum_p \sum_{m} \frac{1}{m} \sum \frac{1}{p_1 \ldots p_k \log_4 x}
\]
where the outer sum is over all \( p_i < (\log_4 x)^{1-t} \) \( (1 \leq i \leq k) \) and the inner sum is over \( m \leq x/p_1 \ldots p_k \) satisfying (ii) and (iii). Erdős' proof of (1) shows that
\[
\sum_{m} \frac{1}{m} = \frac{[1 + o(1)] e^{-\gamma}}{(p_1 p_2 \ldots p_k \log_4 x)}
\]
and as the product \( p_1 \ldots p_k \) is obtained \( k \) times in the \( k \)-fold outer sum
of (3), we get
\[
H_k(x) = \frac{[1 + o(1)] x e^{-\gamma} (\log_4 x)^k}{k \log_4 x}
\]
proving the lemma.

We are now ready to prove our theorem.

Proof of theorem. We shall give the proof for \( k = 1 \) and then sketch the modifications needed for general \( k \). Write
\[
A(x) = A_1(x) + A_2(x)
\]
where \( A_1(x) \) counts the contribution of squarefree \( n \) to \( A(x) \) and \( A_2(x) \) counts the remaining \( n \). First we estimate \( A_1(x) \). If \( n \) is not squarefree and \( V(n, \varphi(n)) = 1 \) then certainly \( n = p^a m \) \( (a \geq 2) \) with \( (p, m) = 1 \) and
\[
(m, \varphi(m)) = 1. \text{ The number of such } n \leq x \text{ with } p \geq \log_4 x = y \text{ (say) is clearly}
\]
\[
\ll \sum_{p \geq y} \sum_{m \leq x/p} \frac{x}{p^a} \ll \frac{x}{\log_4 x},
\]
and the number of remaining non-squarefree \( n \) in \( A_1(x) \) is
\[
\ll \sum_{p \geq x} \sum_{m \leq x/p} A(x/p) \ll \frac{x}{\log_4 x},
\]
using (1). Thus, we have
\[
A_1(x) \ll x \log_4 x.
\]
If \( n \) is squarefree, then \( V(n, \varphi(n)) = 1 \) implies that
\[
\sum_{n \leq x} A_1(x) = \sum_{n \leq x} A_1(x) \ll \frac{x}{\log_4 x}.
\]
Let \( \varepsilon > 0 \) be fixed. Then, we write
\[
A_1(x) = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4
\]
where the sums are over those \( n \leq x \) in \( A_1(x) \) of the form (7) and
\[
in \Sigma_1, \quad p > (\log_4 x)^{1-t};
\]
\[
in \Sigma_2, \quad (\log_4 x)^{1-t} \leq p \leq (\log_4 x)^{1-t} ;
\]
\[
in \Sigma_3, \quad p < (\log_4 x)^{1-t} \text{ and at least one prime divisor of } m \text{ is}
\]
\[
< (\log_4 x)^{1-t};
\]
\[
in \Sigma_4, \quad p < (\log_4 x)^{1-t} \text{ and all the prime divisors of } m \text{ are } > (\log_4 x)^{1-t}.
\]
Clearly, we have by Lemma 1,
\[
\Sigma_1 \ll \sum_{x/p} \sum_{p \geq x} \frac{x}{p^a} \ll \sum_{p < x} \frac{x}{p^{\varepsilon}} \ll \frac{x}{(\log_4 x)^{1-t}}.
\]
Also, we get from (1) that
\[
\Sigma_2 \ll \sum_{x/p} \sum_{p \geq x} A(x/p) \ll \sum_{p \geq x} \frac{x}{p} \ll x \log_4 x.
\]
where all the sums are over \( p \) in the range indicated for \( \Sigma_2 \). Now from Lemma 2, the number of \( m \leq x \) \( (m, \varphi(m)) = 1 \) which have a prime divisor
\[
< (\log_4 x)^{1-t}
\]
is
\[
o(\sigma/\log_4 x)^{1-t} = o(x/\log_4 x).
\]
Hence,
\[
\Sigma_3 = o \left( \frac{x}{\log_4 x} \sum_{p} \frac{1}{p} \right) \ll \frac{x}{\log_4 x}.
\]
where the sum is over \( p < (\log x)\gamma \). For \( \Sigma_1 \), we write

