Proceedings of the International Conference on
Algebraic Groups and Arithmetic, December 2001,
TIFR, Mumbai

Summatory Functions of Elements in Selberg's Class*

Wentang Kuo and M. Ram Murty

Abstract

Let $F(s)$ be a Dirichlet series, $F(s) = \sum_{n=1} a_n n^{-s}$, $\Re s > 1$. Define the the summatory function $S(x)$ to be $\sum_{n\le x} a_n$. We assume that $F(s)$ satisfies the following conditions. First, for all $\epsilon > 0$, $|a_n| = O(n^\epsilon)$. In addition, it admits analytic continuation and functional equation. More precisely, there is a function $\Delta(s) = Q^s \prod \Gamma(\alpha_i s + \gamma_i)$, $Q > 0$, $\alpha_i > 0$, $\Re\gamma_i > 0$, such that $F(s)\Delta(s) = \omega\overline{F}(1-s)\overline{\Delta}(1-s)$, $|\omega| = 1$. Furthermore, assume that $F(s)$ is entire. This paper derives an estimation of $S(x)$ without extra conditions. The trivial estimation is $S(x) = O(x^{1+\epsilon})$, $\forall\epsilon > 0$. Let

$$\theta = \frac{d}{d+2} < 1, \ d = 2\sum \alpha_i.$$

The main theorem is

$$S(x) = O(Q^{1-\theta} x^{\theta+\epsilon}), \ \forall\epsilon > 0.$$

In honour of Professor M.S. Raghunathan

1 Introduction

Let $F(s)$ be a Dirichlet series, i.e., $F(s) = \sum_{n\ge1} a_n n^{-s}$. The summatory function $S(x)$ is defined as follows

$$S(x) = \sum_{n\le x} a_n, \ x \ge 1.$$

We suppose that $F(s)$ extends to a meromorphic functions for $\Re(s) \ge 1$. If $F(s)$ has only a simple pole at $s = 1$ on the line of $\Re(s) = 1$ and satisfies

*Research of the second author was partially supported by an NSERC grant.

some conditions on coefficients, the celebrated Tauberian theorem (see [5]) gives us

$$S(x) = \kappa x + R(x),\ R(x) = o(x), \qquad \text{for some constant } \kappa.$$

To find the exact order of the error term $R(x)$ is, in general, a difficult problem.

It is difficult to deal with general Dirichlet series. However, almost all interesting Dirichlet series admit analytic continuation or meromorphic continuation to the entire complex plane and satisfy a functional equation. More precisely, we assume that there is a function

$$\Delta(s) = Q^s \prod_i \Gamma(\alpha_i s + \gamma_i),\ Q > 0,\ \alpha_i > 0,$$

such that

$$F(s)\Delta(s) = \omega \overline{F}(1-s)\overline{\Delta}(1-s),$$

where ω is a complex number with $|\omega| = 1$, $\overline{F}(s) = \overline{F(\overline{s})}$, and $\overline{\Delta}(s) = \overline{\Delta(\overline{s})}$.

In [1], Chandrasekharan and Narasimhan proved an O-theorem and an Ω-theorem for the error term $R(x)$. The second author in [4] used such theorems to get the an estimation of eigenvalues of Hecke operators.

Now we assume that $F(s)$ is entire and satisfies the Ramanujan hypothesis, i.e, for all $\epsilon > 0$, $|a_n| = O(n^\epsilon)$. The main term in the summatory function disappears. By the Ramanujan hypothesis, the trivial estimation of $S(x)$ is $O(x^{1+\epsilon})$, $\forall \epsilon > 0$. Define τ_F to be the smallest value of ξ such that

$$S(x) = O(x^{\xi+\epsilon}),\ \forall \epsilon > 0.$$

To determine τ_F is not an easy job. If we can find some upper bound θ of τ_F less than 1, then the Dirichlet series $\sum_{n\geq 1} a_n n^{-s}$ will converge at $\Re(s) > \theta$ (cf. Lemma 5.1.) The importance of this fact is that we can estimate the size of $F(s)$ inside the the critical strip, which cannot be inferred from the functional equations.

