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# SOME RECENT DEVELOPMENTS IN NUMBER THEORY

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In 1916, Srinivasa Aiyangar Ramanujan wrote two seminal papers that have shaped the development of modern number theory. The first paper [23] modestly entitled, "On certain arithmetical functions," dealt with his celebrated $\tau$-function and conjectures relating to it as well as numerous unexplained congruences that emerged from his work. The second paper [24] entitled "On the expression of a number in the form $ax^2 + by^2 + cz^2 + du^2$" determined all natural numbers $a, b, c, d$ for which the quadratic form $ax^2+by^2+cz^2+du^2$ represents all natural numbers. He gave a complete list which included $(a, b, c, d) = (1, 1, 1, 1)$ that corresponds to the celebrated theorem of Lagrange that every natural number can be written as a sum of four squares. In our brief survey of some recent developments in number theory, we will describe how these two papers gave birth to two lines of development, one in the theory of quadratic forms, and the other in the theory of Galois representations, both of which are now central themes in modern number theory. We begin with the second paper first.

## Quadratic forms

Ramanujan's paper addresses a special case of a more general question: which positive definite quadratic forms with integral coefficients represent all natural numbers? Given a quadratic form $Q(\underline{x})$, we can write it as a matrix equation $\underline{x}^t A\underline{x}$ with $A$ a symmetric matrix. In 1993, Conway and Schneeberger [8] proved a surprising theorem: suppose that $A$ is positive definite with integer entries. If the associated quadratic form represents all natural numbers up to 15, then it represents **all** natural numbers. The original proof was complicated and never published. In 2000, Manjul Bhargava [2] published a much simpler proof.

Conway conjectured that if we consider integer valued quadratic forms (instead of $A$ being integral), then a similar result should hold with 15 replaced by 290. In 2005, Manjul Bhargava and Jonathan Hanke announced a proof of this conjecture. Their work will soon be published in *Inventiones Math*.

If we turn our attention to indefinite quadratic forms, then a theorem of A. Meyer proved in 1884 asserts that any indefinite form $Q$ with rational coefficients in $n$ variables ($n \geq 5$) represents zero non-trivially. In other words, there is a non-zero integral vector $\underline{x}$ such that $Q(\underline{x}) = 0$. It is the best possible theorem in terms of the number of variables since the example

$$Q(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 - p(x_3^2 + x_4^2)$$

with $p$ a prime congruent to 3 (mod 4) shows that $Q(\underline{x}) = 0$ implies $\underline{x} = 0$. Indeed, if there were a solution, then reducing (mod $p$) shows that $x_1^2 \equiv -x_2^2 (\mathrm{mod}\, p)$. If $x_2$ is not divisible by $p$, then we deduce that $-1$ is a square (mod $p$), a contradiction. Thus $x_2$ is divisible by $p$ and a fortiori,

$x_1$ is divisible by $p$. Similarly, one deduces that $x_3$ and $x_4$ are divisible by $p$. Thus, by a descent argument, we see that $\underline{x} = 0$. If we put

$$m(Q) = \inf\{Q(\underline{x}) : \underline{x} \in \mathbb{Z}^n, \underline{x} \neq 0\},$$

then Meyer's theorem is equivalent to $m(Q) = 0$ for any (non-degenerate) indefinite quadratic form which is a multiple of a form with rational coefficients for $n \geq 5$.

If we consider a real (non-degenerate) indefinite quadratic form $Q$ which is not a multiple of a rational form, then Alexander Oppenheim conjectured in 1929 that $m(Q) = 0$ for $n \geq 5$. It was later noted by Davenport and Heilbronn that the conjecture should hold for $n \geq 3$. Building on the 1934 work of Sarvadaman Chowla [6], Davenport [9] [10] wrote a series of papers with Birch, Heilbronn, Lewis and Ridout attacking the Oppenheim conjecture with the main tool being the circle method of Ramanujan. In this way, it was shown that Oppenheim's conjecture is true for $n \geq 21$. In the 1970's, M.S. Raghunathan gave a reformulation of the Oppenheim conjecture in terms of homogeneous group actions. Armed with this new perspective on an old conjecture, Margulis resolved the matter in 1986 using a combination of methods from Lie theory, ergodic theory and number theory. We refer the reader to his highly readable exposition [21].

