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Unimodal sequences: From Isaac Newton to June Huh

M. Ram Murty

Department of Mathematics and Statistics, Queen's University
Kingston, Canada ON K7L 3N6
murty@queensu.ca

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In this paper, we present a simple proof of a famous result of Newton on unimodal sequences associated to polynomials with real roots. We then give a brief exposition of the recent work of Huh on such sequences arising from combinatorial geometries.

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1. Introduction

Newton (1643–1727) is well known for his contributions to physics, notably the laws of universal gravitation and optics, as well as his discovery of the calculus (or the method of fluxions, as he called it). However, he has many important contributions to pure mathematics that are less-known. In this paper, we highlight his theorem about unimodal sequences arising from the coefficients of polynomials in $\mathbb{R}[x]$ with real zeros. In light of such sequences appearing in the recent award-winning work of Huh [30] regarding combinatorial geometries, it seems fitting to review the elementary (and yet profound) work of Newton [38] which will require nothing beyond Rolle's theorem (discovered in 1691), one of the most basic results in calculus. We then discuss briefly the work of Huh on unimodal sequences associated to chromatic polynomials and give an axiomatic description of the new ideas introduced by him. At the end, we review connections to the classical Riemann hypothesis.

At the outset, it is best to say that this paper is not an exhaustive review of unimodal sequences. Many interesting themes are left out. We refer the reader to [15, 46] for wider overviews from a combinatorial perspective. Our motivation here is to see how this concept influences number theory, especially in the context of the Riemann hypothesis.

First, we discuss Newton's theorem. Let $\alpha_1, \alpha_2, ..., \alpha_n$ be real numbers and let σ_k denote the kth elementary symmetric polynomial in $\alpha_1, ..., \alpha_n$ given explicitly by

$$\sigma_1 = \sum_{1 \le i \le n} \alpha_i, \quad \sigma_2 = \sum_{1 \le i < j \le n} \alpha_i \alpha_j, \ldots, \quad \sigma_n = \alpha_1 \alpha_2, \ldots, \alpha_n.$$

Writing

$$\binom{n}{k} a_k := \sigma_k,$$

we have the following theorem.

Theorem 1.1 ([38]). *With the a_k defined as above, we have*

$$a_k^2 \ge a_{k-1} a_{k+1}, \quad \text{for all } 1 < k < n. \tag{1}$$

Sequences of numbers a_k satisfying the inequality (1) are called log-concave sequences. A unimodal sequence is one for which there exists an i such that

$$a_1 \le a_2 \le \cdots \le a_i \ge a_{i+1} \ge \cdots \ge a_n. \tag{2}$$

The sequence is said to be symmetric if $a_i = a_{n+1-i}$ for all $1 \le i \le [(n-1)/2]$. It is said to be without internal zeros if the indices of the nonzero elements are consecutive integers. In other words, there is no index j such that $a_j = 0$ with $a_{j-1} a_{j+1} \ne 0$.

It is relatively simple to see that any log-concave sequence is unimodal. Indeed, if not, there is a k such that $a_{k-1} \ge a_k \le a_{k+1}$ with at least one of the inequalities being strict. But then, $a_k^2 < a_{k-1} a_{k+1}$, a contradiction.

The most familiar example of a unimodal symmetric sequence is the sequence of binomial coefficients $\binom{n}{k}$. The sequence is in fact log-concave because

$$\binom{n}{k}^2 = \binom{n}{k-1}\binom{n}{k+1}\left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) > \binom{n}{k-1}\binom{n}{k+1},$$

for all $1 < k < n$.

We can rewrite Theorem 1.1 in terms of the elementary symmetric functions σ_k. Thus, (1) becomes

$$\sigma_k^2 \binom{n}{k-1}\binom{n}{k+1} \ge \sigma_{k-1}\sigma_{k+1}\binom{n}{k}^2,$$

which is equivalent to

$$\sigma_k^2 \ge \sigma_{k-1}\sigma_{k+1}\left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right).$$

The following corollary of Theorem 1.1 is now evident.

Corollary 1.2. *If $\sum_{k=0}^n b_k x^k$ is a polynomial in $\mathbb{R}[x]$ with only real roots, then*

$$b_k^2 \ge b_{k-1} b_{k+1}, \quad \text{for } 1 < k < n.$$

Newton's theorem is often stated in this form also in the literature, but it is clear that the theorem is stronger than the corollary.

Since the characteristic polynomial of any real symmetric matrix has only real roots, a wealth of examples emerge for which Newton's theorem is applicable. In particular, the adjacency matrices of undirected graphs (where we allow loops and multiple edges) are real and symmetric. By Newton's theorem, the coefficients of the characteristic polynomial of the adjacency matrix of an undirected graph are log-concave and hence unimodal.

2. Proof of Theorem 1.1

Our proof is an aesthetic variation of the one found on Hardy *et al.* [27, p. 104]. We write

$$P(x) := \sum_{j=0}^{n} \binom{n}{j} a_j x^j$$

and from our hypothesis, this polynomial has all its roots real. We first suppose that the roots are distinct and indicate at the end how this assumption can be eliminated.

Since $P(x)$ has n distinct roots, we see from the graph of the polynomial that the roots of the derivative $P'(x)$ interlace the roots of $P(x)$. Proceeding inductively, we see that any derivative of $P(x)$ also has all its roots real and distinct. The trick to proving Newton's theorem is to homogenize the polynomial $P(x)$. Thus, we consider

$$F(x,y) = \sum_{j=0}^{n} \binom{n}{j} a_j x^j y^{n-j}.$$

We may view this as a polynomial in either of the variables x, y. By our remark above, any partial derivative is a homogenous polynomial with real, distinct roots. In particular,

$$\frac{\partial^{n-2}}{\partial x^{k-1}\partial y^{n-k-1}} F(x,y) = \sum_{j=0}^{n} \binom{n}{j} a_j \left[\frac{\partial^{n-2}}{\partial x^{k-1}\partial y^{n-k-1}} (x^j y^{n-j})\right],$$

has all its roots real. The partial derivative in the summand is zero unless $k-1 \le j$ and $n-k-1 \le n-j$. In other words, we obtain $k-1 \le j \le k+1$ and we have only three terms. These are easily computed to be

$$\frac{\partial^{n-2}}{\partial x^{k-1}\partial y^{n-k-1}} F(x,y) = \frac{n!}{2}\left[a_{k-1}y^2 + 2a_k xy + a_{k+1}x^2\right].$$

By our observation concerning roots of partial derivatives of our homogeneous polynomial, the binary quadratic form on the right-hand side has only real roots. Thus the discriminant satisfies

$$4(a_k^2 - a_{k-1}a_{k+1}) \ge 0,$$

which is the desired result in the case the roots are all distinct. If the roots are not distinct, we perturb each α_i by a small amount ϵ_i to make them all distinct. Since the coefficients that ensue are continuous functions of the ϵ_i, we again deduce our result by taking $\epsilon_i \to 0$. □

We present a generalization of Newton's theorem as well as provide an alternate proof in Theorem 14.1.