Let $d = 2\sum \alpha_i$ be the degree of $F(s)$ and Q the conductor appearing in the functional equation. The main theorem of this paper is

Theorem (Theorem 4.1) *Keep all assumptions in this section. We have*

$$S(x) = O(Q^{1-\theta} x^{\theta+\epsilon}),\ \theta = \frac{d}{d+2} = 1 - \frac{2}{d+2} < 1, \quad \forall \epsilon > 0.$$

We should mention here that for $d \geq 2$, the method in [1] can produce a better result of exponent in x; more precisely, for all $\epsilon > 0$,

$$S(x) = O(x^{\theta'+\epsilon}), \quad \theta' = \frac{d-1}{d+1}.$$

However, our method has no restriction on d and is different and simpler than their method. In addition, we include the dependence of the conductor in our result. In [2], the first author works out the computation of the conductor independence by the method in [1]. The result is, for all $\epsilon > 0$,

$$S(x) = O(Q^{1-\theta'} x^{\theta'+\epsilon}), \quad \theta' = \frac{d-1}{d+1}.$$

Therefore, our method can produce a better estimate on the exponent of conductors. The interesting phenomenon is that the sum of two exponents is always equal to one.

The organization of this paper is as follows. We set up the notations in the second section and state some results and tools in the third section. And then we prove our main theorem in the fourth section. In the final section we discuss some applications and variations.

Acknowledgments. The first author would like to thank Queen's University for its hospitality during the summer of 2001 when this work was started. The authors also thank the referee for useful remarks.

2 Notation

In this paper, we consider a set S' of complex functions $F(s)$ satisfying the following conditions:

(i) (Dirichlet series)

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

a Dirichlet series absolutely convergent for $\Re s > 1$;

(ii) (Ramanujan hypothesis) $\forall \epsilon > 0$, $a_n = O(n^\epsilon)$ and the constant only depending on F and ϵ;

(iii) (Analytic continuation) $F(s)$ extends to an entire function of finite order;

(iv) (Functional equation) it has a functional equation of the form:

$$\Phi(s) = \omega \overline{\Phi}(1-s),$$

where $\omega \in \mathbb{C}$, $|\omega| = 1$ and

$$\Phi(s) = \Delta(s)F(s), \ \Delta(s) = Q^s \prod_{i=1}^{m} \Gamma(\alpha_i s + \gamma_i),$$

and $\overline{\Phi}(s) = \overline{\Phi(\overline{s})}$. Here $Q > 0$, $\alpha_i > 0$, $\Re(\gamma_i) \geq 0$, and m is a natural integer. The number Q is called the *conductor*.

Remark Here our conditions are the same as the Selberg class S defined in [8], except for the Euler product condition and the admission of poles at $s = 1$.

Let d_F be *the degree of* F defined by

$$d_F = 2\sum_{i=1}^{m} \alpha_i.$$

Define the *summatory function* $S_F(x)$ of $F(s)$ by

$$S_F(x) = \sum_{n \leq x} a_n.$$

Sometimes we drop the subscript F if there is no confusion.

Let $F(s)$ be defined as above. We define a new Dirichlet series $F_\zeta(\sigma)$, $\sigma > 1$ by

$$F_\zeta(\sigma) = \sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma}.$$

Obviously we have $|F(\sigma + it)| \leq F_\zeta(\sigma)$, $\sigma > 1$.

We shall adopt the following notation:

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s)ds$$

will be abbreviated to

$$\int_{(c,T)} f(s)ds.$$

If T takes values ∞, we just simply drop it and write it as $\int_{(c)}$.

3 Preliminaries

In this section we state the results and tools needed in our proof. The main tool is the method of contour integration.

For $x > 0$, define a function $\delta(x)$ as follows

$$\delta(x) = \begin{cases} 0 & \text{if } 0 < x < 1, \\ \frac{1}{2} & \text{if } x = 1, \\ 1 & \text{if } x > 1. \end{cases}$$

The following lemma is deduced from standard contour integration (see for example, [5]).