In the subsequent years, Marina Ratner, motivated by conjectures of Raghunathan, proved in 1990 a major theorem [25] concerning unipotent flows on homogeneous spaces. Once this theorem is available, Oppenheim's conjecture can be deduced without too much difficulty. We will give a short description of how this is done.

If $V$ is a vector space over a field $k$ and $f$ is a bilinear form on $V$, we denote by $O(f)$ the elements of $GL(V)$ which preserve the form. In our context, the matrix $A$ associated with the quadratic form $Q$ gives rise to a bilinear form and we can consider the group of transformations $H$ such that $Q(h\underline{x}) = Q(\underline{x})$ for all $h \in H$. Thus, to prove the Oppenheim conjecture, it suffices to show that for any $\epsilon > 0$, $Q$ takes values in $[-\epsilon, \epsilon]$ at a point of the form $h\underline{x}$ with $\underline{x} \neq 0$, $h \in H$ and $\underline{x} \in \mathbb{Z}^n$. For instance, if we can show that $\{h\underline{x} : h \in H, x \in \mathbb{Z}^n, \underline{x} \neq 0\}$ contains zero in its closure, then the Oppenheim conjecture follows. The advantage of this perspective is that we have moved from the standard lattice, namely $\mathbb{Z}^n$ to the $H$-orbits of the standard lattice. This viewpoint slowly allows us to translate the problem into a problem of homogeneous spaces.

A *lattice* in $\mathbb{R}^n$ is the set of $\mathbb{Z}$-linear combinations of $n$ linearly independent vectors. It is not hard to see that every lattice is of the form $g\mathbb{Z}^n$ for some $g \in GL_n(\mathbb{R})$. Let $\mathcal{L}_n$ be the set of lattices in $\mathbb{R}^n$ and let $\mathcal{L}_n(\epsilon)$ be the set of lattices that contain $v \in \mathbb{R}^n$ with $||v|| < \epsilon$. The lattice $\mathbb{Z}^n$ can be thought of as a point of $\mathcal{L}_n$. The Oppenheim conjecture would follow if we can show the $H$-orbit of $\mathbb{Z}^n$ intersects non-trivially with $\mathcal{L}_n(\epsilon)$, which is a dynamical reformulation of the conjecture. To any lattice, we can associate an element $X$ of $GL_n(\mathbb{R})$ simply by taking the basis of the lattice for its columns. It is not hard to see that $X$ and $X'$ give rise to the same lattice if and only if $X' = AX$ for $A \in GL_n(\mathbb{Z})$. Thus, the space of lattices can be identified with the coset space $GL_n(\mathbb{R})/GL_n(\mathbb{Z})$. Since all the elements of $O(f)$ have determinant 1, it is more convenient to move to the coset space $\mathcal{K}_n = SL_n(\mathbb{R})/SL_n(\mathbb{Z})$. These coset spaces inherit topologies from $SL_n(\mathbb{R})$ and $GL_n(\mathbb{R})$ and can be given the structure of manifolds. Our interest now is to consider how the $H$ orbit of $\mathbb{Z}^n$ sits in this homogeneous space. A famous criterion of Mahler says that a subset $K$ of $\mathcal{K}_n$ is bounded if it does not intersect $\mathcal{K}(\epsilon) = \mathcal{L}(\epsilon) \cap \mathcal{K}_n$. In other words, our goal is to show that the orbit $H[\mathbb{Z}^n]$ is unbounded in $\mathcal{K}_n$.

An essential feature here is that we are dealing with $n \geq 3$ and that our form is indefinite, not commensurate with a rational form. This means that $H$ contains unipotent elements (that is, elements of $SL_n(\mathbb{R})$ for which all eigenvalues are 1).