3. Applications

There are several immediate applications of the theorem to the study of some well-known sequences. For instance, the absolute values of the Stirling numbers $|s(n,k)|$ of the first kind enumerate the number of permutations of the symmetric group S_n with exactly k cycles in its unique factorization as a product of disjoint cycles. It is known that (see [20, p. 38])

$$\sum_{k=0}^{n} |s(n,k)| x^k = x(x+1)\cdots(x+n-1).$$

Newton's theorem in the form of Corollary 1.2 implies that the sequence $|s(n,k)|$ is log-concave and hence unimodal.

Another application is to the Stirling numbers of the second kind $S(n,k)$. These enumerate the number of partitions of an n element set into k blocks. The polynomial

$$p_n(x) = \sum_{k=0}^{n} S(n,k) x^k,$$

satisfies the recurrence relation

$$p_n(x) = x\left(p_{n-1}(x) + p'_{n-1}(x)\right),$$

which follows from the easily derived recurrence (see [20, p. 39])

$$S(n,k) = S(n-1,k-1) + kS(n-1,k).$$

Letting $Q_n(x) = e^x p_n(x)$, we then see that $Q_n(x) = xQ'_{n-1}(x)$. Thus, by induction we deduce that $Q_n(x)$ has all its roots real. Consequently, all the roots of the polynomial $p_n(x)$ are also real and we can apply Theorem 1.1 to deduce log concavity and unimodality of the Stirling numbers of the second kind.

4. The Hermite Polynomials

Another important class of polynomials with all real zeros are the Hermite polynomials $\mathcal{H}_n(x)$ defined by the generating function

$$e^{2xt-t^2} := \sum_{n=0}^{\infty} \mathcal{H}_n(x)\frac{t^n}{n!} = 1 + 2xt + (4x^2-2)\frac{t^2}{2!} + (8x^3-12x)\frac{t^3}{3!} + \cdots.$$

Differentiating this with respect to t leads to the recursion

$$\mathcal{H}_{n+1}(x) = 2x\mathcal{H}_n(x) - 2n\mathcal{H}_{n-1}(x), \quad n \geq 1. \tag{3}$$

On the other hand,

$$\sum_{n=0}^{\infty} e^{-x^2}\mathcal{H}_n(x)\frac{t^n}{n!} = e^{-(x-t)^2},$$

so that differentiating this equation with respect to x and comparing coefficients of t^n leads to

$$\frac{d}{dx}\left(e^{-x^2}\mathcal{H}_n(x)\right) = e^{-x^2}[2x\mathcal{H}_n(x) - 2n\mathcal{H}_{n-1}(x)] = e^{-x^2}\mathcal{H}_{n+1}(x), \tag{4}$$

by (3). Since the exponential function never vanishes, an easy induction argument now reveals that the roots of $\mathcal{H}_{n+1}(x)$ are all real and interlace those of $\mathcal{H}_n(x)$.

A polynomial with real coefficients and all its zeros real is called hyperbolic in the literature. Thus, the Hermite polynomials form an important family of hyperbolic polynomials that are ubiquitous in mathematics. For instance, they appear later in our discussion of the Riemann hypothesis below.

5. The q-Binomial Theorem

Another sequence to which Newton's theorem applies is the sequence of q-binomial coefficients (see for example [46, p. 501] or [35, p. 598]). These are defined as follows. Let $\mathbb{F}_q$ be the finite field of q elements. The number of ordered sequences of linearly independent vectors $v_1, ..., v_k$ in $\mathbb{F}_q^n$ is clearly

$$(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1}),$$

because the number of choices for v_1 is $q^n - 1$ since $v_1 \neq 0$, the number of choices for v_2 is $q^n - q$ since v_2 should not lie in the one-dimensional space spanned by v_1, and so on. In particular, the number of ordered bases of $\mathbb{F}_q^n$ is

$$(q^n - 1)(q^n - q)\cdots(q^n - q^{n-1}).$$

Since

$$\#\{\text{number of } k\text{-dimensional subspaces } V \text{ of } \mathbb{F}_q^n\}\#\{\text{ordered bases of } V\}$$
$$= \#\{k\text{-tuples of linearly independent vectors in } \mathbb{F}_q^n\},$$

we have

$$\#\{\text{number of } k\text{-dimensional subspaces } V \text{ of } \mathbb{F}_q^n\}$$
$$= \frac{(q^n - 1)(q^n - q)\cdots(q^n - q^{k-1})}{(q^k - 1)(q^k - q)\cdots(q^k - q^{k-1})}.$$

The left-hand side is often denoted by the symbol $\begin{bmatrix} n \\ k \end{bmatrix}_q$ and called the q-binomial coefficient (or sometimes the Gaussian binomial coefficient). Factoring the above expression, we see that the number of k-dimensional subspaces of $\mathbb{F}_q^n$ is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q^n - 1)(q^{n-1} - 1)\cdots(q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1)\cdots(q - 1)}. \tag{5}$$

This is often seen as a generalization of the classical binomial coefficient since

$$\lim_{q\to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k},$$

by a simple application of l'Hôpital's rule. As in the case of the binomial coefficients, we interpret $\begin{bmatrix} n \\ k \end{bmatrix}_q$ to be zero if $k > n$. We will henceforth write $\begin{bmatrix} n \\ k \end{bmatrix}$ dropping the subscript q for the sake of simplicity.

It is clear that the right-hand side of (5) is a rational function $A(q)/B(q)$ with $A(x), B(x)$ being monic polynomials with integer coefficients. By the division algorithm in $\mathbb{Q}[x]$, we can write

$$A(x) = B(x)C(x) + R(x),$$

where the degree of R is strictly less than the degree of B. If N is the least common multiple of the denominators of the coefficients of $C(x)$, we see that $NA(q)/B(q) - NC(q) = NR(q)/B(q)$ is an integer for all prime powers q. But as the degree of R is strictly less than the degree of B, we can make this arbitrarily small, a contradiction unless $R(x)$ is identically zero. This argument shows that (5) is in fact a polynomial in q. By an application of Gauss's lemma regarding contents of polynomials, we deduce further that $C(x) \in \mathbb{Z}[x]$. Therefore, the q-binomial coefficient is a polynomial in q with integer coefficients.

Many combinatorial identities involving binomial coefficients often have q-analogues that signal a profound analogy between subsets of an n-element set and subspaces of $\mathbb{F}_q^n$. For instance, the familiar recurrence for binomial coefficients that results in the Pascal triangle has a q-analogue:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} + q^{n-k+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}. \tag{6}$$

This can be verified directly by noting that

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = \frac{q^{n+1} - 1}{q^k - 1}\begin{bmatrix} n \\ k-1 \end{bmatrix},$$

so that

$$\begin{aligned} \begin{bmatrix} n+1 \\ k \end{bmatrix} - q^{n-k+1}\begin{bmatrix} n \\ k-1 \end{bmatrix} &= \begin{bmatrix} n \\ k-1 \end{bmatrix}\left(\frac{q^{n+1} - 1}{q^k - 1} - q^{n-k+1}\right) \\ &= \begin{bmatrix} n \\ k-1 \end{bmatrix}\left(\frac{q^{n-k+1} - 1}{q^k - 1}\right) = \begin{bmatrix} n \\ k \end{bmatrix}. \end{aligned}$$

This allows us to prove the following theorem.