Lemma 3.1 *For any $x, c > 0$, we have*

$$\delta(x) = \int_{(c)} \frac{x^s}{s} ds.$$

Furthermore, if we use the truncated integration $\int_{(c,T)} (x^s/s) ds$, $T > 0$, then the error term can be controlled as follows

$$\left| \int_{(c,T)} \frac{x^s}{s} ds - \delta(x) \right| \ll \begin{cases} x^c \min\left(1, \frac{1}{T|\log x|}\right) & \text{if } x \neq 1, \\ \frac{c}{T} & \text{if } x = 1. \end{cases}$$

An immediate corollary of the previous lemma is

Corollary 3.2 (Perron's formula) *Let $f(s) = \sum_{n \geq 1} a_n n^{-s}$ be a Dirichlet series absolutely convergent for $\Re s > 1$. Then for $x > 1$ not an integer and $c > 1$, $T > 0$,*

$$S(x) = \sum_{n \leq x} a_n = \int_{(c,T)} f(s) \frac{x^s}{s} ds + O\left(\sum_{n=1}^{\infty} \left(\frac{x}{n}\right)^c |a_n| \min\left(1, \frac{1}{T|\log \frac{x}{n}|}\right) \right).$$

If we use $\Gamma(s)$ instead of $1/s$ as the kernel of integration, we have:

Lemma 3.3 *For any $\sigma > 1$ and $x > 0$,*

$$e^{1/x} = \int_{(\sigma)} x^w \Gamma(w) dw.$$

Similar to Perron's formula, we have

Corollary 3.4 *Let $f(s) = \sum_{n=1} a_n n^{-s}$ be a Dirichlet series absolutely convergent for $\Re s > 1$. For any c and $s \in \mathbb{C}$ so that $c > 1 - \Re(s)$, we have*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{-n/x} = \int_{(c)} f(s+w) x^w \Gamma(w) dw.$$

In 1957, Rademacher [6] proved a sharper version of the Phragmén-Lindelöf theorem. The significance of this result is that it provides precise inequalities satisfied by a general L-function inside the critical strip. The version that he proved is the following:

Proposition 3.5 (Rademacher [7], Theorem 2) *Let $f(s)$ be analytic in the strip $S(a,b) = \{s \in \mathbb{C} | a < \Re s < b\}$ and of finite order. Suppose moreover that*

$$\begin{cases} |f(a+it)| \le E|P+a+it|^{\alpha}, \\ |f(b+it)| \le F|P+a+it|^{\beta}, \end{cases}$$

with

$$P + a > 0, \quad \alpha \ge \beta.$$

Then in the strip $S(a,b)$

$$|f(s)| \le (E|P+s|^{\alpha})^{\frac{b-\sigma}{b-a}} (F|P+s|^{\beta})^{\frac{\sigma-a}{b-a}},$$

where $s = \sigma + it$.

In order to get the control of growth of $F(s)$, we need to know the asymptotic behavior of Γ functions. Fortunately, we have the Stirling formula as follows:

$$|\Gamma(\sigma+it)| \sim e^{-\frac{1}{2}\pi|t|} |t|^{\sigma-\frac{1}{2}} \sqrt{2\pi},$$

as $|t| \to \infty$, uniformly in $-\infty < \lambda_1 \le \sigma \le \lambda_2 < \infty$.

Lemma 3.6 *For all $\sigma > 1, t \in \mathbb{R}$,*

$$|F(1-\sigma+it)| \le C' Q^{2\sigma-1} (|t|+2)^{\frac{d}{2}(2\sigma-1)} F_{\zeta}(\sigma),$$

where C' is a constant that only depends on α_i and γ_i.