\[
\Sigma_1 = \Sigma_1' + \Sigma_1''
\]

where in the first sum, all the prime divisors of \( m \) are \( (\log x)^{\gamma} \), and the second sum contains the remaining \( n \) of \( \Sigma_1 \). Thus, for the \( n \) in \( \Sigma_1'' \), there is a prime divisor \( q \) (say) with

\[
(\log x)^{\gamma} < q < (\log x)^{\gamma+1}
\]

so

\[
\Sigma_1'' \leq \sum_p \sum_q A_0(p|q) \leq \frac{x}{\log x} \sum_p \frac{1}{pq}
\]

where the sum over \( q \) is in the range (12) and the sum over \( p \) is in the range specified for \( \Sigma_1 \). The sum over \( q \) is clearly \( < \epsilon \) so we get

\[
\Sigma_1'' < o(x \log x / \log_2 x).
\]

Finally, recalling the definition of \( H_k(x) \) from Lemma 3, noting that our \( n \) are now squarefree, and that in the range of \( \Sigma_1 \), every number in (7) satisfies \( V(n, \varphi(n)) = 1 \), we get

\[
H_1(x) \geq \Sigma_1' \geq H_1(x) - T(x)
\]

where \( T(x) \) is the number of \( pm = n \leq x \) such that \( m \) has no prime divisor \( \equiv 1 \pmod{p} \) and \( p \) is in the range specified for \( \Sigma_1 \). Lemmas 2 and 3 imply that

\[
\Sigma_1' = \frac{[1 + o(1)]a e^{-\gamma} \log_2 x}{\log x}
\]

so that combining (5), (6), (8)-(11), (13) and (14), and noting that \( \epsilon > 0 \) was arbitrary, the proof for \( k = 1 \) is completed.

Now we sketch the modifications needed in the above proof, for general \( k \). As before, we write \( A_k(x) = A'_k(x) + A''_k(x) \) using the notation as in (5). Recalling \( y = \log x \), we get

\[
A'_k(x) \leq \sum_p \sum_{a \leq x/p^k} A_{k-1} \left( \frac{x}{p^k} \right) + \frac{x}{y} \leq \frac{a(x \log x)^{k-1}}{\log x}
\]

by induction. To estimate \( A''_k(x) \), we write the \( n \leq x \) that are counted, in the form

\[
n = q_1 \cdots q_k m, \quad (m, \varphi(m)) = 1 \quad \text{and} \quad (n, \varphi(n)) = q_1 \cdots q_k.
\]

Then as before,

\[
A''_k(x) = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4
\]

where now, the sums are over those \( n \leq x \) of the form (16) and, in \( \Sigma_1 \), some \( p_i > (\log x)^{\gamma+1} \), in \( \Sigma_2 \), some \( p_i \) satisfies \( (\log x)^{\gamma} < p_i < (\log x)^{\gamma+1} \), in \( \Sigma_3 \), all \( p_1 < (\log x)^{\gamma} \) and at least one prime divisor of \( m \) is \( > (\log x)^{\gamma} \), in \( \Sigma_4 \), all \( p_i < (\log x)^{\gamma} \) and all the prime divisors of \( m \) are \( > (\log x)^{\gamma} \).

For \( \Sigma_1 \), the estimate (8) holds as before, and also

\[
\Sigma_1 = o(x \log x, \quad \Sigma_1 = o(x \log x)^{k} \log x),
\]

by simple modifications in (9) and (10). Finally, writing \( \Sigma_4 = \Sigma_4' + \Sigma_4'' \) in the same notation as in (11), we find again by a simple modification that

\[
\Sigma_4' = o(x \log x)^{k}/\log x)
\]

and

\[
H_k(x) \geq \Sigma_4' \geq H_k(x) - \sum_{j=0}^{k-1} \frac{A_j(x)}{k!}
\]

so that by Lemma 3, we get from (19) that

\[
\Sigma_4' = \frac{1 + o(1)}{k! \log x}
\]

and combining (15), (8), (17), (18) and (20), the proof of the theorem is complete.

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References