Theorem 5.1 (The *q*-binomial theorem). *If n is a natural number,*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2} x^k = \prod_{k=0}^{n-1} (1 + q^k x). \tag{7}$$

Proof. We induct on n. For $n = 1$, the result is clear. By the induction hypothesis,

$$\prod_{k=0}^{n} (1 + q^k x) = (1 + q^n x) \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2} x^k \right).$$

The coefficient of x^k on the right-hand side is

$$\begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2} + \begin{bmatrix} n \\ k-1 \end{bmatrix} q^{n+(k-1)(k-2)/2} = q^{k(k-1)/2} \left(\begin{bmatrix} n \\ k \end{bmatrix} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix} \right)$$

and the expression in brackets on the right-hand side is by (6) equal to $\begin{bmatrix} n+1 \\ k \end{bmatrix}$ completing the proof of (7). □

We can apply the q-binomial theorem to deduce the log concavity and hence the unimodality of the Gaussian binomial coefficients. Indeed, we can re-write (7) as

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2} - \binom{n}{2}} x^k = \prod_{k=0}^{n-1} (x + q^{-k}).$$

Since the roots of the polynomial on the left are patently real, Newton's theorem gives the stronger result:

$$\begin{bmatrix} n \\ k \end{bmatrix}^2 \geq \begin{bmatrix} n \\ k-1 \end{bmatrix} \begin{bmatrix} n \\ k+1 \end{bmatrix} q, \quad 1 < k < n.$$

This can also be verified directly as we did in the case of binomial coefficients. Further properties of the Gaussian binomial coefficients can be found in [20, 35].

6. Combinatorics and Commutative Algebra

Suppose n distinct objects, each available in r identical copies, are distributed among n persons in such a way that each person receives exactly r objects. How many such distributions are there? This was the question asked in 1966, in a seminal paper written by Anand *et al.* [3]. Denoting by $H_n(r)$ the number of such distributions, they conjectured that (henceforth referred to as the ADG conjecture):

(1) $H_n(r) \in \mathbb{C}[r]$;
(2) $\deg H_n = (n-1)^2$;
(3) $H_n(j) = 0$ for $j = -1, -2, ..., -(n-1)$ and H_n satisfies the functional equation

$$H_n(-n-r) = (-1)^n H_n(r).$$

According to [9], the first two parts of the conjecture were proved by Stein and Stein in 1970 and seem to have been re-discovered by Stanley who also proved (3) and extended the conjecture (see below).

The number $H_n(r)$ is also the number of $n \times n$ magic squares with line sum r where we view a magic square of rank n as an $n \times n$ matrix with non-negative integers and whose row sums, column sums and the two diagonal sums are all equal. (This is different from a classical magic square of rank n which is an $n \times n$ matrix with entries from 1 to n^2 with row sum, column sum and the two diagonal sums all equal. The reader may refer to [9] for a friendly introduction to magic squares).

If we now form the power series

$$F(t) := \sum_{r=0}^{\infty} H_n(r) t^r,$$

the ADG conjecture is equivalent to

$$F(t) = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^{(n-1)^2+1}}, \quad d = n^2 - 3n + 2,$$
$$h_0 + h_1 + \cdots + h_d \neq 0 \quad \text{and} \quad h_i = h_{d-i}, \quad i = 0, 1, ..., d.$$

To this list of three conjectures, Stanley [45] added two more:

(4) $h_i \geq 0$ for all $0 \leq i \leq d$;
(5) h_i is unimodal and we have $h_0 \leq h_1 \leq \cdots \leq h_{[d/2]}$.

It would not be an exaggeration to say that these conjectures inspired the rapid growth of the application of commutative algebra to combinatorial problems beginning with the work of Stanley [45] who proved conjectures (1) to (4). Conjecture (5) was settled by Athanasiadis [7] in 2005. The reader can find further details in nice survey papers by Bruns [16] and Verma [47]. The log-concave property of the h-sequence is a recent theorem of Iarrobino [32].

To give a synopsis of these developments, we review some classical work of Hilbert on graded rings and modules. Let us begin with a prototypical example. Let F be a field and consider the polynomial ring $R = F[x_1, ..., x_n]$ in n variables. A monomial $x_1^{a_1} \cdots x_n^{a_n}$ is said to have total degree $a_1 + \cdots + a_n$. We can decompose our ring as

$$R = \bigoplus_{i=0}^{\infty} R_i, \tag{8}$$

where $R_0 = F$ and each R_i is the vector space over F spanned by monomials of total degree i. Furthermore, we have $R_i R_j \subset R_{i+j}$. This can be studied abstractly as follows.

Let F be a field and R a commutative F-algebra with identity element. Suppose that the additive group R decomposes as (8) with $R_i R_j \subset R_{i+j}$. We then call R

a graded ring. Elements of R_i are called homogeneous elements of degree i. The Hilbert series of R is defined as the formal power series

$$F(R,t) := \sum_{i=0}^{\infty} (\dim R_i) t^i.$$

The Hilbert–Serre theorem states that if R is generated by homogeneous elements $x_1, ..., x_m$ where $\deg x_i = e_i$, then

$$F(R,t) = f(t) / \prod_{i=1}^{m} (1 - t^{e_i}),$$

where $f(t) \in \mathbb{Z}[t]$ (see for example, [33, p. 442]).

Stanley's solution of the ADG conjecture begins by interpreting $F(t)$ as the Hilbert series of a graded F-algebra where F is an arbitrary field. By further refining this connection, he resolved the first part of the ADG conjecture (1) to (4). We refer the reader to [16, 47] for excellent surveys on the ADG conjecture.

7. Log-Concave and Unimodal Sequences

The recent work of Huh [30] amplifies the ubiquitous nature of log-concave and unimodal sequences arising in combinatorics. The examples described in the previous section are prototypical of a larger phenomenon. We give a brief description.

Let G be a finite graph with n vertices. We allow loops and multiple edges. A proper coloring of G is an assignment of colors to each vertex such that no two distinct adjacent vertices receive the same coloring. In 1912, Birkhoff [10] defined $\chi_G(\lambda)$ to be the number of proper colorings of G using λ colors. The deletion-contraction relation can be used to show that $\chi_G(\lambda)$ is a polynomial in λ of degree n in the following way. If e is an edge, the graph $G \backslash e$ is the graph obtained from G by deleting e and G/e is the contraction of the same edge e. Since any proper coloring of G can be obtained by extending a proper coloring of $G \backslash e$, it is evident that

$$\chi_G(\lambda) = \chi_{G \backslash e}(\lambda) - \chi_{G/e}(\lambda),$$

because the last term enumerates those extended proper colorings of $G \backslash e$ in which the two vertices joined by e received the same coloring. A simple induction argument now shows that $\chi_G(\lambda)$ is a polynomial of degree n. A similar induction argument also shows that the coefficients of the chromatic polynomial alternate in sign. (We leave both assertions as easy exercises for the reader).