Proof By the functional equation and Stirling's formula, we have

$$\begin{aligned} |F(1-\sigma+it)| &= \frac{Q^{\sigma}}{Q^{1-\sigma}} \cdot \prod_{i=1}^{m} \frac{|\Gamma(\alpha_i(\sigma+it)+\overline{\gamma_i})|}{|\Gamma(\alpha_i(1-\sigma+it)+\gamma_i)|} \cdot |F(\sigma-it)| \\ &\le Q^{2\sigma-1} \cdot C'(|t|+2)^{\sum_{i=1}^{m} \alpha_i(2\sigma-1)} \cdot F_{\zeta}(\sigma) \\ &= C' Q^{2\sigma-1} (|t|+2)^{\frac{d}{2}(2\sigma-1)} F_{\zeta}(\sigma). \end{aligned}$$

Now we apply the sharper Phragmén-Lindelöf theorem on $F(s)$.

Lemma 3.7 *Let $\sigma > 1$. For all $t \in \mathbb{R}$ and $1-\sigma \le \eta \le \sigma$,*

$$|F(\eta+it)| \le C\big(Q(|t|+2)^{\frac{d}{2}}\big)^{\sigma-\eta} F_\zeta(\sigma),$$

where C is a constant that only depends on α_i and γ_i.

Proof From the previous lemma, we know

$$|F(1-\sigma+it)| \le CQ^{2\sigma-1}(|t|+2)^{\frac{d}{2}(2\sigma-1)} F_\zeta(\sigma),$$

where C is a constant that only depends on d, α_i and γ_i. On the other hand,

$$|F(\sigma+it)| \le F_\zeta(\sigma).$$

If we apply Proposition 3.5, we get the result.

Remark If $F_\zeta(\sigma)$ has meromorphic continuation to $\Re(\sigma) \ge 1$ with only a pole at $\sigma = 1$ of order k, we can get a better estimation. Since we know

$$F_\zeta(\sigma) \sim \frac{C}{(\sigma-1)^k},$$

we choose

$$\sigma - 1 = k \cdot \Big(\log\big(Q(|t|+2)^{\frac{d}{2}}\big)\Big)^{-1}.$$

Then

$$\begin{aligned} |F(\eta+it)| &\le C\big(Q(|t|+2)^{\frac{d}{2}}\big)^{\sigma-\eta} F_\zeta(\sigma) \\ &\le C\big(\frac{e}{k}\big)^k \big(Q(|t|+2)^{\frac{d}{2}}\big)^{1-\eta}\Big(\log\big(Q(|t|+2)^{\frac{d}{2}}\big)\Big)^k. \end{aligned}$$

This is the formula appearing in [4]. Note that the choice of σ here is best.

The Lindelöf hypothesis on S' is a conjecture which predicts the behavior of $F(s)$ in the critical strip. The statement is the following:

Let $F(s) \in S'$. For all $\epsilon > 0$ and $\eta \ge 1/2$,

$$|F(\eta+it)| = O(t^\epsilon Q^\epsilon),$$

where the constants depends only on α_i, γ_i and ϵ.

This is still a major open question in the theory.

4 The Main O-theorem

Now we state the main theorem.

Theorem 4.1 *Let $F(s)$ be the Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ that satisfies the conditions in §2. Then $S(x) = \sum_{n\le x} a_n = O(Q^{1-\theta}x^{\theta+\epsilon})$, $\forall\epsilon > 0$. Here $\theta = \frac{d}{d+2} < 1$.*

Proof Given $c > 1$ and $0 < \eta < c$. By Corollary 3.2 (Perron formula) and the entireness of $F(s)$, we have

$$
\begin{aligned}
S(x) =& \int_{(c,T)} F(s)\frac{x^s}{s}ds + O\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^c |a_n| \min\left(1, \frac{1}{T|\log\frac{x}{n}|}\right)\right) \\
=& -\int_{(\eta,T)} F(s)\frac{x^s}{s}ds + \frac{1}{2\pi i}\left(\int_{\eta-iT}^{c-iT} F(s)\frac{x^s}{s}ds - \int_{\eta+iT}^{c+iT} F(s)\frac{x^s}{s}ds\right) \\
&+ O\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^c |a_n| \min\left(1, \frac{1}{T|\log\frac{x}{n}|}\right)\right) \\
=& E_1 + E_2 + E_3.
\end{aligned}
$$