Now we can explain Ratner's theorem and how the Oppenheim conjecture can be deduced from it. Let $H \subset SL_n(\mathbb{R})$ be generated by one-parameter unipotent subgroups. Very briefly, Ratner's theorem is that the closure of the orbit of $H[\mathbb{Z}^n]$ inside $\mathcal{K}_n$ is of the form $H'[\mathbb{Z}^n]$ for a closed subgroup $H' \supseteq H$. Moreover, there exists an $H'$-invariant probability measure on $H'[\mathbb{Z}^n]$.

Now the group $H = SO(f)$ is maximal inside $SL_n(\mathbb{R})$. So Ratner's theorem implies that $H[\mathbb{Z}^n]$ is closed or dense in $\mathcal{K}_n$. The former possiblity can occur only if $Q$ is a multiple of a rational form, which it isn't. Thus, the orbit is dense, and this completes the proof of the Oppenheim conjecture. (Note that we have proved something stronger than the conjecture.) A readable and more detailed exposition can be found in Venkatesh's paper [30]. Refinements of Ratner's theorems have found applications in other questions of number theory such as in the work of Vatsal [29] settling a conjecture of Mazur in the theory of elliptic curves.

**Higher degree forms**

The results on the Oppenheim conjecture expand our understanding of quadratic forms and the values they assume at integer lattice points. The situation is not the same when we move to cubic forms or higher degree forms. Binary quadratic forms are the easiest to study. They have a long and venerable history. Already in the work of Brahmagupta in sixth century (C.E.) India, we find the equation

$$(x_1^2 - dy_1^2)(x_2^2 - dy_2^2) = (x_1x_2 + dy_1y_2)^2 - d(x_1y_2 + x_2y_1)^2$$

which is an example of a "composition law." In the 1801 work *Disquisitiones Arithmeticae* of Gauss, we find the complete generalization of this in the form

$$(a_1x_1^2 + b_1x_1y_1 + c_1y_1^2)(a_2x_2^2 + b_2x_2y_2 + c_2y_2^2) = AX^2 + BXY + CY^2,$$

where $X, Y$ are linear functions of $x_1x_2, x_1y_2, y_1x_2, y_1y_2$ and $A, B, C$ are determined as functions of $a_1, b_1, c_1, a_2, b_2, c_2$. This is the celebrated law of composition of binary quadratic forms. For binary quadratic forms $ax^2+bxy+cy^2$ with discriminant $b^2-4ac$ fixed, Gauss's composition law serves to establish a one-to-one correspondence between ideal classes of the quadratic number field with discriminant $D$ and binary quadratic forms of discriminant $D$. This correspondence allows us to define a group law on the set of binary quadratic forms of a fixed discriminant. This is the essence of Gauss's theorem. If we identify the binary quadratic form $ax^2 + bxy + cy^2$ with the integer triple $[a, b, c]$ as a lattice point in $\mathbb{R}^3$, then Gauss's theorem is that certain lattice points may be put in one-to-one correspondence with quadratic number fields and their ideal class groups. Viewed in this way, it is natural to ask if there are other lattice points in higher dimensional Euclidean spaces that could be made to correspond in a "natural way" to higher degree number fields and their ideal class groups. In 1964, Delone and Fadeev, building on earlier work of Hermite discovered a non-trivial lattice correspondence betweeen integral binary cubic forms and cubic rings.

It is precisely this question that is addressed in the Princeton doctoral thesis of Manjul Bhargava [3]. In particular, Bhargava finds new composition laws that allow one to study ideal class groups of quartic and quintic extensions. This work has applications to a folklore conjecture regarding the enumeration of algebraic number fields with absolute discriminant below a given bound. This conjecture predicts that the number of algebraic number fields $K/\mathbb{Q}$ with $[K : \mathbb{Q}] = n$ and Galois closure $\widetilde{K}$ satisfying $\mathrm{Gal}(\widetilde{K}/\mathbb{Q}) \simeq S_n$ and discriminant $d_K$ satisfying $|d_K| \leq X$ is asymptotically $c_n X$ as $x$ tends to infinity. For $n = 2$, this is an easy exercise. For $n = 3$, it follows from the work of Davenport and Heilbronn [11]. The cases $n = 4$ and $5$ were recently completed by Bhargava [4] [5]. A good exposition of this work can be found in the Séminaire Bourbaki article [1].