Since $\lambda = 0$ is clearly a root of $\chi_G(\lambda)$, we may write

$$\chi_G(\lambda) = \lambda(a_0 \lambda^{n-1} - a_1 \lambda^{n-2} + \cdots + (-1)^{n-1} a_{n-1}).$$

In 1968, Read [41] conjectured that the absolute values of the coefficients of the chromatic polynomial are unimodal. In 1974, Hoggar [29] made the stronger conjecture that they are in fact, log-concave. These conjectures were finally proved by Huh [31] in 2012.

Roots of chromatic polynomials are called chromatic roots and there are many beautiful results as well as open questions in this area of research [18]. For example, it is not always the case that the zeros of chromatic polynomials are all real. In fact, Cameron and Morgan (see [18, Theorem 1.2]) showed that non-real chromatic roots are dense in the complex plane. Thus, the idea of using Newton's theorem to prove unimodality or log-concavity of the absolute values of coefficients of chromatic polynomials is not a viable approach.

Some interesting results are known about the location of real chromatic roots. For instance, there are no chromatic roots in $(-\infty, 0) \cup (0, 1) \cup (1, 32/27]$ and they are dense in the interval $[32/27, \infty)$. In particular, there are no negative chromatic roots (see [18, Theorem 1.1]). If α is an algebraic integer, it is conjectured that for some natural number n, the number $\alpha + n$ is a chromatic root. In the special case that α has degree two, this conjecture is proved in [18].

Let us note that Stirling numbers of the first kind arise as coefficients of the chromatic polynomial of the complete graph K_n on n vertices:

$$\chi_{K_n}(\lambda) = \lambda(\lambda - 1) \cdots (\lambda - (n - 1)).$$

Thus, the log concavity of the Stirling numbers of the first kind is a special case of this more general result.

A graph is called planar if it can be drawn on the plane in such a way that no two edges intersect (except at the end points). The four color conjecture (now a theorem due to Appel and Haken [4], see also [5, 6]) states that any planar graph can be properly colored using four colors. Birkhoff [11, 12] originally hoped to prove the four color conjecture using his chromatic polynomial. It is somewhat curious that he succeeded in showing that if G is planar, then

$$\chi_G(\lambda) \geq \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-3},$$

for any natural number $\lambda \geq 5$. Thus, he gave a new proof of the five color theorem. If he could have proved this inequality for $\lambda = 4$, he would have proved the four color conjecture!.

8. From Graphs to Matroids

The concept of a matroid was introduced by Hassler Whitney in 1935 as an abstraction of the notion of linear independence that arises in the theory of vector spaces. There are several equivalent ways of defining a matroid (see [8] for others). Here is one way. A matroid M is a pair $(E, \mathscr{F})$ where E is a finite set and $\mathscr{F}$ is a collection of subsets of E (called independent sets) satisfying the following three axioms called the independence axioms:

(1) The empty set $\emptyset$ belongs to $\mathscr{F}$;
(2) Every subset of an independent set is independent;
(3) If A and B are independent with $|A| < |B|$, then there is an $x \in B \backslash A$ such that $A \cup \{x\} \in \mathscr{F}$ (exchange property).

If G is a finite graph and $\mathscr{F}$ is the set of all subsets of edges of G which do not contain a cycle, then it is easily verified that $\mathscr{F}$ satisfies the independence axioms. We call this the matroid associated to G and denote it as $M(G)$. Matroids arising from graphs in this way are called graphic matroids. From this viewpoint, graph theory can be viewed as a part of matroid theory.

Given a matroid $M = (E, \mathscr{F})$, a subset of E which is not independent is called dependent. A minimal dependent set is called a circuit and a maximal independent set is called a basis. Similar to the case of linear algebra, one can show that all bases of M have the same cardinality which is called the rank of the matroid and denoted by $r(M)$. More generally, if A is a subset of E, we denote by $r(A)$ to be the maximal size of an independent subset of A and call it the rank of A. This allows us to define the characteristic polynomial of a matroid M as follows:

$$\chi_M(\lambda) := \sum_{A \subseteq E} (-1)^{|A|} \lambda^{r(M)-r(A)}, \tag{9}$$

where the summation is over all subsets A of E.

If G is a graph, then the characteristic polynomial of the matroid $M(G)$ is equal to $\chi_G(\lambda)/\lambda^c$ where c is the number of connected components of G.

From (9), we see that the characteristic polynomial of M is monic of degree $r = r(M)$ and we can write

$$\chi_M(\lambda) = w_0(M)\lambda^r + w_1(M)\lambda^{r-1} + \cdots + w_r(M), \tag{10}$$

with $w_0(M) = 1$ and $w_k(M) \in \mathbb{Z}$. The numbers $w_k(M)$ are called Whitney numbers of the first kind and are seen as a generalization of the Stirling numbers of the first kind discussed earlier. The unimodality of the absolute values of these numbers was first conjectured by Rota [43] in 1970. The stronger conjecture of the log concavity of the coefficients was proposed by Welsh [48] and Heron [28] in 1970's. Both conjectures are now theorems due to the work of Adiprasito *et al.* [1].

If we let $f_i(M)$ be the number of independent sets of M of size i, then Mason [37] conjectured that these numbers are log-concave and again, this is a consequence of the work [1]. This is deduced from the solution of the Rota *et al.* conjectures by first showing that the $f_i(M)$ coincide with the absolute value of the coefficient of λ^{r-i} of the reduced characteristic polynomial of another matroid M' of rank $r+1$ constructed from M by Brylawski [17] in 1977 and again re-discovered by Lenz [36] in 2013.

As mentioned earlier, the notion of a matroid is inspired by the concept of independence in linear algebra. There is an equivalent way of defining a matroid using the concept of closure which can be seen as an abstraction of linear span. Here is the formal definition. Let $\mathscr{P}(E)$ denote the power set of E (that is, the set of all subsets of E). A matroid M is then defined as a set E together with a closure map $X \mapsto \overline{X}$ on $\mathscr{P}(E)$ satisfying the four span axioms: For all $X, Y \in \mathscr{P}(E)$ and $x, y \in E$, we have

(1) $X \subseteq \overline{X}$;
(2) if $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$;
(3) $\overline{\overline{X}} = \overline{X}$;
(4) if $y \in \overline{X \cup \{x\}}$ and $y \notin \overline{X}$, then $x \in \overline{X \cup \{y\}}$.