Here

$$
\begin{aligned}
E_1 &= -\int_{(\eta,T)} F(s)\frac{x^s}{s}ds, \\
E_2 &= \frac{1}{2\pi i}\left(\int_{\eta-iT}^{c-iT} F(s)\frac{x^s}{s}ds - \int_{\eta+iT}^{c+iT} F(s)\frac{x^s}{s}ds\right), \\
E_3 &= O\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^c |a_n| \min\left(1, \frac{1}{T|\log\frac{x}{n}|}\right)\right).
\end{aligned}
$$

We deal with each term separately. For E_1,

$$
\begin{aligned}
|E_1| &= \left|\int_{(\eta,T)} F(s)\frac{x^s}{s}ds\right| \le \frac{1}{2\pi}\int_{-T}^{T} |F(\eta+it)|\frac{x^\eta}{|\eta+it|}dt \\
&\le \frac{C}{2\pi}F_\zeta(c)Q^{c-\eta}x^\eta \int_{-T}^{T}(|t|+2)^{\frac{d}{2}(c-\eta)}|\eta+it|^{-1}dt \\
&\le \frac{C}{2\pi}F_\zeta(c)\eta^{-1}Q^{c-\eta}x^\eta \int_{-T}^{T}(|t|+2)^{\frac{d}{2}(c-\eta)-1}dt \\
&\le \frac{2C}{2\pi}F_\zeta(c)\eta^{-1}Q^{c-\eta}x^\eta(T+2)^{\frac{d}{2}(c-\eta)} = C_1 x^\eta\left(Q(T+2)^{\frac{d}{2}}\right)^{c-\eta},
\end{aligned}
$$

where

$$C_1 = \frac{C}{\pi} F_\zeta(c) \eta^{-1}.$$

For E_2, by Lemma 3.7, we have

$$|E_2| \le \frac{1}{2\pi}\left(\left|\int_{\eta+iT}^{c+iT} F(s)\frac{x^s}{s}ds\right| + \left|\int_{\eta-iT}^{c-iT} F(s)\frac{x^s}{s}ds\right|\right)$$

$$\le \frac{2C}{2\pi} F_\zeta(c) \int_\eta^c Q^{c-\lambda}(T+2)^{\frac{d}{2}(c-\lambda)} x^\lambda |\lambda+iT|^{-1} d\lambda$$

$$\le \frac{C}{\pi} F_\zeta(c)\eta^{-1} Q^c (T+2)^{\frac{dc}{2}-1} \int_\eta^c \left(\frac{x}{Q(T+2)^{\frac{d}{2}}}\right)^\lambda d\lambda$$

$$\le \frac{C}{\pi} F_\zeta(c)\eta^{-1} Q^c (T+2)^{\frac{dc}{2}-1} \left|\log\left(\frac{x}{Q(T+2)^{\frac{d}{2}}}\right)\right|^{-1} \left|\left(\left(\frac{x}{Q(T+2)^{\frac{d}{2}}}\right)^c - \left(\frac{x}{Q(T+2)^{\frac{d}{2}}}\right)^\eta\right)\right|$$

$$\le \begin{cases} \frac{C}{\pi}(\log\frac{3}{2})^{-1} F_\zeta(c)\eta^{-1}\Big(\frac{x^c}{T+2} + x^\eta \big(Q(T+2)^{\frac{d}{2}}\big)^{c-\eta} T^{-1}\Big) & \text{if } \frac{x}{Q(T+2)^{\frac{d}{2}}} \ge \frac{3}{2} \text{ or } \frac{x}{Q(T+2)^{\frac{d}{2}}} \le \frac{2}{3}, \\ \frac{3C}{\pi} F_\zeta(c)\eta^{-1}(c-\eta) Q^c (T+2)^{\frac{dc}{2}-1} & \text{if } \frac{2}{3} \le \frac{x}{Q(T+2)^{\frac{d}{2}}} \le \frac{3}{2}, \end{cases}$$

