Irrationality of zeros of the digamma function

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Dedicated to the memory of Professor Eduard Wirsing

Abstract We prove that all the zeros of the digamma function with at most one possible exception are real and irrational.

Key words: Digamma function, irrationality

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1 Introduction

The digamma function $\psi(z)$ is the logarithmic derivative of the Γ-function $\Gamma(z)$. Thus,

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty} \left(\frac{1}{n+z} - \frac{1}{n} \right), \tag{1}$$

the second equality arises from the logarithmic differentiation of the Hadamard factorization of the Γ-function. Here, γ denotes Euler's constant. Note that $\psi(z)$ has poles at the non-positive integers $z = 0, -1, -2, \ldots$ and when $z = 1$, the series on the right hand side of (1) telescopes and we deduce $\psi(1) = -\gamma$. More generally, if m is a natural number, the right hand side of (1) again telescopes and

$$\psi(m) = -\gamma + \sum_{j=1}^{m-1} \frac{1}{j}.$$

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In an earlier paper, the author and Saradha [5] studied transcendental values of the digamma function. In particular, we showed that for $q > 1$, the values

$$\psi(a/q) + \gamma, \qquad \text{with} \qquad (a, q) = 1, \qquad 1 \leq a < q, \tag{2}$$

are all transcendental. More precisely, we showed that these numbers are all non-vanishing $\overline{\mathbb{Q}}$-linear forms of logarithms of algebraic numbers and hence transcendental by Baker's theory [1]. This paper can be seen as extending the results of [5] and [6] in another direction.

These results are related to a celebrated theorem of Baker, Birch and Wirsing [2]. Chowla asked the question whether there exitsts a rational valued function $f(n)$, periodic with prime period p, such that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0.$$

Baker, Birch and Wirsing [2] showed that such a function must be identically zero. In fact, they proved a stronger theorem: if f is a non-vanishing function defined on the integers with algebraic values and period q (not necessarily prime) such that

(a) $f(r) = 0$ for r satisfying $1 < (r, q) < q$;
(b) the q-th cyclotomic polynomial is irreducible over the field $\mathbb{Q}(f(1), ..., f(q))$,
(c) $\sum_{r=1}^{q} f(r) = 0$,

then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

In [5], we showed that under these conditions,

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = -\frac{1}{q} \sum_{a=1}^{q} f(a)\psi(a/q),$$

and thus the Baker-Birch-Wirsing theorem is related to the digamma function. In fact, the right hand side is a linear form in logarithms of algebraic numbers and its non-vanishing shows that it is a transcendental number.

Following [4], it is convenient to define a *Baker period* as a $\overline{\mathbb{Q}}$-linear form of logarithms of algebraic numbers and a *general Baker period* as any algebraic number plus a non-zero Baker period. Baker's theorem [1] then says that if $\alpha_1, ..., \alpha_m$ are non-zero algebraic numbers such that

$$\log \alpha_1,\ ...,\ \log \alpha_m$$

are linearly independent over $\mathbb{Q}$, then

$$1,\ \log \alpha_1,\ ...,\ \log \alpha_m$$

are linearly independent over $\overline{\mathbb{Q}}$. In particular, if α is an algebraic number and Λ is a non-zero Baker period, then $\alpha + \Lambda$ is transcendental. This remark can be applied to our study of the digamma function.

By virtue of the two functional equations

$$\psi(z+1) = \psi(z) + \frac{1}{z}, \qquad \psi(1-z) = \psi(z) + \pi \cot \pi z, \tag{3}$$

the result (2) can be extended to all rational numbers as follows.

Theorem 1.1 *For an arbitrary rational number x which is not an integer, $\psi(x) + \gamma$ is a general Baker period which is transcendental.*

As a consequence, we will deduce:

Theorem 1.2 *All the zeros of $\psi(x) + \gamma$ are real and irrational except for $x = 1$.*

Several questions arise. What are the zeros of $\psi(z)+\gamma$ and what is their arithmetic nature? Are they transcendental? By our theorem, they are certainly not rational. What are the zeros of the digamma function and are they transcendental? In this paper, we address these questions.