It is not hard to show that our two definitions are equivalent. Indeed, given a matroid satisfying the independence axioms, we define the closure of a set $X \in \mathscr{P}(E)$ to be

$$\overline{X} := \{x \in E : r(X \cup \{x\}) = r(X)\}.$$

One easily checks the closure operator satisfies the span axioms above. Conversely, given a matroid satisfying the span axioms, we define a subset A to be independent if and only if $x \in A$ implies $x \notin \overline{A \backslash \{x\}}$.

The span axioms allow us to define a flat of E as a closed subset. That is, X is called a flat if $X = \overline{X}$. A loop is an element of the closure of the empty set. An element y is said to be parallel to x if $y \in \overline{x}$. By (4) above, taking $X = \emptyset$, we see that $y \in \overline{\{x\}}$ implies $x \in \overline{y}$ so that y parallel to x implies x parallel to y. A matroid is called simple if it has no loops or parallel elements. A simple matroid is also called a combinatorial geometry in the literature. We will denote by $\mathcal{F}$ the collection of flats of E.

Dual to the concept of Whitney numbers of the first kind, one can define Whitney numbers of the second kind, $W_k(M)$ to be the number of flats of M of rank k. These generalize the Stirling numbers of the second kind. It is still an open problem to show that the Whitney numbers of the second kind form a log-concave sequence.

9. Posets and Lattices

A partially ordered set (or poset for short) is a pair $(P, \leq)$ where P is a set and $\leq$ is a binary relation on P such that

(1) $x \leq x$ for all $x \in P$;
(2) $x \leq y$ and $y \leq x$ implies $x = y$;
(3) $x \leq y$ and $y \leq z$ implies $x \leq z$.

Sometimes we write $x < y$ if $x \leq y$ and $x \neq y$. A prototypical example of a poset is the set of all subsets of a set partially ordered by set inclusion. With this example in mind, we sometimes say that y contains x if $x \leq y$.

Given a poset P and $y \in P$, we say $z \in P$ is an upper bound for y if $y \leq z$. Similarly, we say $x \in P$ is a lower bound for y if $x \leq y$. An element z is said to be a Least Upper Bound (LUB) for y if it is contained in every upper bound of y. Similarly, x is said to be a Greatest Lower Bound (GLB) for y if it contains every other lower bound of y. It is clear that when these exist, they are unique. A poset is said to have a minimal element (usually denoted as 0), if $0 \le x$ for all $x \in P$. It is said to have a maximal element (usually denoted as 1) if $x \le 1$ for all $x \in P$. We denote by $[x, z]$ the set $\{y : x \le y \le z\}$ and call it the interval $[x, z]$. A poset is called locally finite if every interval $[x, z]$ is finite. If the poset has a minimal element 0, we say x is an atom if the interval $[0, x]$ has only two elements.

A (combinatorial) lattice is a poset $(P, \le)$ for which any two elements x, y have a least upper bound (denoted $x \vee y$) and greatest lower bound (denoted $x \wedge y$). For example, the set of all subsets of a given set S is a lattice under usual union (LUB) and intersection (GLB) with minimal element being the empty set and the maximal element being S. Another less familiar example is the set of natural numbers partially ordered by divisibility in which $x \vee y$ and $x \wedge y$ are the least common multiple and greatest common divisor of x and y (respectively). This poset has a minimal element but no maximal element. It is however locally finite.

In 1964, Rota [42] initiated a rich combinatorial theory of locally finite posets by introducing the concept of the Möbius function which underlies many combinatorial problems. For example, one can define the Möbius function $\mu : P \times P \to \mathbb{Z}$ recursively via the relations: $\mu(x, x) = 1$ for all $x \in P$ and

$$\mu(x, z) = -\sum_{x \le y < z} \mu(x, y), \quad \text{if} \quad x < z. \tag{11}$$

It is easily seen that $\mu(x, z)$ depends only on the order structure of the interval $[x, z]$ and not on the rest of P. Introducing the Kronecker delta function $\delta(x, y) = 1$ if $x = y$ and zero otherwise, we can write Eq. (11) as

$$\sum_{x \le y \le z} \mu(x, y) = \delta(x, z).$$

It is not difficult to see that we could have equivalently defined the Möbius function via the recurrence

$$\sum_{x \le y \le z} \mu(y, z) = \delta(x, z) \quad \text{for } x \le z. \tag{12}$$

The fundamental property of the Möbius function is the following inversion formula.

Theorem 9.1 ([42]). *Let P be a locally finite poset and let f and g be functions on P with values in any ring. Then,*

$$g(y) = \sum_{x \le y} f(x),$$

if and only if

$$f(y) = \sum_{x \le y} \mu(x, y) g(x).$$

Proof. We have

$$\sum_{x\le y}\mu(x,y)g(x)=\sum_{x\le y}\mu(x,y)\sum_{z\le x}f(z)=\sum_{z\le y}f(z)\left(\sum_{z\le x\le y}\mu(x,y)\right)$$

and the inner sum by (12) equals $\delta(z,y)$ from which the result follows. □

A lattice L is said to be geometric if every element is the join of atoms and if there exists a rank function $r: L\to\mathbb{R}$ such that $r(x\vee y)+r(x\wedge y)\le r(x)+r(y)$ for all $x,y\in L$. We will henceforth assume our lattices are finite having minimal and maximal elements 0 and 1, respectively. The characteristic polynomial of a geometric lattice L is then defined as

$$p_L(\lambda):=\sum_{x\in L}\mu(0,x)\lambda^{r(1)-r(x)}.$$

Given a matroid M, the collection of its flats $\mathcal{F}$ forms a poset under set inclusion. In fact, it is a geometric lattice with the rank function as defined in the previous section having minimal element the empty set Ø and maximal element E.

If L is a geometric lattice, a special role is played by modular elements and modular pairs which are defined as follows. A pair $a,b\in L$ is called a modular pair if $r(a\wedge b)+r(a\vee b)=r(a)+r(b)$. An element $a\in L$ is called a modular element if a,x is a modular pair for all $x\in L$. These notions are motivated by group theory and suggestions of the second isomorphism theorem in the theory of finite groups are evident in the following lemma.

Lemma 9.2. *If a is a modular element of a finite geometric lattice L, then there is an isomorphism between $[a\wedge b,a]$ and $[b,a\vee b]$ for all $b\in L$ given by $z\mapsto z\vee b$. Consequently, for a modular element a, we have*

$$r(a\wedge b)+r(a\vee b)=r(a)+r(b),$$

for every $b\in L$.

Proof. See [2, p. 59]. □

The following application of the Möbius inversion formula is useful.