$$\le \begin{cases} C_2\Big(\frac{x^c}{T+2} + x^\eta \big(Q(T+2)^{\frac{d}{2}}\big)^{c-\eta} T^{-1}\Big) & \text{if } \frac{x}{Q(T+2)^{\frac{d}{2}}} \ge \frac{3}{2} \text{ or } \frac{x}{Q(T+2)^{\frac{d}{2}}} \le \frac{2}{3}, \\ C_2(c-\eta) Q^c (T+2)^{\frac{dc}{2}-1} & \text{if } \frac{2}{3} \le \frac{x}{Q(T+2)^{\frac{d}{2}}} \le \frac{3}{2}. \end{cases}$$

Here

$$C_2 = \frac{3C}{\pi} F_\zeta(c)\eta^{-1}.$$

For E_3, we choose $x \in \mathbb{N} + \frac{1}{2}$. Then

$$\begin{aligned}
|E_3| &\le B\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^c |a_n| \min\left(1, \frac{1}{T|\log\frac{x}{n}|}\right)\right) \\
&\le B\left(\sum_{n=1}^{\infty}\left(\frac{x}{n}\right)^c |a_n| \left(\frac{1}{T|\log\frac{x}{n}|}\right)\right) \\
&\le B(\log 2)^{-1}\left(\frac{x^c}{T}\sum_{n=1}^{\infty}\frac{|a_n|}{n^c} + (\log 2)^{-1}2^c \sum_{\frac{1}{2}x\le n\le 2x} |a_n| |\log\frac{x}{n}|^{-1}\right) \\
&\le B(\log 2)^{-1}\left(F_\zeta(c)\frac{x^c}{T} + 10\cdot 2^c\frac{1}{T}\sum_{\frac{1}{2}x\le n\le 2x}|a_n|\frac{n}{|x-n|}\right)
\end{aligned}$$

$$\le B(\log 2)^{-1}\left(F_\zeta(c)\frac{x^c}{T} + 2(\log 2)^{-1}\cdot 2^c\frac{x}{T}\cdot\max\{|a_n|, \frac{1}{2}x\le n\le 2x\}\sum_{x+\frac{1}{2}\le n\le 2x}\frac{1}{x-n}\right)$$

$$\le B(\log 2)^{-1}\left(F_\zeta(c)\frac{x^c}{T} + 2(\log 2)^{-1}\cdot 2^c\frac{x}{T}\cdot\max\{|a_n|, \frac{1}{2}x\le n\le 2x\}\sum_{0\le j\le x-\frac{1}{2}}\frac{1}{\frac{1}{2}+j}\right)$$

$$\le B(\log 2)^{-1}\left(F_\zeta(c)\frac{x^c}{T} + 2(\log 2)^{-2}\cdot 2^c\frac{x\log x}{T}\max\{|a_n|, \frac{1}{2}x\le n\le 2x\}\right)$$

$$\le 2B\left(F_\zeta(c)\frac{x^c}{T} + 8C_\epsilon\cdot 2^c\frac{x^{1+\epsilon}\log x}{T}\right)$$

Here C_ϵ is a constant depending on ϵ by Ramanujan hypothesis and B is the constant from Corollary 3.2. We choose $\epsilon = (c-1)/2$. Then

$$\begin{aligned}
|E_3| &\le 2B\left(F_\zeta(c)\frac{x^c}{T} + 8C_\epsilon\cdot 2^c\frac{x^c\cdot(x^{(1-c)/2}\log x)}{T}\right) \\
&\le 2B\left(F_\zeta(c)\frac{x^c}{T} + 8C_c\cdot 2^c\frac{x^c}{T}\right) \\
&\le C_3\frac{x^c}{T}.
\end{aligned}$$