The first observation to make is that taking imaginary parts of (1), we have for $z = x + iy$, with $x, y \in \mathbb{R}$,

$$\Im(\psi(z)) = \frac{y}{x^2 + y^2} + \sum_{n=1}^{\infty} \frac{y}{(x+n)^2 + y^2},$$

which vanishes if and only if $y = 0$. In other words, all the zeros of $\psi(z)$ and $\psi(z)+\gamma$ are real. Since

$$\psi'(x) = \frac{1}{x^2} + \sum_{n=0}^{\infty} \frac{1}{(x+n)^2}$$

is positive, $\psi(x)$ is a **strictly increasing** function of x in each of the intervals $I_n = (-n, -n+1)$ for $n = 1, 2, \ldots$. For notational convenience, we let $I_0 = (0, \infty)$. Thus, there is a unique real zero x_n in each of the intervals I_n for $n = 0, 1, 2, \ldots$ In particular, there is a unique positive zero x_0. This observation is a special case of a general theorem of Laguerre (see Theorem 2.8.1 on page 23 of [3]). This theorem states the following. If $f(z)$ is an entire function, not a constant, which is real for real z and has only real zeros, and is of order 0 or 1, then the zeros of $f''(z)$ are also real and are separated by the zeros of $f(z)$. The result above for the digamma function follows from considering $f(z) = 1/\Gamma(z)$.

In this paper, we will prove:

Theorem 1.3 *All the zeros of $\psi(x)$ defined as x_n are irrational for $n \geq 0$, with at most one exception.*

We conjecture that they are all in fact transcendental numbers. But we are unable to prove this using our present state of knowledge. What our proof shows is that all the x_n's are irrational if γ is not a general Baker period.

2 Preliminary results on the digamma function

In this section, we will review various results needed in the proofs of Theorems 1.2 and 1.3. The first is a famous formula of Gauss (see page 300 of [5]) discovered in 1813. It gives an explicit formula for $\psi(a/q)$ as a linear form in logarithms of algebraic numbers alluded to earlier.

Proposition 1.1 (Gauss, 1813)
For $1 \leq a < q$*, with* $(a, q) = 1$*, we have*

$$\psi(a/q) + \gamma = -\log 2q - \frac{\pi}{2}\cot\frac{\pi a}{q} + 2\sum_{0<j\leq q/2}\left(\cos\frac{2\pi a j}{q}\right)\log\sin\frac{\pi j}{q}.$$

Let us first note that $\cot \pi a/q$ is algebraic for any rational a/q. Using the principal branch of the logarithm and noting that $2\log i = \pi$, we see that $\psi(a/q) + \gamma$ is a $\overline{\mathbb{Q}}$- linear form in logarithms. Baker's theory tells us that such a linear form is either zero or transcendental. By the monotonicity of the digamma function, $\psi(a/q) + \gamma < \psi(1) + \gamma = 0$ and so the linear form above is non-zero and hence transcendental. (This is a simpler proof of the transcedence of $\psi(a/q) + \gamma$ than the one given in [5].)

We will show below that there is at most one rational number x for which $\psi(x)$ is zero.

3 The digamma function at rational arguments

Let us consider all the rational numbers in the interval $(0, 1]$. We will first prove:

Theorem 1.4 *For any two distinct rational numbers* $x, x' \in (0, 1)$*, the difference* $\psi(x') - \psi(x)$ *is a non-zero Baker period and hence transcendental. In particular, for any rational number* $x \in (0, 1]$*,* $\psi(x)$ *is transcendental with at most one possible exception.*

Proof Suppose we have two distinct rational numbers $x, x' \in (0, 1)$ such that both $\psi(x)$ and $\psi(x')$ are algebraic. Without any loss of generality, we may suppose $x < x'$. By the monotonicity of the digamma function,

$$0 < \psi(x') - \psi(x) = (\psi(x') + \gamma) - (\psi(x) + \gamma)$$

and the two terms in brackets on the right hand side are Baker periods by Gauss's formula (Proposition 1.1). As we have a strict inequality, their difference is a non-zero Baker period and hence transcendental, contradicting the algebraicity of both $\psi(x)$ and $\psi(x')$. Thus, there can be at most one rational x in $(0, 1]$ for which $\psi(x)$ is algebraic. This proves the theorem. $\square$

We want to extend the above result to all rational numbers. To do this, we define an equivalence relation on the set of rational numbers

$$S = \mathbb{Q} \backslash \{0, -1, -2, \ldots\}.$$

We will say that two rational numbers of S are equivalent if they are equal modulo 1. This partitions S into equivalence classes. Our theorem now implies:

Corollary 1.1 *For any two distinct non-equivalent $x, x' \in S$, the difference $\psi(x) - \psi(x')$ is a (non-vanishing) general Baker period and hence transcendental. In particular, there is at most one rational number x (up to equivalence) for which $\psi(x)$ is algebraic.*

Proof Suppose we have two distinct rational numbers x, x' such that $\psi(x)$ and $\psi(x')$ are both algebraic. If x, x' both lie in I_n, the proof of Theorem 1.4 extends to give the desired result by the monotonicity of the digamma function in I_n. Thus, we can assume x and x' lie in two distinct intervals. Let $s, s' \in (0, 1)$ such that $x \equiv s \pmod 1$ and $x' \equiv s' \pmod 1$. Then

$$\psi(x) = \psi(s) + \text{rational number}, \qquad \psi(x') = \psi(s') + \text{rational number}$$

so that

$$\text{algebraic number } = \psi(x) - \psi(x') = \psi(s) - \psi(s') + \text{rational number}.$$

As x and x' are non-equivalent, $s \not\equiv s' \pmod 1$ and by Theorem 1.4, $\psi(s) - \psi(s')$ is a non-zero Baker period. By Baker's theorem, any algebraic number added to a non-zero Baker period is again transcendental. This contradiction proves the theorem. □

4 Zeros of $\psi(x) + \gamma$ and proofs of Theorems 1.1 and 1.2

Gauss's theorem (Proposition 1.1), along with the first of the two functional equations (3) immediately implies that for any rational number x unequal to an integer, the number $\psi(x) + \gamma$ is a general Baker period. Thus, by Baker's theorem, it is transcendental.

Let y_n be a zero of $\psi(x) + \gamma$ lying in the interval $(-n, -n + 1)$. Let us assume it is rational. Using the first of the two functional equations (3), we see that

$$0 = \psi(y_n) + \gamma = [\psi(w_n) + \gamma] - \sum_{j=0}^{n-1} \frac{1}{y_n + j}$$

with w_n a positive rational number in the interval $(0, 1)$ equivalent to y_n. As we are assuming y_n is rational, we see that the sum on the right hand side above is a rational number. Moreover, $w_n < 1$ because y_n is not an integer. By Gauss's formula, $\psi(w_n) + \gamma$ is a non-zero Baker period. By Baker's theorem, the entire expression on the right hand side is transcendental, which is a contradiction because it equals zero. Thus, y_n is irrational. This proves both Theorems 1.1 and 1.2.

5 Proof of Theorem 1.3

By Corollary 1.1, if $\psi(x) = \psi(x') = 0$ for two distinct rational numbers in S, they must be equivalent. In other words, x and x' differ by an integer. Without any loss of generality, let us suppose that $x < x'$ so that $x + n = x'$ for some positive integer n, with n being the smallest such integer with this property. Thus, $x + n - 1 < 0$. Then, by the first of the two functional equations (3)

$$0 = \psi(x') = \psi(x+n) = \psi(x) + \sum_{j=0}^{n-1} \frac{1}{x+j} = \sum_{j=0}^{n-1} \frac{1}{x+j}.$$

If $x' < 0$, then $x < 0$ and each term $x + j < x + n = x' < 0$ in the summation, is strictly negative and hence cannot sum to zero. If $x' > 0$, then $x + n - 1 < 0$ and again the same argument applies. This completes the proof.

6 Concluding remarks

It would be of interest to show that x_0 is irrational but this looks quite difficult. The positive zero $x_0 = 1.461632...$ can be written as $x_0 = 1 + \frac{a}{q}$ with $(a, q) = 1$. Thus, $0.45 < a/q < 0.5$. By the first of the two functional equations (3), we have

$$0 = \psi(1 + a/q) = \psi(a/q) + \frac{q}{a}.$$

On the other hand, by Gauss's formula, we deduce that γ is a Baker period:

$$\gamma = \frac{q}{a} - \log 2q - \frac{\pi}{2} \cot \frac{\pi a}{q} + 2 \sum_{0<j\le q/2} \left(\cos \frac{2\pi a j}{q}\right) \log \sin \frac{\pi j}{q}. \tag{4}$$

Thus, if x_0 is rational, we deduce that γ is a general Baker period, which is highly unlikely.

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