Lemma 9.3. *Let x be a fixed element of the lattice L and let $v\in L$. Then*

$$\mu(0,v)=\sum_{y\le x,z\wedge x=0,y\vee z=v}\mu(0,y)\mu(0,z).$$

Proof. Let $f(v)$ denote the right-hand side. Then,

$$\sum_{u\le v}f(u)=\sum_{u\le v}\sum_{y\le x,z\wedge x=0,y\vee z=u}\mu(0,y)\mu(0,z).$$

Since $y \vee z = u$ and $u \leq v$, we see that $y \leq v$. As $y \leq x$, we must have $y \leq x \wedge v$. Therefore,

$$\sum_{u \leq v} f(u) = \sum_{y \leq x \wedge v} \sum_{z \leq v, z \wedge x = 0} \mu(0,y)\mu(0,z) = \delta(0, x \wedge v) \sum_{z \leq v, z \wedge x = 0} \mu(0,z).$$

Either $\delta(0, x \wedge v) = 0$ or $x \wedge v = 0$ in which case the sum on the right ranges over all $z \leq v$. Thus,

$$\sum_{u \leq v} f(u) = \delta(0, v).$$

Inverting this yields the desired result. □

Stanley [44] discovered an important factorization theorem for characteristic polynomials using modular elements.

Theorem 9.4 ([44]). *If x is a modular element of a finite geometric lattice L, then*

$$p_L(\lambda) = p_{[0,x]}(\lambda) \left(\sum_{z \in L : z \wedge x = 0} \mu(0,z) \lambda^{r(1) - r(x) - r(z)} \right).$$

Proof. The right-hand side equals

$$\sum_{y \leq x} \sum_{z \wedge x = 0} \mu(0,y)\mu(0,z)\lambda^{r(1) - r(y) - r(z)}.$$

Since x is a modular element, x, z is a modular pair for any $z \in L$ and $[x \wedge z, z] \simeq [x, x \vee z]$. For z in the inner sum, we have $z \wedge x = 0$. Consequently $y \wedge z = 0$ so that $[0, z] = [y \wedge z, z] \simeq [y, y \vee z]$. Thus, $r(z) = r(y \vee z) - r(y)$ and the sum becomes

$$\sum_{y \leq x} \sum_{z \wedge x = 0} \mu(0,y)\mu(0,z)\lambda^{r(1) - r(y \vee z)}.$$

Interchanging summation, we see that

$$p_L(\lambda) = \sum_{v \in L} \lambda^{r(1) - r(v)} \sum_{y \leq x, z \wedge x = 0, y \vee z = v} \mu(0,y)\mu(0,z).$$

As $y \leq x$, the inner sum is over y, z such that $y \vee z = v, y \wedge z = 0$ which works out to $\mu(0, v)$ by Lemma 9.3. The proof is now complete. □

We can apply Stanley's factorization theorem above to determine the characteristic polynomial of any modular lattice in which every element is modular. It turns out that the roots of this polynomial are all real and hence, Newton's theorem applies to show log concavity of the coefficients. Indeed, if the lattice is of rank n, one can take a maximal chain of modular elements $0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$. Let α_i be the number of atoms below x_i but not below x_{i-1}. It turns out that the roots of the characteristic polynomial are $\alpha_1, \ldots, \alpha_n$. This can be deduced by choosing x to be x_{n-1} in Theorem 9.4 and noting that $z \wedge x_{n-1} = 0$ implies $z = 0$

or z is an atom. Thus, apart from $z = 0$, only rank 1 elements contribute to the sum on the right-hand side in Theorem 9.4. The number of elements of rank 1 in the sum is easily seen to be α_n. Factoring this and then applying induction gives the result.

In the argument above, only the existence of a maximal chain consisting of modular elements was used. Thus, Stanley [44] defines a supersolvable lattice to be a geometric lattice in which such a maximal chain exists and deduces a similar factorization theorem for these lattices as well.

10. Mixed Volumes and Log-Concave Sequences

An interesting log-concave sequence arises in the study of mixed volumes of convex sets. Recall that a convex set S in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ such that for any two $\mathbf{a}, \mathbf{b} \in S$, the combination $\lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ lies in S for every $0 \le \lambda \le 1$. A compact convex set is called a convex body. Given two convex bodies K and L in $\mathbb{R}^n$, the Minkowski sum is the set

$$xK + yL := \{x\mathbf{a} + y\mathbf{b} : \mathbf{a} \in K, \mathbf{b} \in L\}.$$

If Vol denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$, then Minkowski proved that there are non-negative real numbers $V_j(K,L)$ such that

$$\mathrm{Vol}(xK + yL) = \sum_{j=0}^{n} \binom{n}{j} V_j(K,L) x^{n-j} y^j.$$

The number $V_j(K,L)$ is called the jth mixed volume of K and L. If $x = 1$ and $y = 0$, we get $V_0(K,L) = \mathrm{Vol}(K)$ and if $x = 0, y = 1$, we get $V_n(K,L) = \mathrm{Vol}(L)$. Thus, the numbers $V_j(K,L)$ may be viewed as interpolating between $V(K)$ and $V(L)$. In 1936, Aleksandrov and Fenchel independently proved that the sequence

$$V_0(K,L), V_1(K,L), \ldots, V_n(K,L),$$

is a log-concave sequence.

A nice application of this result to combinatorics is presented in [46]. Let P be a finite poset with n elements and fix $v \in P$. Let N_i be the number of order preserving bijections $f : P \to \{1, 2, \ldots, n\}$ such that $f(v) = i$. Then, the sequence $N_1, \ldots, N_n$ is log-concave which settles a conjecture of Chung *et al.* [19]. This is proved by realizing the numbers N_i as mixed volumes of certain convex polytopes.

11. Some Linear Algebra

A useful perspective for the study of combinatorial problems presented by Stanley [46] is the *linear algebra paradigm*. Suppose that we want to study a sequence $a_0, a_1, \ldots, a_n$ of non-negative natural numbers that we want to prove is unimodal, symmetric or log-concave. The problem gains an added dimension if we can find vector spaces $V_0, V_1, \ldots, V_n$ (over $\mathbb{C}$ say) and linear maps $\phi_k : V_k \to V_{k+1}$ such that

(1) $\dim V_i = a_i$, for $0 \le i \le n$;
(2) ϕ_k is injective for $0 \le k \le [(n-1)/2]$;
(3) $V_i \simeq V_{n-i}$.

Then, the sequence $a_0, a_1, \ldots, a_n$ is clearly symmetric and unimodal.

Sometimes, it is convenient to construct a single graded vector space

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_n$$

and a single linear transformation ϕ with $\phi(v) = \phi_k(v)$ for $v \in V_k$ for $0 \le k < n$ and $\phi(v) = 0$ if $v \in V_n$. But this is simply a change of notation from the above paradigm.