Here

$$C_3 = 2B(F_\zeta(c) + 8C_c\cdot 2^c), \quad \text{and } C_\epsilon = C_c.$$

To sum up,

$$|S(x)| \le |E_1| + |E_2| + |E_3|$$
$$\le \begin{cases} C_1 x^{\eta}\left(Q(T+2)^{\frac{d}{2}}\right)^{c-\eta} + C_2\left(\frac{x^c}{T+2} + x^{\eta}\left(Q(T+2)^{\frac{d}{2}}\right)^{c-\eta}T^{-1}\right) + C_3\frac{x^c}{T} \\ \qquad \text{if } \frac{x}{Q(T+2)^{\frac{d}{2}}} \ge \frac{3}{2} \quad \text{or} \quad \frac{x}{Q(T+2)^{\frac{d}{2}}} \le \frac{2}{3}, \\ C_1 x^{\eta}\left(Q(T+2)^{\frac{d}{2}}\right)^{c-\eta} + C_2(c-\eta)Q^c(T+2)^{\frac{dc}{2}-1} + C_3\frac{x^c}{T} \\ \qquad \text{if } \frac{2}{3} \le \frac{x}{Q(T+2)^{\frac{d}{2}}} \le \frac{3}{2}. \end{cases}$$

Assume $T \ge 2$. We choose our T wisely such that $\frac{x}{Q(T+2)^{\frac{d}{2}}} \ge \frac{3}{2}$ or $\frac{x}{Q(T+2)^{\frac{d}{2}}} \le \frac{2}{3}$. We obtain

$$|S(x)| \le A_1 x^{\eta}\left(Q(T+2)^{\frac{d}{2}}\right)^{c-\eta} + A_2\frac{x^c}{T+2},$$

where

$$A_1 = C_1 + \frac{C_2}{2}, \quad A_2 = C_2 + 2C_3.$$

To minimize the sum, we equalize $x^{\eta}\left(Q(T+2)^{\frac{d}{2}}\right)^{c-\eta}$ and $x^c(T+2)^{-1}$. Set $c-\eta = 1$ and $c = 1+\epsilon$. We have

$$x^{\eta}\left(Q(T+2)^{\frac{d}{2}}\right) = \frac{x^c}{T+2} \Longrightarrow Q(T+2)^{\frac{d}{2}+1} = x \Longrightarrow T+2 = \left(\frac{x}{Q}\right)^{\frac{2}{d+2}}.$$

The main term in the sum is

$$\frac{x^c}{T+2} = \frac{x^{1+\epsilon}}{\left(\frac{x}{Q}\right)^{\frac{2}{d+2}}} = Q^{\frac{2}{d+2}} \cdot x^{\frac{d}{d+2}+\epsilon} = Q^{\frac{2}{d+2}} \cdot x^{\theta+\epsilon}, \ \theta = \frac{d}{d+2}.$$

Finally,

$$|S(x)| \le (A_1 + A_2)Q^{\frac{2}{d+2}} \cdot x^{\theta+\epsilon} \Longrightarrow S(x) = O(Q^{1-\theta}x^{\theta+\epsilon}).$$

This ends the proof.

□

5 Applications

In this section we discuss some applications.

Lemma 5.1 *Let $F(s)$ be a Dirichlet series. Assume that its summatory function $S(x)$ is $O(x^\delta)$. For $\Re(s) > \delta$, the series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges. Note that we do not assume that $F(s)$ converges absolutely for $\Re(s) > \delta$.

Proof Let $F_x(s)$ be $\sum_{n\leq x} \frac{a_n}{n^s}$. It suffices to prove that $F_x(s)$ converges uniformly when x tends to infinity. Let $s = c + it$. By definition,

$$F_x(s) = \int_1^x \frac{1}{t^s} dS(t),$$

where $S(t)$ is the summatory function. By partial summation,

$$\begin{aligned} F_x(s) &= \int_1^x \frac{1}{t^s} dS(t) = \frac{S(x)}{x^s} - \frac{S(1)}{1} + s\int_1^x \frac{S(t)}{t^{s+1}} dt \\ &= s\int_1^\infty \frac{S(t)}{t^{s+1}} dt - S(1) + O(x^{\delta-c}) - \int_x^\infty \frac{S(t)}{t^{s+1}} dt \\ &= A(s) + O(x^{\delta-c}) + O\int_x^\infty \frac{t^\delta}{t^{s+1}} dt = A(s) + O(x^{\delta-c}), \end{aligned}$$