The re-interpretation of questions concerning indefinite quadratic forms allowed us to use the recent advances in the ergodic theory. There is another celebrated conjecture formulated in 1930 by Littlewood that also can be reformulated in dynamical terms. Littlewood conjectured that for any $\alpha, \beta$, we have

$$\liminf_{n\to\infty} n||n\alpha||||n\beta|| = 0.$$

Another way to say this is that for any $\epsilon > 0$, the inequality

$$|x(x\alpha - y)(x\beta - z)| < \epsilon$$

can be solved with $x \neq 0$ and $x, y, z$ integers. The function $L(x, y, z) = x(x\alpha - y)(x\beta - z)$ is a product of three linear forms which admits a two-dimensional torus as a group of automorphisms. This conjecture has received considerable attention recently simply because it fits into this dynamical framework and one feels that the new methods of ergodic theory and Lie theory should resolve the conjecture. Indeed, in a recent paper, Einsiedler, Katok and Lindenstrauss [12] showed that the set of exceptions to Littlewood's conjecture has Hausdorff dimension zero.

**Galois representations and Serre's conjecture**

Let us now turn to the other paper of Ramanujan written in 1916 concerning the $\tau$-function. Ramanujan found many interesting congruences for it. For example,

$$\tau(n) \equiv \sigma_{11}(n) (\mathrm{mod}\, 691),$$

where $\sigma_{11}(n)$ denotes the sum of the 11-th powers of the positive divisors of $n$. Similar congruences were found by Ramanujan for the modulus 2, 3, 5, 7, and 23. To explain the mystery of these congruences, Serre suggested the existence of an $\ell$-adic representation

$$\rho_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F}_\ell)$$

such that if $\mathrm{Frob}_p$ denotes the Frobenius automorphism, then $\rho_\ell(\mathrm{Frob}_p)$ has trace $\tau(p)$ and determinant $p^{11}$ (mod $\ell$). Such a representation was discovered by Deligne (in the context of his work on the Weil conjectures and Ramanujan's conjecture). Serre and Swinnerton-Dyer studied this representation and noted that the special congruences arise from the "ramification" of this representation.

These results inspired Serre [26] to ask if the converse holds. That is, does every such representation "arise" from some modular form? To be precise, suppose that

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{F}_\ell)$$

is a continuous homomorphism such that $\rho$ is simple (that is, there is no basis in which the image of $\rho$ is upper triangular). If $\rho$ is odd (that is $\rho$(complex conjugation)$= -1$) and unramified at all primes unequal to $\ell$, is there a modular form $f$ (of level 1 and weight $k$) such that the trace of $\rho(\mathrm{Frob}_p)$ is equal to $a_f(p)$ (the $p$-th Fourier coefficient of $f$) and its determinant is $p^{k-1}$ (mod $\ell$)? Serre conjectured that the answer is "yes" and this is usually referred to as the level one case of Serre's conjecture. In 2006, Chandrasekhar Khare [16] proved this level one case. Serre [26] also formulated a higher "level" analogue of his conjecture and this was recently settled by Khare and Wintenberger [17].

An interesting application of this work that has a "popular appeal" is to Fermat's Last Theorem. The long and complicated proof of Ribet, Taylor and Wiles is now replaced with a relatively "shorter proof." In fact, all of the conjectural applications given in Serre's paper are now theorems.

Even more astounding about Khare's work is its application to Artin $L$-series attached to odd two-dimensional complex linear representations of the absolute Galois group over $\mathbb{Q}$.