12. A General Formalism

In this solution of the Read–Hoggar conjectures, Huh [30] identified a general formalism inspired by the developments that led to the proof of the Weil conjectures in algebraic geometry and number theory. One begins with a mathematical object X to which we can associate some "dimension" d. For example, X can be an algebraic variety of dimension d defined over a field F. Often, it is possible to construct a graded vector space $A(X)$ over $\mathbb{R}$ that has a decomposition as a graded ring (the "cohomology ring"):

$$A(X) = \bigoplus_{j=0}^{d} A^j(X),$$

which admits a symmetric bilinear pairing $P : A(X) \times A(X) \to \mathbb{R}$ and a graded linear map $L : A^j(X) \to A^{j+1}(X)$ which is symmetric with respect to P. The linear map L usually comes as a member of a convex cone $K(X)$ in the space of linear operators on $A(X)$. The letters P, L and K have been chosen deliberately to suggest the names "Poincaré", "Lefschetz" and "Kähler". The following properties are assumed:

(PD) The bilinear pairing $A^j(X) \times A^{d-j}(X) \to \mathbb{R}$ given by $(x, y) \mapsto P(x, y)$ is non-degenerate (Poincaré duality);

(HL) For any $L_1, ..., L_{d-2j} \in K(X)$, the linear map

$$A^j(X) \to A^{d-j}(X), \quad x \mapsto \left(\prod_{i=1}^{d-2j} L_i \right) x,$$

is bijective (the hard Lefschetz theorem for X);

(HR) For any $L_1, ..., L_{d-2j} \in K(X)$, the bilinear form

$$A^j(X) \times A^j(X) \to \mathbb{R}, \quad (x, y) \mapsto (-1)^j P\left(x, \left(\prod_{i=1}^{d-2j} L_i\right) y\right),$$

is positive definite on the kernel of the linear map

$$A^j(X) \to A^{d-j+1}(X), \quad x \mapsto \left(\prod_{i=1}^{d-2j} L_i\right) x$$

(the Hodge–Riemann relations for X).

13. The Chow Ring of a Matroid

Let M be a matroid of rank r and set $d = r - 1$. The Chow ring $A(M)$ of a loopless matroid $M = (E, \mathcal{F})$ is defined as follows. For each $F \in \mathcal{F}' = \mathcal{F} \backslash \{\emptyset, E\}$, we associate an indeterminate x_F and consider the polynomial ring $\mathbb{R}[x_F]_{F \in \mathcal{F}'}$ modulo the relations:

(1) for $a, b \in E$, we have

$$\sum_{F \in \mathcal{F}' : a \in F} x_F = \sum_{F \in \mathcal{F}' : b \in F} x_F;$$

(2) $x_F x_{F'} = 0$ whenever F and F' are incomparable in the poset $\mathcal{F}'$.

If $A^j(M)$ denotes the span of degree j monomials in $A(M)$, we have the decomposition

$$A(M) = \bigoplus_{j=0}^{d} A^j(M).$$

The formalism implies there is a unique linear isomorphism called the "degree map",

$$\deg : A^d(M) \to \mathbb{R},$$

which maps $x_{F_1} \cdots x_{F_d}$ to 1 for every maximal chain $F_1 \subset F_2 \subset \cdots \subset F_d$ of non-empty proper flats. The authors of [1] show that the Chow ring of a matroid satisfies the formalism outlined in the previous section. A consequence of this is that for any $u, v \in K(M)$, the real symmetric matrix

$$\begin{pmatrix} \deg(\eta u^2) & \deg(\eta uv) \\ \deg(\eta uv) & \deg(\eta v^2) \end{pmatrix}, \quad \text{where } \eta = \prod_{i=1}^{d-2} L_i,$$

has exactly one positive eigenvalue. Consequently, the determinant is negative. In other words,

$$(\deg(\eta uv))^2 \geq \deg(\eta u^2) \deg(\eta v^2).$$

In [1], the authors consider two distinguished elements of $A^1(M)$ as follows. Fix $j \in E$ and define

$$\alpha = \sum_{j \in F} x_F, \quad \beta = \sum_{j \notin F} x_F.$$

The two elements are shown not to depend on the choice of j and that they both belong to $K(M)$. Moreover, they are numerically effective (nef) elements of $A^1(M)$.

The above inequality is extended as follows. Suppose that $\alpha, \beta \in A^1(M)$ and that β is nef. Then,

$$\deg(\alpha^2\beta^{d-2})\deg(\beta^2\beta^{d-2}) \leq \deg(\alpha\beta^{d-1})^2. \tag{13}$$

Letting $\chi_M(\lambda)$ be the chromatic polynomial of M, and writing

$$\frac{\chi_M(\lambda)}{\lambda - 1} = e_0(M)\lambda^d - e_1(M)\lambda^{d-1} + \cdots + (-1)^d e_d(M),$$

it transpires that

$$e_i(M) = \deg(\alpha^i\beta^{d-i}), \quad \text{for every } i.$$

Thus,

$$e_1(M) = \deg(\alpha\beta^{d-1}), \quad e_0(M) = \deg(\beta^2\beta^{d-2}), \quad e_2(M) = \deg(\alpha^2\beta^{d-2}).$$

The log-concave property for this triple is equivalent to

$$\deg(\alpha^2\beta^{d-2})\deg(\beta^2\beta^{d-2}) \leq \deg(\alpha\beta^{d-1})^2,$$

which is (13). The other inequalities are derived via an induction process of "truncation" of the matroid.

14. Generalizations and Connections with the Riemann Hypothesis

The following generalization of Newton's theorem is due to Laguerre (see [13, p. 24]). Recall that an entire function $f(z)$ is said to have order 1 if for any $\epsilon > 0$, there is a constant $K(\epsilon)$ such that

$$|f(z)| \leq K(\epsilon)\exp(|z|^{1+\epsilon}).$$

For example, Stirling's approximation for $\Gamma(z)$ shows that $1/\Gamma(z)$ is an entire function of order 1. The Riemann ξ-function associated to the ζ-function,

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \tag{14}$$

is also of order 1.

Theorem 14.1. *Let $f(z)$ be an entire function of order less than or equal to* 1. *Suppose that for real z, $f(z)$ is real valued and that all its roots are real. Then, $f'(z)$ is again real valued for real z and the zeros of $f'(z)$ are also real. Writing*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

we have for $k \geq 1$,

$$a_{k-1}a_{k+1}\left(1 + \frac{1}{k}\right) \leq a_k^2.$$

Proof. Let R be the multiset of nonzero roots of $f(z)$ which may be finite or infinite. Since $f(z)$ is of order r with $r = 0$ or 1, we can write by the Hadamard factorization theorem

$$f(z) = Ce^{Az}z^d \prod_{\alpha \in R} \left(1 - \frac{z}{\alpha}\right) e^{rz/\alpha},$$

for certain constants A, C and d. Without any loss of generality, we may suppose that $d = 0$ (for otherwise, we simply consider $f(z)/z^d$. Since $f(z)$ is real when z is real and all the roots are real, the infinite product above is real valued when z is real. Thus, it is easily seen that both A and C are real simply by considering $f(z_1)$ and $f(z_2)$ for two distinct real numbers z_1 and z_2 and solving for A and C. Taking the logarithmic derivative, we get

$$\frac{f'(z)}{f(z)} = A + \sum_{\alpha \in R} \left(\frac{1}{z - \alpha} + \frac{r}{\alpha}\right). \tag{15}$$