where

$$A(s) = s\int_1^\infty \frac{S(t)}{t^{s+1}} dt - S(1).$$

Therefore, $F_x(s)$ converges to $A(s)$ uniformly when x tends to infinity. This proves our assertion.

Immediately, we have the following corollary.

Corollary 5.2 *If $F(s) \in S'$, then for $\Re s > \theta = d/(d+2)$, the series*

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges.

This corollary gives the information inside the critical strip. There is the other kind of summatory functions using the Γ kernel. Let $F(s) \in S'$. For all $x > 0$ and $s \in \mathbb{C}$, we define the exponential summatory function $S^x(s)$ as follows:

$$S^x(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{-n/x}.$$

By Corollary 3.4 and the entireness of $F(s)$, for any $c > 0$, we obtain

$$S^x(s) = \int_{(c)} f(s+w) x^w \Gamma(w) dw.$$

If we move the line passing 0, we will pick up the residues at 0 and obtain the following lemma:

Lemma 5.3 *Let $F(s) \in S'$. For any $0 < \eta < 1$,*

$$S^x(s) = F(s) + \int_{(-\eta)} F(s+w) x^w \Gamma(w) dw.$$

Apply Lemma 5.3 above on the case for $s = 1$ and $\eta = 1/2$. By the Stirling formula and the Lindelöf hypothesis, we can easily obtain the following corollary.

Corollary 5.4 *Let $F(s) \in S'$. Assume the Lindelöf hypothesis. For $s = 1$ and $\eta = 1/2$, we have*

$$\sum_{n=1}^{\infty} \frac{a_n}{n} e^{-n/x} = S^x(1) = F(1) + O(x^{-1/2} Q^{\epsilon}).$$

Here the constant implied by the 'O' symbol is absolute.

These results will be useful in the study of average values of families of elements of the Selberg class. We hope to undertake this study in forthcoming research.

References

[1] K. Chandrasekharan and R. Narasimhan, *Functional equations with multiple gamma factors and the average order of arithmetical functions*, Ann. Math. **76** (1962), 93–136.

[2] W. Kuo, *Summatory functions of elements in Selberg's class II*, C.R. Math. Acad. Sci. Soc. R. Can. **25** (2003), no. 2, 55–62.

[3] L. Mai and M. Ram Murty, *The Phragmén-Lindelöf theorem and modular elliptic curves*, in *The Rademacher Legacy to Mathematics* (University Park, 1992), pp. 335-340, Contemp. Math. **166**, American Mathematical Society, 1994.

[4] M. Ram Murty, *On the estimation of eigenvalues of Hecke operators*, Rocky Mountain J. Math. **15** (1985), 521–533.

[5] M. Ram Murty, *Problems in Analytic Number Theory*, Springer, 2001.

[6] H. Rademacher, *The Phragmén-Lindelöf theorem and subharmomic functions*, Collected Papers, II, 476–485.

[7] H. Rademacher, *On the Phragmén-Lindelöf theorem and some applications*, Math. Z., **72** (1959), 192–204.

[8] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, Proceed. Amalfi Conf. *Analytic Number Theory*, (Maiori, 1989), 367–385, Salerno, 1992 (*Collected Papers, Vol.II*, pp. 47–63, Springer, 1991).

Wentang Kuo, Department of Mathematics and Statistics, Queen's University, Kingston, Ontario, Canada K7L 3N6
E-mail: wtkuo@mast.queensu.ca

M. Ram Murty, Department of Mathematics and Statistics, Queens University, Kingston, Ontario, Canada K7L 3N6
E-mail: murty@mast.queensu.ca