These non-abelian $L$-series generalize the classical Riemann $\zeta$-function and the Dirichlet $L$-functions to the non-abelian Galois setting. Artin conjectured that each of his non-abelian $L$-series attached to an irreducible representation $\rho$ extends to an entire function. One of the principal goals of the program of Langlands is to prove Artin's conjecture. Indeed, if the image of $\rho$ is a finite solvable group of $GL_2(\mathbb{C})$, then Langlands [18] and Tunnell [28] proved Artin's conjecture using the full theory of the Langlands program for $GL_2$. This was the starting point of Wiles's [31] celebrated proof of Fermat's Last Theorem. As a consequence of his work on Serre's conjecture, Khare was able to show the full Artin conjecture for all odd 2-dimensional representations. One can view this as a 2-dimensional version of the classical Artin reciprocity law (which includeds the well-known law of quadratic reciprocity).

The study of Galois representations and their properties has led to other advances in number theory and this short survey cannot do justice to these new results. The most notable among these is the resolution of the Sato-Tate conjecture in the theory of elliptic curves due to Clozel, Harris, Shepherd-Baron and Taylor [7]. A short survey of this work along with a generalization related to the Chebotarev density theorem can be found in [22].

The theory of modular forms is a special case of the larger universe of automorphic representations and the Langlands program. Central to this program is the fundamental lemma or the "fundamental matching conjecture" recently proved by Ngo [19]. Surely this work will have significant consequences for the theory of $L$-function in the coming years.

**Other developments and future directions**

We have not been able to discuss the recent advances in additive combinatorics, especially the work of Green and Tao [14]. Their theorem is quite elementary to state. It is that the sequence of prime numbers contains arbitrarily long arithmetic progressions. That is, for every natural number $k$, there is a $k$-term arithmetic progression of primes. Here again, the new ideas consist of a combination of methods from analytic number theory and ergodic theory. What is interesting in this work is the use of some classical techniques from analytic number theory involving truncted von Mangoldt functions:

$$\Lambda_R(n) := \sum_{d|n, d\leq R} \mu(d) \log(R/d),$$

where $\mu$ denotes the Möbius function. These functions also appear in work of Goldston, Pintz and Yildirim [13] who showed that

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0,$$

which was a famous conjecture for a long time.

Another important theme inspired by quantum mechanics and number theory is the quantum unique ergodicity conjecture. In a special case, this conjecture states that if $f(z)$ is a holomorphic cuspidal Hecke eigenform of weight $k$ (like $\Delta(z)$, with $k = 12$ for example) then for any smooth bounded function in the fundamental domain $\mathcal{D}$ for the standard action of $SL_2(\mathbb{Z})$ on the upper half-plane, we have

$$\lim_{k\to\infty} \int_{\mathcal{D}} y^k |f(z)|^2 g(z) \frac{dxdy}{y^2} \to \int_{\mathcal{D}} g(z) \frac{dxdy}{y^2}.$$

This conjecture was recently proved by Holowinsky and Soundararajan [15]. A nice corollary of this result is that the zeros of holomorphic Hecke eigenforms become equidistributed in the fundamental domain as the weight tends to infinity. An essential ingredient in their proof is the recurrent theme of "breaking convexity" in the theory of $L$-functions.

The analog of this conjecture for Maass forms (which form the non-holomorphic counterpart of the theory of classical modular forms) is that if $f(z)$ is a Maass form which is an eigenfunction for all the Hecke operators as well as the non-Euclidean Laplacian, (with corresponding eigenvalue $\lambda$) then for any smooth $g(z)$,

$$\int_{\mathcal{D}} |f(z)|^2 g(z) \frac{dxdy}{y^2} \to \int_{\mathcal{D}} g(z) \frac{dxdy}{y^2}$$

as $\lambda \to \infty$. Lindenstrauss [20] proved using ergodic methods that the limit is as expected, upto a scalar factor of some constant $c$ with $0 \le c \le 1$. Recently, Soundararajan [27] showed that $c = 1$.

These are some of the highlights in number theory in the recent decades. Surely, it is impossible to faithfully record all of the accomplishments. However, we hope that in this short survey, we have been able to give some flavour of the developments that have emerged in the recent past and some that are yet to come.

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