Since the right-hand side is real for real values of z, we immediately see that $f'(z)$ is real for real z. Taking imaginary parts of both sides, we get by writing $z = x + iy$ with $y \in \mathbb{R}$,

$$\mathrm{Im}\left(\frac{f'(z)}{f(z)}\right) = -y\left(\sum_{\alpha \in R} \frac{1}{|z - \alpha|^2}\right).$$

It is now patently clear that the right-hand side cannot vanish unless $y = 0$. In other words, the zeros of $f'(z)$ are real. This proves the first assertion. By induction, the zeros of all the derivatives of f are real. Differentiating (15), we see that

$$\frac{d}{dz}\left(\frac{f'(z)}{f(z)}\right) = -\sum_{\alpha \in R} \frac{1}{(z - \alpha)^2},$$

is negative for any real z. That is, for any real z,

$$f(z)f''(z) - f'(z)^2 \leq 0.$$

In particular, we can apply this to $z = 0$ and deduce $2a_0a_2 \leq a_1^2$. Since this applies to all the higher derivatives as well, we deduce

$$f^{(k-1)}(0)f^{(k+1)}(0) \leq f^{(k)}(0)^2.$$

Thus,

$$a_{k-1}a_{k+1}(k-1)!(k+1)! \leq a_k^2 k!^2,$$

which simplifies to

$$a_{k-1}a_{k+1}\left(1 + \frac{1}{k}\right) \leq a_k^2. \qquad \square$$

We can apply Laguerre's theorem to the Riemann ζ-function in the following way. Putting $s = \frac{1}{2} + iz$ in (14), the functional equation $\xi(s) = \xi(1 - s)$ becomes

$$\xi(1/2 + iz) = \xi(1/2 - iz),$$

which means that $\xi(1/2 + iz)$ is an entire function of order 1 which is real valued for real z. The Riemann hypothesis is the assertion that all the roots of $\xi(1/2+iz)$ are real. If that is the case, then writing

$$\xi(1/2 + iz) = \sum_{k=0}^{\infty} \gamma_k z^k.$$

Laguerre's theorem would predict that

$$\gamma_{k-1}\gamma_{k+1} \leq \gamma_k^2.$$

But this is trivially true since $f(z) = \xi(1/2+iz)$ is an even function by the functional equation and so $\gamma_{2k+1} = 0$ for $k \geq 0$. Thus, it makes sense to consider

$$G(z) = \sum_{k=0}^{\infty} \gamma_{2k} z^k,$$

so that $f(z) = G(z^2)$. The Riemann hypothesis implies that all the roots of $f(z)$ are real and positive and so Laguerre's theorem predicts

$$\gamma_{2k}^2 \geq \gamma_{2k-2}\gamma_{2k+2}, \quad \forall k \geq 1.$$

This was conjectured by Pólya [39] in 1927. In 1966, Grosswald [26] showed that this holds for sufficiently large k and then in 1986, Csordas *et al.* [21] established it for all values of k. Thus, we have some confirmation towards the Riemann hypothesis.

In his commentary of Pólya's collected papers, Marden (see[40, p. 426]) writes that Pólya's 1927 paper [39] was the outcome of Pólya's

> "findings on examining the "Nachlass" of the Danish mathematician J. L. W. V. Jensen who died in 1925. Fourteen years earlier Jensen had announced that he would publish a paper regarding his algebraic function theoretic research on the Riemann ξ-function. In view of Jensen's well-known interest in the zeros of polynomials and entire functions, expectations were high the Jensen would contribute to the solution of the Riemann hypothesis problem regarding the zeros of the ξ-function. However, this paper was never published, and so on Jensen's death it was a matter of paramount importance to have his papers examined by an expert in this area. Professor Pólya undertook this task, but after an arduous examination he found no clue to any progress that Jensen may have made towards the Riemann hypothesis."

It was in this context that Pólya wrote his 1927 paper [39]. In it, he derived a necessary and sufficient condition for an entire function of order at most 1 to have all its zeros real. This gave rise to what are now called the Jensen–Pólya polynomials which we now define.

Given any sequence $\mathbf{a} = \{a_k\}$ of real numbers, we define the Jensen–Pólya polynomial of degree d and shift n by

$$J_{\mathbf{a}}^{d,n}(x) := \sum_{k=0}^{d} \binom{d}{k} a_{k+n} x^k.$$

Thus,

$$J_{\mathbf{a}}^{0,n}(x) = a_n, \quad J_{\mathbf{a}}^{1,n}(x) = a_n + a_{n+1}x,$$

$$J_{\mathbf{a}}^{2,n}(x) = a_n + 2a_{n+1}x + a_{n+2}x^2, \quad J_{\mathbf{a}}^{3,n}(x) = a_n + 3a_{n+1}x + 3a_{n+2}x^2 + x^3$$

and so on. Pólya [39] proved that the power series

$$\sum_{k=0}^{\infty} a_k x^k,$$

has real zeros if and only if all the polynomials $J_{\mathbf{a}}^{d,n}(x)$ have real zeros for every d, n. Observe that the hyperbolicity (that is, all roots being real) of $J_{\mathbf{a}}^{2,n}(x)$ is equivalent to the log-concave property of the sequence.

In the case of the sequence $\mathbf{\Gamma} = \{\gamma_{2k}\}$ associated with the Riemann zeta function, positivity of the roots of $J_{\mathbf{\Gamma}}^{2,n}(x)$ is the theorem in [21]. In 2010, Dimitrov and Lucas [22] showed that $J_{\mathbf{\Gamma}}^{3,n}(x)$ has all its roots real. The striking theorem that for all sufficiently large d, the polynomials $J_{\mathbf{\Gamma}}^{d,n}(x)$ have all roots real was proved by Griffin *et al.* [24] in 2019. These results were made effective in [25] where the authors showed (in particular) that the polynomials $J_{\mathbf{\Gamma}}^{d,n}(x)$ are hyperbolic for $d \leq 9 \times 10^{24}$ giving rise to optimism that the Riemann hypothesis was within reach. Indeed, Bombieri [14] announced these developments as a "major breakthrough". But as Farmer [23] points out, "Jensen polynomials are not a plausible route to proving the Riemann hypothesis" because by a result of Kim [34], any entire function $f(z)$ of order less than 2, which is real on the real axis, and which has all zeros in a strip $|\operatorname{Im}(z)| \leq A$ has for any fixed $R > 0$, the property that $f^{(n)}(z)$ has only real zeros in $|z| < R$ for n sufficiently large. The essential result in [24] is a precise asymptotic for $\xi^{(2n)}(1/2)$ and a new limit formula relating the Jensen–Pólya polynomials to the Hermite polynomials, which we have already seen are hyperbolic. In particular, it is not surprising that $J_{\mathbf{\Gamma}}^{d,n}(x)$ has all its roots real for d sufficiently large. The fugitive Riemann hypothesis is still an elusive mystery!.

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ORCID

M. Ram Murty https://orcid.org/0000-0003-2086-102X

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